ATTRIBUTIONS

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“Arithmetic for College Readiness, 3rd Edition” by Scottsdale Community College

“Basic Arithmetic Student Workbook, 2nd Ed.” by Scottsdale Community College

“College Mathematics, 1st Edition” by Scottsdale Community College

“Math in Society” by David Lippman
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UNIT 1 – SYSTEMS OF MEASURE AND UNIT CONVERSIONS

INTRODUCTION

In this Unit, we will begin our study of Geometry by investigating what it means to measure an object, and what attributes of an object we can measure. We will learn to measure objects in various ways, compare measurements, and convert between different units and systems of measure.

Section 1.1: Distinguish between 1, 2, and 3 dimensional measures

Section 1.2: Measure length

Section 1.3: Measure area

Section 1.4: Measure volume

Section 1.5: Convert U.S. measurements using tables

Section 1.6: Convert simple U.S. measurements using dimensional analysis

Section 1.6: Convert multi-unit U.S. measurements

Section 1.6: Convert multi-step U.S. measurements

Section 1.7: Convert simple metric measurements using a table or dimensional analysis

Section 1.8: Convert between U.S. and metric systems using dimensional analysis

Section 1.9: Convert rates using dimensional analysis
UNIT 1 – MEDIA LESSON

When we want to communicate the size of an object, we talk about its measure. Most objects have many different attributes that we can measure. For example, a 1-dimensional attribute of an object is its length (the distance between two points). A 2-dimensional attribute of an object is its area (the size of the surface of the object). In 3-dimensions, we talk about an object’s volume (the holding capacity of the object, or how much space it takes up).

So what does it mean to measure an object? First we need to know what attribute that we plan to measure. For example, suppose that you are designing shelves in your garage to hold storage boxes. What attributes of the box would we want to measure to help with the design?

In order to plan for the depth of our shelves, we would need to know the length of the box. If we want to know how many boxes will fit on the shelf, it would be helpful to know how much space the base of the box takes up, so we’d want to measure the area of the box’s base. If we’re thinking about how much we can store in each of the boxes, then we might want to know the volume of the boxes. Once we know what attribute that we want to measure, we can compare the attribute of the object to a known quantity of the same attribute.

SECTION 1.1: UNDERSTANDING DIMENSION

We say that an object is 1-dimensional if at each location, there is only 1 independent direction to move within the object. For example, in a 1-dimensional world, a creature could only move forward/backward. Some examples of 1-dimensional objects are: line segments, the outer edge of a circle, the line segments making up a rectangle, or the edge where two walls meet.

We say that an object is 2-dimensional if at each location in the object, there are 2 independent directions along which to move within the object. For example, in a 2-dimensional world, a creature could move forward/backward, or right/left. Some examples of 2-dimensional objects are: a piece of paper, the inside of a circle, the inside of a rectangle, the surface of a wall, or the surface of the base of a box.

We say an object is 3-dimensional if at each location, there are 3 independent directions along which to move within the object. For example, our world is 3-dimensional. We can move forward/backward, right/left, or up/down. Some examples of 3-dimensional objects are: the earth, the inside of a box, the feathers that fill a pillow, the contents of a soda bottle.
**Problem 1: Worked Example - Understanding Dimension**

Determine whether the following describe a 1-dimensional, 2-dimensional, or 3-dimensional measure.

**Solution:** It is useful to think of the object and determine whether you can

1) Draw as line segments or curvy lines:

2) Cover the object with squares or a curvy shape that has an inside:

3) Fill the object with cubes or shape that covers a bottom and fills “up”.

---

a) The amount of tile needed for the bathroom floor
   “Cover” with tile: 2 – dimensional

b) The length of the baseboard:
   “Length” can be drawn with lines: 1 – dimensional

c) The amount of paint needed to paint the bathroom
   Paint can may be “filled up”: 3 – dimensional

d) A footprint on a bathtub mat
   “Cover” with a foot: 2 – dimensional

e) The amount of water that a bathtub will hold
   Bathtub can be “filled up” with water: 3 – dimensional
**Problem 2: You Try - Understanding Dimension**

Determine whether the following describe a 1-dimensional, 2-dimensional, or 3-dimensional measure.

**a)** The distance from home to campus: _______________________________

**b)** The diagonal of your cell phone screen: __________________

**c)** The top surface of your cell phone screen: _______________________________

**d)** The amount of energy drink that a bottle will hold: _______________________________

**e)** Describe one-dimensional, two dimensional, and three-dimensional aspects of a swimming pool. What are some practical reasons for wanting to know these measurements?

**SECTION 1.2: MEASURING LENGTH**

*Length* can be thought of as the distance between two points. We measure length to answer the question “how long”, “how far”, or “how wide”? In order to measure the length of our box, we simply compare it to some known length. There are many tools that can be used to measure length; the most common tool is a ruler. Some standard units of length that we might use for comparison are inches, feet, or centimeters. These are units of length that are understood by everyone. But we really could measure our box by comparing it to any known length. Once we choose our measurement unit, then we need to determine how many times as large the length of the box is compared to the known length that we are using for comparison. The most direct way to measure a length is to count how many of the units are in the quantity to be measured.

*A system of measurement* is a collection of standard units. In the U.S. there are two systems of measurement that are commonly used: U.S. Customary system and the Metric System. The U.S. Customary System is derived from the British system of measure and will be familiar to you. The Metric system is more commonly used around the world, and is much easier to understand and to convert between units since it is based on the decimal system of numbers.
In the metric system units are created in a uniform way. For any quantity to be measured, there is a base unit (meter, liter, gram), then the base unit is paired with a prefix that indicates the unit’s relationship to the base unit. For example, the prefix *kilo* means *thousand*, so a *kilometer* is a thousand meters. Many of the metric prefixes are only used in scientific contexts. The table below lists some of the commonly used metric prefixes.

### Metric Prefixes

<table>
<thead>
<tr>
<th>Prefix</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nano</td>
<td>Billionth</td>
</tr>
<tr>
<td>Micro</td>
<td>Millionth</td>
</tr>
<tr>
<td>Milli</td>
<td>Thousandth</td>
</tr>
<tr>
<td>Centi</td>
<td>Hundredth</td>
</tr>
<tr>
<td>Deci</td>
<td>Tenth</td>
</tr>
<tr>
<td>Base Unit</td>
<td>One</td>
</tr>
<tr>
<td>Deka</td>
<td>Ten</td>
</tr>
<tr>
<td>Hecto</td>
<td>Hundred</td>
</tr>
<tr>
<td>Kilo</td>
<td>Thousand</td>
</tr>
<tr>
<td>Mega</td>
<td>Million</td>
</tr>
<tr>
<td>Giga</td>
<td>Billion</td>
</tr>
</tbody>
</table>

### Standard Units of Length

<table>
<thead>
<tr>
<th>U.S. Customary System</th>
<th>Metric System</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Unit</strong></td>
<td><strong>Abbreviation</strong></td>
</tr>
<tr>
<td>inch</td>
<td>in</td>
</tr>
<tr>
<td>foot</td>
<td>ft</td>
</tr>
<tr>
<td>yard</td>
<td>yd</td>
</tr>
<tr>
<td>mile</td>
<td>mi</td>
</tr>
</tbody>
</table>

You are likely familiar with the size of the units in the US Customary System, but it good to have some sense of the size of the common metric measures. For example, a millimeter is about the size of the width of a dime. A centimeter is about the width of a small fingernail (there are approximately two and a half cm in an inch). A meter is about a yard. A kilometer is 0.6 mi – so a little more than half of a mile.
Unit 1 – Systems of Measure and Conversions

**Problem 3: Worked Example – Measuring Length**

a) Measure the length of line segment AB using the ruler below. The unit of measurement is centimeters.

Solution: Note that the line segment spans the length from 2 cm to 6 cm. So the length is $6 \text{ cm} - 2 \text{ cm} = 4 \text{ cm}$.

Length: **4 cm**

b) You want to install baseboard on the floors of your living room. Measure the distance around the edge of the living room in the diagram to the right to determine the length of baseboard required.

Solution: You need to measure all 6 sides of the room and add the results. In the diagram below all six sides are measured by finding the 1-dimensional length of the sides. Notice you are counting the *sides of the squares* and not the squares (2-dimensional).

Once you find the length of each side, find the sum of the sides. Make sure to use a systematic method to ensure you count all of the sides and do not double count any of them.

Sum: $7 \text{ ft.} + 3 \text{ ft.} + 3 \text{ ft.} + 5 \text{ ft.} + 4 \text{ ft.} + 8 \text{ ft.} = 40 \text{ ft.}$

Note: 6 sides and 6 measurements added

Length: **40 feet**

Answer: You will need 40 linear feet of baseboard.
c) Determine what units would be appropriate to use to measure the following lengths.

<table>
<thead>
<tr>
<th>Item</th>
<th>U.S. customary unit</th>
<th>Metric unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>i. The distance from home to campus</td>
<td>miles</td>
<td>kilometers</td>
</tr>
<tr>
<td>ii. The height of a bedroom</td>
<td>feet</td>
<td>meters</td>
</tr>
<tr>
<td>iii. The length of an ant</td>
<td>inches</td>
<td>millimeters or centimeters</td>
</tr>
</tbody>
</table>

**Problem 4: You Try – Measuring Length**

a) Find the length of the ladybug to the nearest millimeter.

Length: ______________

b) The figure below represents a 3-Dimensional box. We will call the length the distance from left to right. We will call the height the distance from bottom to top. We will call the width the distance from front to back. The side length of each small square in the grid represents 1 cm.

Find the length and height of the box. Include units in your answers.

Length: ______________

Height: ______________
c) Determine what units would be appropriate to use to measure the following lengths

<table>
<thead>
<tr>
<th>Item</th>
<th>U.S. customary unit</th>
<th>Metric unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>i. The distance from AZ to NY</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ii. The depth of a swimming pool</td>
<td></td>
<td></td>
</tr>
<tr>
<td>iii. The diagonal of your cellphone</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**SECTION 1.3: MEASURING AREA**

*Area* can be thought of as the amount of space within the boundaries of a 2-dimensional shape. We measure area when we are trying to answer questions like, “how much material will it take to make this”, or “how much space do I need on my shelf to fit this”? In order to measure area, we must compare our object to a known unit of area, and we determine how many units (including partial units) it would take to cover the object without gaps or overlaps.

Some standard units of area are square inches (in² - a square that has 1-in long sides), square feet (ft² – a square that has 1-foot long sides), and square centimeters (cm² – a square that has 1-cm long sides). Once we decide on the unit area that we will use, we need to determine how many times larger our object's area is than the unit area is. More simply, we could count how many of the units it takes to completely cover our object.

**Units of Area**

**U.S. Customary System**

<table>
<thead>
<tr>
<th>Unit</th>
<th>Abbreviation</th>
<th>Relationships</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square inch</td>
<td>in²</td>
<td></td>
</tr>
<tr>
<td>Square foot</td>
<td>ft²</td>
<td>1 ft² = 12² in² = 144 in²</td>
</tr>
<tr>
<td>Square yard</td>
<td>yd²</td>
<td>1 yd² = 3² ft² = 9 ft²</td>
</tr>
<tr>
<td>Square mile</td>
<td>mi²</td>
<td></td>
</tr>
<tr>
<td>Acre</td>
<td></td>
<td>1 acre = 43,560 ft²</td>
</tr>
</tbody>
</table>
Notice that each unit of length has an associated unit of area. The area unit is the square with the given side length. For example, a square inch looks like a square whose side lengths are 1 inch long.

**Problem 5: Worked Example – Measuring Area**

a) Find the area of the shaded figure to the right. The side of each square represents 1 cm.

Solution: Since each square has a side length of 1 cm, each square has an area of 1 square centimeter. Count the number of squares that are shaded in the figure.

In the diagram below, each square is labeled with a number to find the total number of squares. This is not the only way to label the squares, but you should use a systematic method when counting so you do not miss any or double count.

Area: 15 square centimeters
b) Find the area of the front of the box in Figure A. Each square has a side length of 1 inch.

Solution: The front of the box is a 2-dimensional surface area. It is covered with squares each with an area of 1 square inch. We can count the number of squares.

In Figure B, the squares are counted and labeled by rows. There are other ways to label and count the squares. Just be sure to use a systematic method so you do not miss any or count any twice.

Note you may also use patterns to find the number of squares. There are 4 rows with 9 squares each or \(4 \cdot 9 = 36\) total squares. This corresponds with the area of a rectangle in which you may be familiar.

Area: 36 square inches

<table>
<thead>
<tr>
<th>Item</th>
<th>U.S. customary unit</th>
<th>Metric unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>i. The area of Scottsdale</td>
<td>square miles</td>
<td>square kilometers</td>
</tr>
<tr>
<td>ii. The area of a sheet of paper</td>
<td>square inches</td>
<td>square centimeters</td>
</tr>
<tr>
<td>iii. The area of the side of a building</td>
<td>square feet</td>
<td>square meters</td>
</tr>
</tbody>
</table>
Problem 6: You Try – Measuring Area

a) Find the area of the shaded figure below. The side of each square represents 1 yard.

![Shaded Figure]

Area: ___________________________

b) Create and shade two different shapes in the grids below that cover 9 square units.

![Grids with Shapes]

c) What units would be appropriate to use to measure the following?

<table>
<thead>
<tr>
<th>Item</th>
<th>U.S. customary unit</th>
<th>Metric unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>i. The area of Arizona</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ii. The area of your cellphone’s screen</td>
<td></td>
<td></td>
</tr>
<tr>
<td>iii. The area of your kitchen table</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
SECTION 1.4: MEASURING VOLUME

**Volume** is the space taken up by a 3-dimensional object. We measure volume when we want to answer questions like “how many sugar cubes would it take to fill this box”, “how much air is in this room”, or “how much water will it take to fill the pool”? In order to measure volume, we must compare our object to a known unit of volume, and we determine how many units (including partial units) it would take to completely fill the object.

Some standard units of volume are cubic inches (in$^3$ - a cube that has 1-in long sides), cubic feet (ft$^3$ – a cube that has 1-foot long sides), and cubic centimeters (cm$^3$ – a cube that has 1-cm long sides). Once we decide on the unit of volume that we will use, we need to determine how many times larger our object’s volume is than the unit volume is. More simply, we could count how many of the units it takes to completely fill our object.

**Units of Volume and Capacity (liquid volume)**

**U.S. Customary System**

<table>
<thead>
<tr>
<th>Unit</th>
<th>Abbreviation</th>
<th>Relationships</th>
</tr>
</thead>
<tbody>
<tr>
<td>cubic inch</td>
<td>in$^3$</td>
<td></td>
</tr>
<tr>
<td>cubic foot</td>
<td>ft$^3$</td>
<td>$1 \text{ ft}^3 = 12 \text{ in}^3 = 1728 \text{ in}^3$</td>
</tr>
<tr>
<td>cubic yard</td>
<td>yd$^3$</td>
<td>$1 \text{ yd}^3 = 3 \text{ ft}^3 = 27 \text{ ft}^3$</td>
</tr>
<tr>
<td>teaspoon</td>
<td>tsp</td>
<td></td>
</tr>
<tr>
<td>tablespoon</td>
<td>T or tbsp</td>
<td>$1 \text{ T} = 3 \text{ tsp}$</td>
</tr>
<tr>
<td>fluid ounce</td>
<td>fl oz</td>
<td>$1 \text{ fl oz} = 2 \text{ T}$</td>
</tr>
<tr>
<td>cup</td>
<td>c</td>
<td>$1 \text{ c} = 8 \text{ fl oz}$</td>
</tr>
<tr>
<td>pint</td>
<td>pt</td>
<td>$1 \text{ pt} = 2 \text{ c}$</td>
</tr>
<tr>
<td>quart</td>
<td>qt</td>
<td>$1 \text{ qt} = 2 \text{ pt}$</td>
</tr>
<tr>
<td>gallon</td>
<td>gal</td>
<td>$1 \text{ gal} = 4 \text{ qt}$</td>
</tr>
</tbody>
</table>
Metric System

<table>
<thead>
<tr>
<th>Unit</th>
<th>Abbreviation</th>
<th>Relationships</th>
</tr>
</thead>
<tbody>
<tr>
<td>cubic millimeter</td>
<td>mm³</td>
<td></td>
</tr>
<tr>
<td>cubic centimeter</td>
<td>cm³</td>
<td>1 cm³ = 10³ mm³ = 1000 mm³</td>
</tr>
<tr>
<td>cubic meter</td>
<td>m³</td>
<td>1 m³ = 100³ cm³ = 1,000,000 cm³</td>
</tr>
<tr>
<td>cubic kilometer</td>
<td>km³</td>
<td>1 km³ = 1000³ m³ = 1,000,000,000 m³</td>
</tr>
<tr>
<td>milliliter</td>
<td>mL</td>
<td>1 mL = 1 cm³</td>
</tr>
<tr>
<td>centiliter</td>
<td>cL</td>
<td>1 cL = 10 mL = 10 cm³</td>
</tr>
<tr>
<td>liter</td>
<td>L</td>
<td>1 L = 100 cL = 1000 mL = 1000 cm³</td>
</tr>
<tr>
<td>kiloliter</td>
<td>kL</td>
<td>1 kL = 1000 L = 100,000 cL = 1,000,000 mL</td>
</tr>
</tbody>
</table>

Problem 7: Media Example – Measuring Volume

a) The figure below is the front view of a 3 dimensional object made up of stacked cubes. How many cubes make up the volume of this figure including the ones we cannot see?
b) Determine the volume of the toy staircase shown by imagining that it is filled with centimeter cubes.

![Toy staircase diagram]

1 cm

1 cm

10 cm

c) What units would be appropriate to use to measure the following?

<table>
<thead>
<tr>
<th>Item</th>
<th>U.S. customary unit</th>
<th>Metric unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>i. The amount of water in a bathtub</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ii. The amount of coffee in a mug</td>
<td></td>
<td></td>
</tr>
<tr>
<td>iii. The amount of helium in a balloon</td>
<td></td>
<td></td>
</tr>
<tr>
<td>iv. The amount of fluid in single tear of joy</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Problem 8: You Try – Measuring Volume

a) Determine the volume of the following shape by imagining it is filled with centimeter cubes.

![Shape diagram]

1 cm

5 cm

1 cm

5 cm
b) What units would be appropriate to use to measure the following?

<table>
<thead>
<tr>
<th>Item</th>
<th>U.S. customary unit</th>
<th>Metric unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>i. The amount of water in a pool</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ii. The amount of water in a bottle</td>
<td></td>
<td></td>
</tr>
<tr>
<td>iii. The amount of air in a room</td>
<td></td>
<td></td>
</tr>
<tr>
<td>iv. The amount of fluid in an allergy shot</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

SECTION 1.5: INTRODUCTION TO CONVERTING MEASURES

Recall that measurement is just a comparison between the attribute of an object that we want to measure, and a known quantity with the same attribute. For example, if we want to measure the length of a pencil, we compare the length of the pencil with the length of an inch. We ask ourselves the question, “how many copies of an inch would it take to make the length of this pencil”, or, “how many times larger than an inch is this pencil”? But we could have chosen to compare the length of the pencil with the length of a centimeter. Either approach is valid.

Sometimes we know a measurement in a particular unit, but we are interested in the value of the measurement in a different unit. Suppose we know that the length of a table is 7ft, but we want to know what the value of the measurement is in inches. This process of converting a measurement from one unit to another is called unit conversion. We can convert between units within a measurement system or between measurement systems.

Below is a list showing the primary units of measure in the US Customary system of measurement along with conversions between units. This table is a convenient tool when you need to convert between units.
# Unit 1 – Systems of Measure and Conversions

## US Unit Conversions

<table>
<thead>
<tr>
<th>Type of Measure</th>
<th>Unit / Abbreviations</th>
<th>Conversions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Length</strong></td>
<td>Inches = in.</td>
<td>1 ft. = 12 in.</td>
</tr>
<tr>
<td></td>
<td>Feet = ft.</td>
<td>1 yd. = 3 ft.</td>
</tr>
<tr>
<td></td>
<td>Yards = yd.</td>
<td>1 mi. = 5280 ft.</td>
</tr>
<tr>
<td></td>
<td>Miles = mi</td>
<td></td>
</tr>
<tr>
<td><strong>Mass/Weight</strong></td>
<td>Ounces = oz.</td>
<td>1 lb. = 16 oz.</td>
</tr>
<tr>
<td></td>
<td>Pounds = lb.</td>
<td>1 ton = 2000 lb.</td>
</tr>
<tr>
<td></td>
<td>ton</td>
<td></td>
</tr>
<tr>
<td><strong>Area</strong></td>
<td>Square inches = in²</td>
<td>144 in² = 1 ft²</td>
</tr>
<tr>
<td></td>
<td>Square feet = ft²</td>
<td>1 yd² = 9 ft²</td>
</tr>
<tr>
<td></td>
<td>Square yards = yd²</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Square miles = mi²</td>
<td></td>
</tr>
<tr>
<td><strong>Volume</strong></td>
<td>Fluid ounces = oz.</td>
<td>1 c = 8 oz.</td>
</tr>
<tr>
<td></td>
<td>Cups = c</td>
<td>1 pt. = 2 c</td>
</tr>
<tr>
<td></td>
<td>Quarts = qt</td>
<td>1 gal = 4 qt</td>
</tr>
<tr>
<td></td>
<td>Gallons = gal</td>
<td>1728 in³ = 1 ft³</td>
</tr>
<tr>
<td></td>
<td>Cubic feet = ft³</td>
<td>27 ft³ = 1 yd³</td>
</tr>
<tr>
<td></td>
<td>Cubic yards = yd³</td>
<td></td>
</tr>
<tr>
<td><strong>Time</strong></td>
<td>Seconds = sec</td>
<td>1 min = 60 sec</td>
</tr>
<tr>
<td></td>
<td>Minutes = min</td>
<td>1 hr. = 60 min</td>
</tr>
<tr>
<td></td>
<td>Hours = hr.</td>
<td>1 day = 24 hr.</td>
</tr>
<tr>
<td></td>
<td>Days = day</td>
<td>1 wk. = 7 days</td>
</tr>
<tr>
<td></td>
<td>Weeks = wk.</td>
<td>1 yr. = 52 wk.</td>
</tr>
<tr>
<td></td>
<td>Months = mo.</td>
<td>1 yr. = 12 mo.</td>
</tr>
<tr>
<td></td>
<td>Years = yr.</td>
<td>1 yr. = 365 days</td>
</tr>
</tbody>
</table>
Problem 9: Worked Example – Using Tables to Convert Between U.S. Units

Use the table below to write the corresponding values for each unit of measure and find the indicated conversion.

a) Fill in the missing values in the table.

<table>
<thead>
<tr>
<th>Quarts</th>
<th>0 qt.</th>
<th>1 qt.</th>
<th>2 qt.</th>
<th>3 qt.</th>
<th>4 qt.</th>
<th>5 qt.</th>
<th>6 qt.</th>
<th>7 qt.</th>
<th>8 qt.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cups</td>
<td>0 c.</td>
<td>____ c.</td>
<td>____ c.</td>
<td>____ c.</td>
<td>____ c.</td>
<td>____ c.</td>
<td>____ c.</td>
<td>____ c.</td>
<td>____ c.</td>
</tr>
</tbody>
</table>

b) Use the table to find the following conversions.

4 quarts = ____ pints
6 pints = ____ cups
3 quarts = ____ cups
28 cups = ____ quarts

Solution:

a) From the conversion table, we know that 1 quart = 2 pints so we can place this in the table. Since the quart row is increasing by 1 quart, the pint row will increase by 2 pints. So we can fill the pint row by adding 2 pints to the previous amount.

We also know that 1 pint = 2 cups so we can place this in the table. Since the pint row is increasing by 2 pints, the cup row will increase by $2 \cdot 2 = 4$ cups. So we can fill the cup row by adding 4 cups to the previous amount.

b) To find the conversions below, we can find the corresponding measurement from the table.

4 quarts = 8 pints
6 pints = 12 cups
3 quarts = 12 cups
28 cups = 7 quarts
Problem 10: You Try – Using Double Number Lines to Convert Between U.S. Units

Use the table below to write the corresponding values for each unit of measure and find the indicated conversion.

a) Fill in the missing values in the table.

<table>
<thead>
<tr>
<th>Yards</th>
<th>0 yd.</th>
<th>1 yd.</th>
<th>2 yd.</th>
<th>3 yd.</th>
<th>4 yd.</th>
<th>5 yd.</th>
<th>6 yd.</th>
<th>7 yd.</th>
<th>8 yd.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Feet</td>
<td>0 ft.</td>
<td>____ ft.</td>
<td>____ ft.</td>
<td>____ ft.</td>
<td>____ ft.</td>
<td>____ ft.</td>
<td>____ ft.</td>
<td>____ ft.</td>
<td>____ ft.</td>
</tr>
<tr>
<td>Inches</td>
<td>0 in.</td>
<td>____ in.</td>
<td>____ in.</td>
<td>____ in.</td>
<td>____ in.</td>
<td>____ in.</td>
<td>____ in.</td>
<td>____ in.</td>
<td>____ in.</td>
</tr>
</tbody>
</table>

b) Use the table to find the following conversions.

6 yd. = ______ ft.  15 ft. = ______ in.  7 yd. = ______ in.  144 inches = ______ yd.

SECTION 1.6: DIMENSIONAL ANALYSIS AND U.S. CONVERSIONS

One question that students often ask is whether they should multiply or divide to convert between two units of measure. We will use a method called **dimensional analysis** where we always multiply by a conversion factor written in fraction form.

When you multiply by a fraction, you can think of the numerator of the fraction as making copies or multiplying and the denominator of the fraction as cutting into groups or dividing. So multiplying by a fraction is equivalent to the idea of multiplying or dividing to convert between units. However, when we use a conversion factor that is a fraction with our units labeled, we can use dimensional analysis to be certain we are operating in the appropriate way.

Consider the following conversion questions.

How many inches are in 3 feet?  How many feet are in 18 inches?

*Conversion Equation:* 1 foot = 12 inches  
*Conversion Factors:* \( \frac{1 \text{ foot}}{12 \text{ inches}} = \frac{12 \text{ inches}}{1 \text{ foot}} = 1 \)

Notice that the conversion factors are fractions that are both equal to 1. This may seem odd because there are different numbers in the numerator and denominator. However, since 12 inches = 1 foot, dividing one by other equals 1 when we include the units of measure.
Recall that multiplying by 1 does not change the value of a number, but creates an equivalent form. So we can multiply the given numbers by the appropriate conversion factors to change our units.

\[
3 \text{ feet} = \frac{3 \text{ feet}}{1} \times \frac{12 \text{ inches}}{1 \text{ foot}} = 3 \times 12 \text{ inches} = 36 \text{ inches}
\]

Notice we drew a line crossing out feet in the numerator and foot in the denominator leaving only inches in the numerator. Dimensional analysis helps keep track of units until we have the correct unit remaining. For the second conversion, we will use the other conversion factor to make inches cancel to 1 (instead of division) and the units of feet remain.

\[
18 \text{ inches} = \frac{18 \text{ inches}}{1} \times \frac{1 \text{ foot}}{12 \text{ inches}} = \frac{18}{1} \times \frac{1}{12} \text{ feet} = \frac{18}{12} \text{ feet} = 1.5 \text{ feet}
\]

It is true that to change from feet to inches, we multiply by 12 and to change from inches to feet we divide by 12. When you are very comfortable with the units of measure, it is fine to use this process. However, to be certain you are converting correctly, it is highly recommended that you use dimensional analysis to ensure the correct conversion.

**Problem 11: Media Example – Simple U.S. Unit Conversions**

For each problem, write the conversion equation, conversion factors, and conversion multiplication to convert the unit of measure.

<table>
<thead>
<tr>
<th>Convert</th>
<th>Conversion Equation</th>
<th>Possible Conversion Factors</th>
<th>Conversion Process</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 4 lb. to oz.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b) 10 yd. to ft.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c) 2.4 pt. to c.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
d) Sarah needs 1.5 cups of ketchup to make her famous meatloaf recipe. She has a brand new, 20-oz bottle of ketchup in her cupboard. How many ounces of ketchup will she need for her meatloaf?

e) Your new truck weighs 8000 lbs. How many tons is this?

**Problem 12:** You Try – Simple U.S. Unit Conversions

For each problem, write the conversion equation, conversion factors, and conversion multiplication to convert the unit of measure.

<table>
<thead>
<tr>
<th>Convert</th>
<th>Conversion Equation</th>
<th>Possible Conversion Factors</th>
<th>Conversion Process</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 7 qt. to gal.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b) 330 minutes to hours</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c) Your friend Sara writes to you saying that she will be away for 156 weeks. How many years will she be gone?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
d) Carlton ran $4 \frac{1}{2}$ miles this morning. How many feet did he run?

e) Shari is counting the hours until her vacation. She just realized that she has 219 hours to go! How many days before she goes?

**Problem 13:** Media Example – Multi-Unit U.S. Conversions

The following examples illustrate additional basic conversions within the U.S. System. A modified form of the conversion process will be used for these problems.

a) Write 26 inches in feet and inches.

b) Write 5 lbs., 6 oz. in ounces.

c) Write 30 months in months and years.

d) Write 1 min, 20 sec in seconds.
Problem 14: Media Example – Multi-Unit U.S. Conversions

Some conversions require more than one step. See how the single-step conversion process is expanded in each of the following problems.

a) How many minutes are in a week?

b) Bryan needs 10 cups of fruit juice to make Sangria. How many quarts of juice should he buy at the grocery store?

c) Rick measured a room at 9 ft. long by 10 ft. wide to get an area measurement of 90 square feet (area of a rectangle is length times width). He wants to carpet the room with new carpet, which is measured in square yards. Rick knows that 1 yd is equivalent to 3 ft. so he ordered 30 square yards of carpet. Did he order the correct amount?
Problem 15: You Try – Multi-Unit and Multi-Step Conversions

Perform each of the following conversions within the U.S. system. Round to tenths as needed. Show complete work.

a) A young girl paced off the length of her room as approximately 8 feet. How many inches would that be?

b) 18 oz. = ____ lb.

c) 100 yd = ____ ft.

d) 10,235 lb. = ____ tons

e) How many inches are in 6 feet, 8 inches?

f) How many square inches are in 10 square feet?
SECTION 1.7: UNIT CONVERSIONS IN THE METRIC SYSTEM

The strength of the *metric system* is that it is based on powers of ten as you can see in the chart below. Prefixes are the same for each power of ten above or below the base unit. This also makes conversions easy in the metric system. Some common base units are meters for length, liters for capacity, and grams for mass.

**Metric Chart**

<table>
<thead>
<tr>
<th>KILO</th>
<th>HECTO</th>
<th>DEKA</th>
<th>NO PREFIX</th>
<th>DECI</th>
<th>CENTI</th>
<th>MILLI</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000 × Base</td>
<td>100 × Base</td>
<td>10 × Base</td>
<td>Base Unit</td>
<td>0.10 × Base</td>
<td>0.01 × Base</td>
<td>0.001 × Base</td>
</tr>
</tbody>
</table>

**Some Common Metric Conversions**

- 1 centimeter (cm) = 10 millimeters (mm)
- 1 meter (m) = 100 centimeters (cm)
- 1 kilometer (km) = 1000 meters (m)

**Problem 16: Media Example – Simple Metric Unit Conversions**

Use a metric chart to convert the metric units.

a) \[4200 \text{ g} = \underline{_______} \text{ mg}\]

b) \[45 \text{ cm} = \underline{_______} \text{ m}\]

c) \[7,236,137 \text{ ml} = \underline{_______} \text{ kl}\]

d) If a person's pupillary distance (from one pupil to the other) is 61 mm and the distance from their pupil to the middle of their upper lip is 7 cm, which distance is longer?
Problem 17: You Try – Simple Metric Unit Conversions

Use a metric chart to convert the metric units. Show all of your work.

a) 1510 m = _______________ mm

b) 13.50 ml = _______________ l

c) 5 km = _______________ m

SECTION 1.8: CONVERSIONS BETWEEN U.S. AND METRIC MEASURES

Although the U.S. relies heavily on our standard measurement system, we do use some metric units. Therefore, we need to know how to move back and forth between the systems. We will use dimensional analysis, conversion equations, and conversion factors to achieve this process.

A table of some common U.S./Metric conversions are below. Note that many of these conversions are approximations. For example, our table uses the approximation 1 mile = 1.61 km. I googled the conversion equation for miles and kilometers. The result I was given was 1 mile = 1.60934 km. This is an approximation too! I used another calculator online that gave 1 mile = 1.609344 km (one more decimal place than google). The amount of decimal places you use in conversions depends on how accurate you need your measure to be. For our purposes, the chart below will work fine.

<table>
<thead>
<tr>
<th>Length</th>
<th>Mass/Weight</th>
<th>Area</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 mi = 1.61 km</td>
<td>1 lb. = 0.454 kg</td>
<td>1 in² = 6.45 cm²</td>
<td>1 pt. = 0.47 L</td>
</tr>
<tr>
<td>1 yd = 0.9 m</td>
<td>1 oz. = 28.3 g</td>
<td>1 yd² = 0.84 m²</td>
<td>1 qt = 0.95 L</td>
</tr>
<tr>
<td>1 in = 2.54 cm</td>
<td>1 kg = 2.2 lb.</td>
<td>1 mi² = 2.59 km²</td>
<td>1 gal = 3.8 L</td>
</tr>
<tr>
<td>1 km = 0.621 mi</td>
<td>1 g = 0.0353 oz.</td>
<td>1 cm² = 0.16 in²</td>
<td>1 L = 2.1 pt.</td>
</tr>
<tr>
<td>1 m = 1.094 yd</td>
<td>1 metric ton = 1.1 ton</td>
<td>1 m² = 1.2 yd²</td>
<td>1 L = 1.06 qt</td>
</tr>
<tr>
<td>1 cm = 0.394 in</td>
<td></td>
<td>1 km² = 0.39 mi²</td>
<td>1 L = 0.26 gal</td>
</tr>
</tbody>
</table>
Problem 18: Media Example – Conversions between Measurement Systems

Use dimensional analysis to perform the indicated conversions.

a) Express 5 ml in terms of cups.

b) The country of Cambodia is approximately 700 km from N to S. What would this distance be in miles?

c) Soda is often sold in 2-liter containers. How many quarts would this be? How many gallons?

Problem 19: You Try – Conversions between Measurement Systems

Use dimensional analysis to perform the indicated conversions.

a) Your friend Leona is planning to run her first 10km race in a few weeks. How many miles will she run if she completes the race?

b) A roll of Christmas wrapping paper is 3 meters long. How long is this in yards?

c) Although Britain now uses the metric system, they still serve beer in pints. If they switched to the metric system for beer, how many liters of beer would be in 1 pint?
SECTION 1.9: UNIT CONVERSIONS FOR RATES

In this section, we will look at converting rates. A rate is a ratio where the quantities we are comparing are measuring different types of attributes. First notice, that a rate is considered a type of ratio so a rate is also a multiplicative comparison of two quantities. However, the two quantities measure different things. For example,

1) miles per hour (distance over time, which we may also call speed)
2) dollars per hour (money over time, which we may also call rate of pay)
3) number of people per square mile (population over land area, which we may also call population density).

A special type of rate is called a unit rate. A unit rate is a rate where the quantity of the measurement in the denominator of the rate is 1. For example, suppose you are offered a new job after graduation, and your new employer says that you will be paid at a rate of $805 per 25 hours or \( \frac{\$850}{25 \text{ hours}} \). This is indeed a rate of pay, but it is difficult to conceptualize this rate. It may be more useful to know how much you will be paid per 1 hour instead of per 25 hours. This unit rate of pay can be found as shown below. Note that when we write “per hour”, this is equivalent to a denominator of 1 hour if we need to write a quantity in the denominator.

\[
\frac{\$850}{25 \text{ hours}} = \frac{\$850 \div 25}{25 \text{ hours} \div 25} = \frac{\$34}{1 \text{ hour}} \quad \text{or} \quad \$34 \text{ per hour}
\]

In the next problems, we will convert from one rate to another. Since rates involve two types of measurements, we may be required to convert both units to different measures.

Problem 20: Media Example – Converting Rates

Use dimensional analysis to perform the indicated conversions.

a) An arborist measured the growth of an apple tree. The tree grew 19 inches per year. On average, how many millimeters did the tree grow per day?
b) You are biking at a rate of 20 miles per hour. What is your rate of speed in feet per second? Round your answer to two decimal places.

Problem 21: You Try – Converting Rates

Use dimensional analysis to perform the indicated conversions.

a) People’s hair grows at different rates. A fast growth rate is 12 cm of hair growth in 4 months. How many inches is this per year?

b) In September 2016, the conversion rate between Euros and US dollars was 1 U.S. dollar ≈ 0.896 euros. The cost of gasoline in Berlin was around 1.3 euros per liter. What would the equivalent cost be in U.S. dollars per gallon? (Use 1 gallon ≈ 3.785 liters)
UNIT 2 – PERIMETER AND AREA

INTRODUCTION

In this Unit, we will define basic geometric shapes and use definitions to categorize geometric figures. Then we will use the ideas of measuring length and area that we studied to find the perimeter, circumference, or area of various geometric figures.

Section 2.1: Categorize geometric figures using definitions

Section 2.2: Model a context as a geometric shape and find its perimeter

Section 2.2: Find the perimeters of various shapes

Section 2.3: Find the circumference of circles in various contexts using a formula

Section 2.4: Develop strategies for finding area

Section 2.5: Find the formula for the area of a parallelogram

Section 2.5: Apply the formula for the area of a parallelogram

Section 2.5: Find the formula for the area of a triangle

Section 2.5: Find the formula for the area of a trapezoid

Section 2.5: Apply the formula for the area of a trapezoid

Section 2.5: Find the formula for the area of a circle

Section 2.5: Apply the formula for the area of a circle

Section 2.5: Find the area of nonstandard shapes
UNIT 2 – MEDIA LESSON

SECTION 2.1: GEOMETRIC SHAPES AND DEFINITIONS

We will begin by defining some elemental shapes and characteristics of geometric figures. The famous mathematician Euclid set out to define basic geometry terms in his book *The Elements* in approximately 300 B.C. in Alexandria, Egypt. We will refer to some of his work below to show the difficulty in defining some of the simplest terms in geometry.

**Geometric Definitions**

Point - A location in space. Euclid defined a point as “*that which has no part*”. A point is dimensionless, and has no width, length, or height.

![point A](image.png)

Line - A collection of points that extend along a straight path in two directions without end. Euclid defined a line as “*a line is breadthless length*”. A line is one-dimensional.

![line l](image.png)

Line segment - A part of a line that has two endpoints. Line segments can be measured and have a finite length.

![line segment](image.png)

Ray - A part of a line that has one endpoint.

![ray](image.png)
Angle - Two rays that have a common endpoint. We measure an angle by the amount of rotation from one ray to the other ray.

Vertex - The common endpoint of two rays or two line segments

Plane - A flat two dimensional surface that extends infinitely in its two dimensions

Closed figure - A figure that has an inside and outside. You cannot reach the inside from the outside without crossing the figure’s boundary.
Open figure - A figure that is not closed. It does not have an inside and outside.

Polygon - A closed two-dimensional figure with line segments as sides.

Convex polygon - A polygon in which any line segment drawn between two points within the figure does not cross a boundary.

Triangle - A three sided polygon

Quadrilateral - A four sided polygon

Circle - A two dimensional figure that is the set of all points equidistant from a point called the center.
SECTION 2.2: PERIMETER

You may have heard the term perimeter in crime shows. The police will often “surround the perimeter”. This means they are guarding the outside of a building or shape so the suspect cannot escape. In mathematics, the perimeter of a two dimensional figure is the one dimensional total distance around the edge of the figure. We want to measure the distance around a figure, building, or shape and determine its length. Since the perimeter refers to the distance around a closed figure or shape, we compute it by combining all the lengths of the sides that enclose the shape.

In this section, we will introduce the concept of perimeter and learn why it is useful. We will find the perimeters of many different types of shapes and develop a general strategy for finding the perimeter so we don’t have to rely on formulas.

**Problem 2: Worked Example - The Perimeter in Context**

Joseph does not own a car so he bikes everywhere he goes. On Mondays, he must get to school, to work, and back home again. His route is pictured below.

![Route Diagram]

a) Joseph starts his day at home. Complete the chart below by determining how far he has biked between each location and the total amount he has biked that day at each point. Include units in your answers.

<table>
<thead>
<tr>
<th>Location</th>
<th>Starts at Home</th>
<th>Arrives at School</th>
<th>Arrives at Work</th>
<th>Returns Home</th>
</tr>
</thead>
<tbody>
<tr>
<td>Miles traveled from previous location</td>
<td>0 miles</td>
<td>3 miles</td>
<td>6 miles</td>
<td>6 miles</td>
</tr>
<tr>
<td>Total miles traveled since leaving home</td>
<td>0 miles</td>
<td>3 miles</td>
<td>9 miles</td>
<td>15 miles</td>
</tr>
</tbody>
</table>
b) Based on the information in your chart. What is the total distance Joseph Biked on Monday? Write your answer as a complete sentence.

Answer: Joseph traveled a total distance of 15 miles on Monday

c) Another way to work with this situation is to draw a shape that represents Joseph’s travel route and label it with the distance from one location to the next as shown below. Find the perimeter of this shape.

Solution:

Computation: 3 miles + 6 miles + 6 miles = 15 miles

The perimeter of the figure is 15 miles

Result: The perimeter of the geometric figure is equivalent to the distance Joseph traveled. However, in part c, we modeled the situation with a geometric shape and then applied a specific geometric concept (perimeter) to compute how far Joseph traveled.

Notes on Perimeter:

- The Perimeter is a one-dimensional measurement that represents the distance around a closed geometric figure or shape (no gaps).
- To find the perimeter, add the lengths of each side of the shape.
- If there are units, include units in your final result. Units will always be of single dimension (i.e. feet, inches, yards, centimeters, etc...)
Problem 3: Media Example - Finding the Perimeter of a Figure

Find the perimeter for each of the shapes below. Label any sides that aren't labeled and justify your reasoning. Show all of your work and include units in your answer.

a) Keith bought a square board for a school project. What is the perimeter of the board?

Computation:

The perimeter of the board is ____________

b) Judy is planting flowers in a rectangular garden. How many feet of fence does she need to fence in the garden?

Computation:

The perimeter is ____________

c) Dana cut out the figure to the right from cardboard for an art project. What is the perimeter of the figure?

Computation:

The perimeter is ____________

d) Sheldon set up a toy train track in the shape given to the right. Each length is measured in feet. How far would the train travel around the track from start to finish?

Computation:

The perimeter is ____________
Problem 4: You Try – Finding the Perimeter of a Figure

Find the perimeter for each of the problems below. Draw any figures if the shapes are not given. Label any sides that aren't labeled. Show all of your work and include units in your answer.

a) Find the perimeter of a square with side length 2.17 feet.

b) Find the perimeter of a triangle with sides of length 3, 6, 8.

c) Jaik’s band was playing at the club The Bitter End in New York City. A diagram of the stage is given below. What was the perimeter of the stage?

Computation:

Final Answer as a Complete Sentence:

---

d) Steve works at the mall as a security guard. He is required to walk the perimeter of the mall every shift. The mall is rectangular in shape and the length of each side is labeled in the figure below. How far does Steve need to walk to complete this task?

Computation:

Final Answer as a Complete Sentence:
SECTION 2.3: CIRCUMFERENCE

The distance around a circle has a special name called the *circumference*. Since a circle doesn't have line segments as sides, we can't think of the circumference as adding up the sides of a circle. Before we find the formula for the circumference of a circle, we will first need to define a few attributes of a circle.

Mathematically, a *circle* is defined as the set of all points equidistant to its *center*. The *diameter* is the distance across the circle (passing through the center). The *radius* is the distance from the center of a circle to its edge. *Notice that the diameter of the circle is two times as long as the radius of the circle.*

Imagine a circle as a wheel. Now in your mind's eye, roll the wheel one complete turn. The distance the wheel covered in one rotation equals the distance around the circle, or the circumference.

You can probably imagine that the length of the radius or diameter is related to the circumference. The larger the circle, the larger the radius or diameter, the larger the distance that is covered in one rotation. In fact, the circumference of a circle is a *constant multiple* of its radius or diameter. Observing the number lines in the diagram below the circumference we can see the following results.
Unit 2 – Perimeter and Area

1) If we use the circle’s radius as a measuring unit to measure the distance around the circle, we find that it takes just a little more than six copies of the radius to complete the circle.

2) If we use the circle’s diameter as a measuring unit to measure the distance around the circle, we find that it takes just a little more than three copies of the diameter to complete the circle.

3) Since the diameter is twice as large as the radius, it makes sense that the number of diameter length segments to cover the distance is half the size of the number of radius length segments.

4) The constant factor between the diameter and circumference is a special number in mathematics called pi, pronounced “pie”, and written with the Greek letter π.

   Result: The formula for finding the circumference of a circle can be written in terms of either the circle’s radius or diameter. These formulas are given below.

   \[ C = \pi \times d \quad \text{or} \quad C = \pi \times 2 \times r \quad \text{or} \quad C \approx 6.28r \]

   \[ C = \pi d \quad \text{or} \quad C = 2\pi r \quad \text{or} \quad C \approx 3.14d \]

   Where C is the circumference, d is the diameter, and r is the radius.

Problem 5: Media Example – Finding the Circumference of a Circle

Use the given information to solve the problems. Show all of your work and include units in your answer. Write your answers in exact form and in rounded form (to the hundredths place).

a) Anderson rollerbladed around a circular lake with a radius of 3 kilometers. How far did Anderson rollerblade?
b) Liz bought a 14-inch pizza. The server said the 14 inch measurement referred to the diameter of the pizza. What is the circumference of the pizza?

c) Use the diagram of the circle to answer the questions.

i. Are you given the radius or diameter of the circle? How do you know?

ii. Find the circumference of the circle in exact and rounded form.

d) Use the diagram of the circle to answer the questions.

i. Are you given the radius or diameter of the circle? How do you know?

ii. Find the circumference of the circle in exact and rounded form.
Problem 6: You Try – Finding the Circumference of a Circle

Use the given information to solve the problems. Draw a diagram for each problem labeling either the radius or diameter (as given). Show all of your work and include units in your answer.

a) The Earth’s equator is the circle around the Earth that is equidistant to the North and South Poles, splitting the Earth into what we call the Northern and Southern Hemispheres. The radius of the Earth is approximately 3958.75 miles.

What is circumference of the equator? Write your answers in exact form and in rounded form (to the hundredths place).

b) The diameter of a penny is 0.75 inches.

What is circumference of a penny? Write your answers in exact form and in rounded form (to the hundredths place).
SECTION 2.4: STRATEGIES FOR FINDING AREA

In this section, we will learn to find the area of a two dimensional figure. When we studied perimeter, we found the one dimensional or linear distance of the boundary of a two dimensional figure. To find the area of a two dimensional figure, we want to find the two dimensional space inside the figure’s boundaries. Since we are measuring a two dimensional space when we find area, we need a two dimensional measure. Typically, we use square units (as opposed to linear units) to measure area. Our goal is to find how many non-overlapping square units fill up or cover the inside of the figure. In this section, we will begin our study of area by investigating some common strategies for finding area.

Problem 7: Worked Example – Strategies for Finding Areas

a) Find the area of the given shape by counting the square units that cover the interior of the shape. Assume the side of each small square is 1 cm.

Original

Solution: The area is 30 square cm

b) Alice, Ben, Chris, and Dana were given the task of finding the area of the shape below, but their teacher didn't give them the grid with the squares to count. Each student knew how to find the area of a rectangle; you multiply the length times the width or equivalently, the number of rows times the number of columns. They each came up with a different strategy for finding the area of this shape.
Alice’s Strategy: Alice noticed that if she cut the figure in two pieces as shown by the red dashed line, that each piece would be a rectangle. She found the width of the Rectangle 1 by subtracting the width of Rectangle 2 from the bottom length of the original figure, 6cm - 3cm = 3cm, and labeled the diagram.

The area of the original figure equals the sum of the small 3 cm by 4cm rectangle and the larger 3 cm by 6 cm rectangle.

Area = 3 cm \cdot 4 cm + 3 cm \cdot 6 cm = 12 cm^2 + 18 cm^2 = 30 cm^2

Ben’s Strategy: Ben noticed that the shape looked like a square with a rectangle cut out of the upper left corner. He drew the red rectangle below to turn the shape into a square. He found the lengths of the square by using the larger length of 6 cm and subtracting off the opposite pieces.

The area of the original figure equals the area of the large 6 cm by 6 cm square minus the area of the small 2 cm by 3 cm rectangle.

Area = 6 cm \cdot 6 cm - 2 cm \cdot 3 cm = 36 cm^2 - 6 cm^2 = 30 cm^2
Chris’ Strategy: Chris noticed that if he cut the figure as shown in Figure A, he would create two rectangles. He also noticed that each rectangle has a width of 3 cm. So he stacked Rectangle 2 on top of Rectangle 1, making one big rectangle.

The area of the original figure equals the area of the large rectangle which is 3 cm by 4 cm + 6 cm or 3 cm by 10 cm.

\[
\text{Area} = 3 \text{ cm} \cdot (4 \text{ cm} + 6 \text{ cm}) = 3 \text{ cm} \cdot 10 \text{ cm} = 30 \text{ cm}^2
\]

Dana’s Strategy: Dana noticed that if she had two copies of the figure, she could rotate one and they would fit together like puzzle pieces and make a rectangle.

The large rectangle is 6 cm by 6 cm + 4 cm or 6 cm by 10 cm. Since this is two of the original figure, the original figure has half the area of this rectangle.

\[
\text{Area} = \frac{1}{2} \cdot 6 \text{ cm} \cdot (4 \text{ cm} + 6 \text{ cm}) = 3 \text{ cm} \cdot 10 \text{ cm} = 30 \text{ cm}^2
\]
RESULTS – Strategy Types for Finding Areas

The list below contains the specific strategies each student used.

1) Alice used an *adding strategy* to find the area of shape.

2) Ben used a *subtraction strategy* to find the area of the shape.

3) Chris used a *move and reattach strategy* to find the area of the shape.

4) Dana used a *double and half strategy* to find the area of the shape.

Each of these strategies is valid, and each of the strategies can be helpful when you need to find the area of a shape. When trying to find area, there are two fundamental principles that you need to follow:

*The moving principle* – you can move a shape and its area doesn’t change

*The additivity principle* – if you combine shapes without stretching or overlapping them, the area of the new shape is the sum of the area of the smaller shapes.

These two principles allow us to find the area of unusual shapes, because we can divide them into pieces and sum the areas of each piece. We can find the area of a rectangle that surrounds our shape, then we can subtract off the area of pieces that are not part of the rectangle. Or we can reattach the pieces to create shapes that we know how to find the area of. All of the strategies that were used in the example are valid because of the moving and additivity principles.

**Problem 8: You Try – Strategies for Finding Areas**

Find the area of the shaded region of the figures using one of the four strategies above. Note which strategy that you used. Show all of your work and include units in your answers. The length of each square in the grid is 1 cm.

a) Show your work below and in the diagram when needed.

   **Strategy:**
b) Find the area. Show your work below and in the diagram when needed.

Strategy:

![Diagram of parallelogram]

---

c) Find the area. Show your work below and in the diagram when needed.

Strategy:

![Diagram of triangle]

---

d) Show your work below and in the diagram when needed.

Strategy:

![Diagram of rectangle with a smaller rectangle inside]
SECTION 2.5: FORMULAS FOR FINDING AREA

For simple shapes, we can often find a formula that will allow us to calculate the area of the shape if we know some measurements of the shape. In this section, we will use the strategies we have learned to develop formulas for some common shapes.

**Problem 9: Worked Example – Finding the Formula for the Area of a Parallelogram**

Use the moving and additivity principles to find the areas of the parallelograms. Then use patterns to find a general formula for parallelograms.

a) Parallelogram 1: Cut off the triangle on the left hand side and reattach it to the right hand side creating a square as shown in Figures A and B below. The Area is $4 \cdot 4 = 16$ square units.

![Parallelogram 1](image1)

![Figure A](image2)

![Figure B](image3)

b) Parallelogram 2: Cut the parallelogram into 3 pieces as shown in Figure A. Rearrange the pieces to create the rectangle shown in Figure B. The Area is $2 \cdot 5 = 10$ square units.

![Parallelogram 2](image4)

![Figure A](image5)

![Figure B](image6)

**Formula for the Area of a Parallelogram:** The area, $A$, of a parallelogram can be found by multiplying its base times its height, where the base is the length of one of the sides, and the height is the perpendicular distance from the base to its opposite side.

$$A = b \cdot h$$
Problem 10: Media Example – Applying the Formula for the Area of a Parallelogram

Use the formula for the area of a parallelogram to find the areas. Make sure to indicate which value is the base and which value is the height. Assume that all measures are given in centimeters. Your answer must include units.

a)

![Parallelogram A](image)

- Base: 
- Height: 
- Area:

b)

![Parallelogram B](image)

- Base: 
- Height: 
- Area:

Problem 11: You Try – Applying the Formula for the Area of a Parallelogram

Use the formula for the area of a parallelogram to find the areas. Make sure to indicate which value is the base and which value is the height. Assume that all measures are given in inches. Your answer must include units.

a)

![Parallelogram C](image)

- Base: 
- Height: 
- Area:

b)

![Parallelogram D](image)

- Base: 
- Height: 
- Area:
Problem 12: Worked Example – Finding the Formula for the Area of a Triangle

a) Triangle 1 – First take one copy of Triangle 1 as shown in Figure A. Make a second copy of Triangle 1 and rotate it as shown. Attach the two triangles to make one rectangle. The rectangle has an area of \(3 \cdot 5 = 15\) square units. Since there are two triangles making up the rectangle, one triangle has an area of \(\frac{1}{2} \times 3 \times 5 = 7.5\) square units.

![Figure A](image1.png)

b) Triangle 2 – First take one copy of Triangle 2 as shown in Figure A. Make a second copy of Triangle 2 and rotate it as shown. Attach the two triangles to make one parallelogram. The parallelogram has an area of \(3 \cdot 5 = 15\) square units. Since there are two triangles making up the parallelogram, one triangle has an area of \(\frac{1}{2} \times 3 \times 5 = 7.5\) square units.

![Figure A](image2.png)

Formula for the Area of a Triangle: The area, \(A\), of a triangle can be found by multiplying \(\frac{1}{2}\) times its base times its height, where the base is the length of one of the sides, and the height is the perpendicular distance to the opposite vertex.

\[
A = \frac{1}{2} b \times h
\]
Problem 13: Media Example – Applying the Formula for the Area of a Triangle

Use the formula for the area of a triangle to find the areas. Make sure to indicate which value is the base and which value is the height. Assume that all measures are given in centimeters. Your answer must include units.

\[ \text{Area} = \frac{1}{2} \times \text{base} \times \text{height} \]

a) 
- Base: 
- Height: 
- Area:

b) 
- Base: 
- Height: 
- Area:

Problem 14: You Try – Applying the Formula for the Area of a Triangle

Use the formula for the area of a triangle to find the areas. Make sure to indicate which value is the base and which value is the height. Assume that all measures are given in feet. Your answer must include units.

\[ \text{Area} = \frac{1}{2} \times \text{base} \times \text{height} \]

a) 
- Base: 
- Height: 
- Area:

b) 
- Base: 
- Height: 
- Area:

c) 
- Base: 
- Height: 
- Area:
Problem 15: Worked Example – Finding the Formula for the Area of a Trapezoid

a) Trapezoid 1 – First take one copy of Trapezoid 1 as shown in Figure A. Make a second copy of Trapezoid 1 and rotate it as shown in Figure B. Attach the two Trapezoids to make one parallelogram. The parallelogram has an area of $4 \cdot 6 = 24$ square units. Since there are two Trapezoids making up the parallelogram, one Trapezoid has an area of $\frac{1}{2} \cdot 4 \cdot 6 = 12$ units$^2$.

b) Trapezoid 2 – First take one copy of Trapezoid 2 as shown in Figure A. Make a second copy of Trapezoid 2 and rotate it as shown in Figure B. Attach the two Trapezoids to make one parallelogram. The parallelogram has an area of $(7 + 3) \cdot 4 = 40$ square units. Since there are two Trapezoids making up the parallelogram, one Trapezoid has an area of $\frac{1}{2} \cdot (7 + 3) \cdot 4 = 20$ units$^2$.

Formula for the Area of a Trapezoid: The area, $A$, of a trapezoid can be found by multiplying $\frac{1}{2}$ times the sum of its bases times its height, where the bases are the opposite parallel sides, and the height is the perpendicular distance between the bases.

$$A = \frac{1}{2} \times (b_1 + b_2) \times h$$
**Problem 16: Media Example – Applying the Formula for the Area of a Trapezoid**

Use the formula for the area of a trapezoid to find the areas. Make sure to indicate the base lengths and the height. Assume that all measures are given in feet. Your answer must include units.

a)

b)

Base 1: 
Base 2: 
Height: 
Area:

**Problem 17: You Try – Applying the Formula for the Area of a Trapezoid**

Use the formula for the area of a trapezoid to find the area. Make sure to indicate the base lengths and the height. Assume that all measures are given in kilometers. Your answer must include units.

Base 1: 
Base 2: 
Height: 
Area:
Problem 18: Media Example – Finding the Formula for the Area of a Circle

Even though a circle looks quite different than the shapes we have been talking about, we can use the move and reattach strategy to derive the formula for finding the area contained within the circle.

a) Figure A is a circle cut into 8 pieces. Figure B is a rearrangement of these pieces. Approximate the lengths of the two line segments labeled with question marks in Figure B in relation to the radius and circumference of Figure A.

If we continue to cut the circle in Figure A into more pieces, we would get the diagrams below. From left to right, the circle is cut into an increasing number of pieces.

b) Describe the change in shape of the resulting figures as they are cut into more pieces.

c) If the last figure is equivalent to the area of the original circle after cutting the circle into really small pieces, what is the area of the circle in terms of its radius and circumference?

d) Write a general formula for the area of a circle in terms of \( \pi \) and the circle's radius.
Problem 19: Media Example – Applying the Formula for the Area of a Circle

Use the given information to solve the problems. Show all of your work and include units in your answer. Write your answers in exact form and in rounded form (to the hundredths place).

a) Liz bought a 14-inch pizza. The server said the 14 inch measurement referred to the diameter of the pizza. What is the area of the pizza?

b) Use the diagram of the circle to answer the questions.

i. Are you given the radius or diameter of the circle? How do you know?

ii. Find the area of the circle in exact and rounded form.

(c) Use the diagram of the circle to answer the questions.

i. Are you given the radius or diameter of the circle? How do you know?

ii. Find the area of the circle in exact and rounded form.
Problem 20: You Try – Applying the Formula for the Area of a Circle

Use the given information to solve the problems. Show all of your work and include units in your answer. Write your answers in exact form and in rounded form (to the hundredths place).

a) Use the diagram of the circle to answer the questions.

i. Are you given the radius or diameter of the circle? How do you know?

ii. Find the area of the circle in exact and rounded form.

b) Use the diagram of the circle to answer the questions.

i. Are you given the radius or diameter of the circle? How do you know?

ii. Find the area of the circle in exact and rounded form.

c) A circular kiddie pool has a diameter of 4.5 feet. What is the area of the bottom of the pool? Use 3.14 for \(\pi\) and round your answer to two decimal places.
Problem 21: Media Example – Finding the Area of Non Standard Shapes

There are no formulas for finding the area of more complicated shapes, however we can use the strategies that were introduced in the beginning of this lesson to help us find areas.

a) Find the area. Break up the areas into shapes that we recognize and add the area values together.

b) Find the area of the given shape. Compute using 3.14 for π and round to the nearest hundredth.
Problem 22: You Try – Finding the Area of Non Standard Shapes

1) Find the area. Break up the areas into shapes that we recognize and add the area values together.

![Diagram](image1)

2) Jackson is putting an above ground swimming pool in his yard. The pool is circular, with a diameter of 12 ft. He wants to put a square deck around the pool that is at least two feet wider than the pool on each edge.

i. How much space will the pool and deck take up in his yard?

![Diagram](image2)

ii. What is the area of the surface of the pool?

iii. What is the area of the deck that he is designing?
Summary of Formulas

Perimeter: The **perimeter** of a two dimensional figure is the one dimensional total distance around the edge of the figure.

To find the perimeter of any polygon (sides are lines)

1) Determine all of the side lengths
2) If one or more side lengths aren't given, use the other side lengths to determine the missing side lengths.
3) Find the sum of all the sides.

Circumference: The **circumference** is distance around the boundary of a circle. The circumference is equivalent to the perimeter of a polygon, but for circles. Since circles do not have lines as sides, we cannot add up the sides, and we need a special formula.

To find the circumference of a circle:

1) Determine either the radius or the diameter. Make sure you know which one you are using.
2) Given the radius, \( r \), \( C = 2 \times \pi \times r = 2\pi r \) (exact) or \( C = 2 \times 3.14 \times r = 6.28r \) (approximate)
3) Given the diameter, \( d \), \( C = \pi \times d = \pi d \) (exact) or \( C = 3.14 \times d = 3.14d \) (approximate)

Area: The **area** is the number of square units that fills the inside of a figure. There are different formulas depending on the shape of the figure.

Area of a Parallelogram = base \( \times \) height
Area of a Rectangle = length \( \times \) width
Area of a Trapezoid = \( \frac{1}{2} \times (\text{base 1} + \text{base 2}) \times \) height
Area of a Square = side \( \times \) side or \( s^2 \)
Area of a Circle: \( \pi \times r^2 \) (exact), \( 3.14 \times r^2 \) (approximate)
Area of a Triangle = \( \frac{1}{2} \times \) base \( \times \) height

Area of Composite Figures: If you need the area of an uncommon shape, you need to cut it into pieces so that you can find the area of the separate pieces with known formulas. Then you can use addition or subtraction to find the area. The principles below describe these methods.

*The moving principle* – you can move a shape and its area doesn't change

*The additivity principle* – if you combine shapes without stretching or overlapping them, the area of the new shape is the sum of the area of the smaller shapes.
UNIT 3 – VOLUME AND THE PYTHAGOREAN THEOREM

INTRODUCTION

In this Unit, we will use the idea of measuring volume that we studied to find the volume of various 3 dimensional figures. We will also learn about the Pythagorean Theorem, one of the most famous theorems in mathematics. We will use this theorem to find missing lengths of right triangles and solve problems.

Section 3.1: Use the concept of stacking cubes to find the volume of a prism
Section 3.2: Use the concept of stacking cubes to find the volume of a cylinder
Section 3.2: Use and develop formulas for finding volumes of prisms and cylinders
Section 3.3: Use formulas to find the volumes of spheres, cones, and pyramids
Section 3.4: Find volumes of composite shapes.
Section 3.5: Find the missing side of a right triangle using the Pythagorean Theorem
Section 3.5: Apply the Pythagorean Theorem to find a missing side of a right triangle or solve an application problem
UNIT 3 - MEDIA LESSON
SECTION 3.1: VOLUME OF PRISMS

In this section, we will learn how to find the volume of a prism. Recall that when we measure the attribute of volume, we are finding the 3 dimensional space that a 3 dimensional object takes up or fills. A prism is a 3 dimensional object where two of its opposite sides are parallel and identical (called the bases), and the sides connecting them are squares, rectangles, or parallelograms. Here are some examples of prisms where the bases are shaded.

Square Prism (cube)  Triangular Prism  Trapezoidal Prism

Let’s look at an example of finding the volume of a rectangular prism.

Example: Suppose you want to build a concrete patio, you will need to order the concrete in units of cubic yards. In Unit 1, we learned that a cubic yard looks like a cube with a length of 1 yd, a width of 1 yd, and a depth of 1 yd. So 1 cubic yard of concrete is the amount of concrete that would fit in the box below.

When we calculate volume, we are finding how many unit cubes will fill up the space in which we are calculating the volume. If the concrete patio has the shape and dimensions of a rectangular prism, we want to know how many cubic yard units will fill up the space.
Consider the geometric diagram of the rectangular prism below as representative of the concrete patio.

When we find the volume of this solid, we are imagining filling the box with cubic yards, or cubes with length of 1 yard, width of 1 yard, and depth of 1 yard. It is a little easier to determine the number of cubic yards in the box if we think of the height representing the number of layers of cubes in our box. Now we might say that there are 3 layers of 4 by 5 arrays of cubes. So the total number of cubes must be $3 \times 4 \times 5$ cubes, or 60 cubic yards.

This strategy will always work when you are finding the volume of a prism. If you know how many cubes are in the bottom layer, then you can multiply that by the number of layers in the solid to find the volume.

Formally, we say that the volume of a prism is equal to the Area of the base times the height of the prism where the height is the distance between the two bases.
Problem 1: Media Example - Volume of a Prism

Write all the indicated measurements and attributes of the given prisms. Then find the volume of the solids. Include units in your answers.

a) The figure to the right is the same shape as the previous example from the text, but rotated a quarter of a turn. Find its volume by using the top of the figure as the base.

Area of the Base:

Height of Prism (distance between two bases):

Volume of the Prism:

How does the volume of this figure compare to the volume of the previous example? Why do you think this relationship holds?

b) Shape of the Base:

Area of the Base:

Height of Prism (distance between two bases):

Volume of the Prism:

c) Shape of the Base:

Area of the Base:

Height of Prism (distance between two bases):

Volume of the Prism:
Problem 2: You Try - Volume of a Prism

Answer the following questions. Include units in all of your answers when appropriate.

a) Shade one of the sides that you are using as one of the bases of your prism (more than one correct answer).

   Area of the Base:

   Height of Prism (distance between two bases):

   Volume of the Prism:

b) Area of the Base:

   Height of Prism (distance between two bases):

   Volume of the Prism:

c) Gloria is making coffee themed gift baskets for her friends. She found some small boxes that she will fill with sugar cubes as one of the items in the basket. The boxes are 5 cm wide, 8 cm long, and 3 cm high. She measures the sugar cubes and finds that they are perfect centimeter cubes! How many sugar cubes will she need to fill each box? What is the volume of the box measured in cm$^3$?
SECTION 3.2: VOLUME OF A CYLINDER

A cylinder is similar to a prism in that they both have two parallel, identical bases. However, a cylinder’s base is a circle, and the sides are not parallelograms, but are smooth like a circle. Some cylinders you may have seen in everyday life are soda cans, a tennis ball container, a paint can, or a candle. Here are some images of cylinders.

We can use the same reasoning that we used when we found the volume of a prism to find the volume of a cylinder. The image below is of the base of a cylinder. The interior, or area, is on grid paper so we can imagine stacking cubes on the base to find a volume.

Radius of Base of Cylinder: 4 units

Area of base of cylinder: \( \pi r^2 = \pi \cdot 16 = 16\pi \approx 50.24 \text{ units}^2 \)

Since the base of the cylinder is a circle, some of the squares in the base are partial squares. However, we can still imagine stacking partial cubes with a base of the size of each of the partial squares and one unit high. For example, if we took 1 cubic yard, and split the bases in half, we would have 2 copies of \( \frac{1}{2} \text{yd}^3 \) as shown in the image below.
So for any partial square in the base of a cylinder, we can stack a partial cube of height 1 with the base of the square and the result is the area of the square times 1 cubic units. This means that even for partial squares in the base, we can stack cubes with a height of 1 unit and attain a measure of volume.

The image to the right is of a cylinder using the base given above and with a height of 6 inches. We'll now use a radius of 4 inches (as opposed to generic units). Notice how the squares and partial squares line up between the top and bottom bases. Now imagine stacking the cubic inches and partial cubic inches from bottom to top. The total number of these cubes will equal the volume of the cylinder.

Radius of Base of Cylinder: 4 inches

Area of base of cylinder: \( \pi r^2 = \pi \times 16 = 16\pi \approx 50.24 \text{ in}^2 \)

Volume of cylinder: \( 6 \times \pi r^2 = 6 \times \pi \times 16 = 96\pi \approx 301.44 \text{ in}^3 \)

In general, like a prism, the volume of a cylinder is the area of its base times its height.

**Problem 3: Media Example – Volume of a Cylinder**

Write all the indicated measurements and attributes of the given cylinders. Then find the volume of the solids. Include units in your answers. Give your answer in exact form (using \( \pi \)) and approximate form using \( \pi \approx 3.14 \).

a) Find the following measures for the figure to the right. The squares in the bases are square feet.

**Area of the Base**

- Exact Form:
- Approximate Form:

**Height of Cylinder (distance between two bases):**

**Volume of the Cylinder**

- Exact Form:
- Approximate Form:
b) Find the following measures for the cylinder to the right.

**Area of the Base**

Exact Form:

Approximate Form:

Height of Cylinder (distance between two bases):

**Volume of the Cylinder**

Exact Form:

Approximate Form:

c) The figure to the right is not a prism or a cylinder, but it has two identical parallel bases. Use the given information and the reasoning from this section to find the following.

**Area of the Base:**

**Height of the Figure:**

**Volume of the Figure:**
Problem 4: You Try – Volume of a Cylinder

a) Find the following measures for the cylinder to the right.

Area of the Base

Exact Form:

Approximate Form:

Height of Cylinder (distance between two bases):

Volume of the Cylinder

Exact Form:

Approximate Form:

b) Donna is making a cylindrical candle. She wants it to fit exactly in her candle holder which has a radius of 5.5 cm. She is going to make the candle 14 cm tall. How many cubic centimeters of wax will Donna need to make the candle? (Use 3.14 for \( \pi \))
SECTION 3.3: VOLUMES OF OTHER SHAPES

It is helpful to know the formula for calculating the volume of some additional shapes. The mathematics for developing these formulas is beyond the scope of this class, but the formulas are easy to use. The chart below shows the formulas to find the volumes of some other basic geometric shapes.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sphere with radius $r$</td>
<td>$V = \frac{4}{3} \pi r^3$</td>
</tr>
<tr>
<td>Cone with height $h$ and base radius $r$</td>
<td>$V = \frac{1}{3} \pi r^2 h$</td>
</tr>
<tr>
<td>Pyramid</td>
<td>$V = \frac{1}{3} l w h$</td>
</tr>
</tbody>
</table>
Problem 5: Media Example – Volumes of Other Shapes

Determine the volume of each of the following solids. Label any given information in the figure. Include units in your final result and round your answers to two decimal places.

a) A basketball has a diameter of approximately 9.55 inches. Find the volume of the basketball.

b) The Great Pyramid of Giza in Egypt has a square base with side lengths of approximately 755.9 feet and a height of approximately 480.6 feet. Find the volume of the pyramid.

c) An ice cream cone has a diameter of 8 cm and a height of 13 cm. What is the volume of the ice cream cone?
Problem 6: You Try – Volumes of Other Shapes

a) Find the volume of the figure given in the diagram below. Write the exact form using \( \pi \) and find the approximate form using the \( \pi \) button on your calculator. Round to two decimal places as needed.

b) Find the volume of the figure given in the diagram below. Round to two decimal places as needed.

c) The planetary object Pluto is approximately spherical. Its diameter is approximately 3300 miles. Find the volume of Pluto. Use 3.14 for \( \pi \). Include units in your final result and round your answers to two decimal places.
SECTION 3.4: VOLUMES OF COMPOSITE FIGURES

When we studied area and perimeter, we sometimes came across shapes that didn’t have a direct formula, but we could break the shapes into multiple different shapes to find the area or perimeter. In this section, we will do the same with volume.

Problem 7: Worked Example – Volumes of Composite Figures

a) A sculpture of a sphere is set on a base that is a rectangular prism as shown in the diagram below. Find the volume of the sculpture.

Solution: The total volume is the sum of the volume of the rectangular prism base and the volume of the sphere. We will find them separately and add the results.

Volume of base = 14 ft. \cdot 18 ft. \cdot 2 ft. = 504 \text{ ft.}^3

Volume of sphere = \frac{4}{3} \times 3.14 \times 6^3 = 904.32 \text{ ft}^3

Total volume = 504 \text{ ft}^3 + 904.32 \text{ ft}^3 = 1408.32 \text{ ft}^3

b) A water glass is 16 cm tall with a radius of 2.5 cm. Three ice cubes are in the glass that have a side length of 1.2 cm. The glass is filled with water up to 2 cm from the top of the glass. How many cubic cm of liquid water are in the glass?

Solution: First we will find the volume of water as if there were no ice cubes. Then we will subtract off the volume of the ice cubes from the result.

Volume of water if there were no ice cubes: We need the radius and the height of the water in the glass to use the cylinder formula.

Radius = 2.5 cm

Height of water in glass: 16 \text{ cm} – 2 \text{ cm} = 14 \text{ cm}

Volume = 3.14 \times 2.5^2 \times 14 = 274.75 \text{ cm}^3

Volume of an ice cube = 1.2^3 = 1.728 \text{ cm}^3

Liquid water in glass = 247.75 \text{ cm}^3 – 3 \times 1.728 \text{ cm}^3 = 242.566 \text{ cm}^3
SECTION 3.5: THE PYTHAGOREAN THEOREM

We have discussed that the perimeter of a shape is equal to the distance around the shape. We can only find the perimeter if we know the length of all of the sides. Sometimes we can use properties of the shape to find unknown side lengths. For example, if we know that the length of one side of a square is 5 inches, then we know that the other three lengths are 5 inches because a square has 4 equal side lengths. The Pythagorean Theorem is a useful formula that relates the side lengths of right triangles.

Consider the right triangle below that has side lengths of 3, 4, and 5.

\[3^2 + 4^2 = 9 + 16 = 25\]
\[5^2 = 25\]
So \[3^2 + 4^2 = 5^2\]

A similar result holds for any right triangle. The diagram below labels a generic right triangle with sides of lengths a, b, and c.

Notice that two of the sides of a right triangle are called legs and we label them with the letters a and b. It actually doesn't matter which we call a and which we call b as long as we are consistent in our computations. However, the third side has a special name called the hypotenuse. It is the side opposite the right angle in the rightmost diagram. When we use Pythagorean Theorem formulas, make sure you only use the hypotenuse for the letter c.
The Pythagorean Theorem:

The mathematician Pythagoras proved the Pythagorean Theorem. The theorem states that given any right triangle with sides a, b, and c as shown below, the following relationship is always true:

\[ a^2 + b^2 = c^2 \]

![Diagram of a right triangle](image)

Notes about the Pythagorean Theorem:
- The triangle must be a right triangle (contains an angle that measures 90°).
- The side c is called the **hypotenuse** and always sits opposite from the right angle.
- The lengths a and b are interchangeable in the theorem but c cannot be interchanged with a or b. In other words, the location of c is very important and cannot be changed.

The following are alternative forms of the Pythagorean Theorem and when you will use them.

Pythagorean Theorem solved for a leg (a or b)

\[ a = \sqrt{c^2 - b^2} \]

\[ b = \sqrt{c^2 - a^2} \]

![Diagram of a right triangle with labels](image)

Use either of these formulas when you are given either leg and the hypotenuse and need to find a missing leg. Again, the labeling of a or b is arbitrary (as long as they are both legs), but once you label your diagram with a specific letter, make sure you use it consistently.

Pythagorean Theorem solved for the hypotenuse (c)

\[ c = \sqrt{a^2 + b^2} \]

Use this formula when you are given both values for the legs and need to find the hypotenuse.
Problem 8: Media Example – Applying the Pythagorean Theorem

Use the Pythagorean Theorem to find the missing length of the given triangles. Round your answer to the tenth's place when needed.

a) Find the unknown side of the triangle.

\[
\begin{array}{c}
\text{c} \\
7 \\
12 \\
\end{array}
\]

b) Find the unknown side of the triangle.

\[
\begin{array}{c}
a \\
17.3 \text{ m} \\
3.8 \text{ m} \\
\end{array}
\]

c) In NBA Basketball, the width of the free-throw line is 12 feet. A player stands at one exact corner of the free throw line (Player 1) and wants to throw a pass to his open teammate across the lane and close to the basket (Player 2). If his other teammate (Player 3 – heavily guarded) is directly down the lane from him 16 feet, how far is his pass to the open teammate? Fill in the diagram below and use it to help you solve the problem. (Source: http://www.sportsknowhow.com).
d) Sara is flying her kite and it gets stuck in a tree. She knows the string on her kite is 17 feet long and she is 6 feet from the tree. How long of a ladder (in feet) will she need to get her kite out of the tree? Round your answer to the nearest hundredth as needed.

![Diagram of a right triangle with one leg of 17 feet and another leg of 6 feet.]

Problem 9: You Try – Applying the Pythagorean Theorem

Use the Pythagorean Theorem to find the missing length of the given triangles. Round your answer to the tenth’s place when needed.

a) Find the unknown side of the triangle.

![Diagram of a right triangle with sides 4 m and 9 m.]

b) Find the unknown side of the triangle.

![Diagram of a right triangle with sides 23 feet and 62 feet.]

c) Given a rectangular field 105 feet by 44 feet, how far is it to walk from one corner of the field to the opposite corner? Draw a picture to represent this situation. Round your answer to the nearest tenth as needed.
UNIT 4 – PERCENTS AND APPLICATIONS

INTRODUCTION

Section 4.1: Identify the usefulness of percents in context
Section 4.2: Change forms between decimals, fractions, and percents
Section 4.3: Find a percent that corresponds to a given amount of a whole in context
Section 4.4: Find an amount given a percent and a whole
Section 4.4: Find an amount given a percent and a whole in context
Section 4.5: Find a whole given a percent and amount
Section 4.5: Find a whole given a percent and amount in context
Section 4.6: Determine a new percent of a whole and multiplicative factor given a percent increase or decrease
Section 4.6: Find a new amount given a whole and a percent increase or decrease
UNIT 4 – MEDIA LESSON

SECTION 4.1: INTRODUCTION TO PERCENTS

A percent represents a ratio with a denominator of 100. Notice that if we think of percent as two words, “per cent” we can think of rates that use the word “per” meaning “for each” or “out of”, and “cent” meaning 100 such as 100 cents in a dollar or 100 years in a century.

In this section, we will introduce percents and learn why they are useful. We will represent percents in multiple ways to connect the idea of percent with other representations such as ratios, fractions, and decimals.

Problem 1: Worked Example - Why Percents?

Sylvia has taken 3 tests in her math class this semester. The values below show the number of points she earned out of the total number of possible points. How can we compare these test scores?

<table>
<thead>
<tr>
<th></th>
<th>Test 1</th>
<th>Test 2</th>
<th>Test 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Points Earned</td>
<td>18</td>
<td>16</td>
<td>42</td>
</tr>
<tr>
<td>Total Points Possible</td>
<td>25</td>
<td>20</td>
<td>50</td>
</tr>
</tbody>
</table>

Since the tests have different total point values, it would be helpful to write them as ratios and then equivalent ratios with a common denominator.

Ratio of Points Earned to Total Points Possible and Equivalent Ratio out of 100

<table>
<thead>
<tr>
<th></th>
<th>Test 1</th>
<th>Test 2</th>
<th>Test 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{18}{25}$</td>
<td>$\frac{16}{20}$</td>
<td>$\frac{42}{50}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{18 \cdot 4}{25 \cdot 4}$</td>
<td>$\frac{16 \cdot 5}{20 \cdot 5}$</td>
<td>$\frac{42 \cdot 2}{50 \cdot 2}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{72}{100}$</td>
<td>$\frac{80}{100}$</td>
<td>$\frac{84}{100}$</td>
</tr>
</tbody>
</table>

Since we wrote the ratios out of 100, we can rewrite them as percents by writing the numerator followed by a percent symbol. The equivalent percents are below.

<table>
<thead>
<tr>
<th></th>
<th>Test 1</th>
<th>Test 2</th>
<th>Test 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>72%</td>
<td>80%</td>
<td>84%</td>
</tr>
</tbody>
</table>

To compare percent values, we need only compare the numbers themselves. Their ordering from least to greatest is 72, 80, and 84. So Sylvia did best on Test 3 with 84%.
SECTION 4.2: CONVERTING DECIMALS, FRACTIONS, AND PERCENTS

We will frequently need to change forms between decimals, fractions, and percents. Let’s generalize one of our examples from the previous section to find a procedure.

On her first test, Sylvia earned 18 out of 25 points. To write this as a percent, we needed to find the equivalent amount of points that corresponded to 100 points as shown in the equation below.

\[
\frac{18}{25} = \frac{\text{percent}}{100}
\]

We did this by finding the equivalent fraction with a denominator of 100. We used the fact that \(25 \cdot 4 = 100\) to achieve this as shown below.

\[
\frac{18 \cdot 4}{25 \cdot 4} = \frac{72}{100} = 72\%
\]

An alternative method is to multiply both sides of the equation by 100 to solve for the percent value.

\[
\frac{\text{percent}}{100} \cdot 100 = \frac{18}{25} \cdot 100
\]

Note that \(\frac{18}{25} \cdot 100 = \frac{18 \cdot 100}{25} = 18 \cdot 4 = 72\). So multiplying the ratio \(\frac{18}{25}\) by 100 achieves the same result as multiplying the numerator 18 by 4 to find an equivalent ratio over 100.

So in general, to find a percent given a ratio or decimal, we multiply by 100.

\[
\text{percent} = \frac{\text{amount}}{\text{whole}} \cdot 100
\]

To reverse this process and write a percent as a ratio, we divide the percent by 100.

\[
\frac{\text{amount}}{\text{whole}} = \frac{\text{percent}}{100}
\]
RULES: Below is an overview of this process.

1) To change a ratio, fraction, or decimal to a percent, multiply by 100%.

2) To change a percent to a ratio, fraction, or decimal, divide by 100%

Problem 2: Worked Example - Changing Forms between Decimals, Fractions, and Percents

a) Fraction to Percent: Rewrite $\frac{2}{5}$ as a percent.

$$\frac{2}{5} \times 100\% = \frac{2 \times 100}{5} \% = \frac{200}{5} \% = 40\%$$

b) Decimal to Percent: Rewrite 0.76 as a percent.

$$0.76 \times 100\% = 76\%$$

c) Percent to Fraction: Rewrite 80% as a fraction.

$$80\% = \frac{80\%}{100\%} = \frac{80}{100} = \frac{4}{5}$$

d) Percent to Decimal: Rewrite 37% as a decimal.

$$37\% = \frac{37\%}{100\%} = \frac{37}{100} = 0.37$$

Note: Changing between decimals and percents is just a matter of moving the decimal point two places. Just make sure to move the decimal point in the correct direction; move the decimal point to the right to change from a decimal to percent and move the decimal point to the left to change from a percent to decimal.
Problem 3: Media Example – Changing Forms between Decimals, Fractions, and Percents

For each problem, you are given either a fraction, decimal, or percent. Find the other two missing forms and place them in the space provided.

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>_______</td>
<td>0.85</td>
</tr>
<tr>
<td>b)</td>
<td>_______</td>
<td>_______</td>
</tr>
<tr>
<td>c) $\frac{17}{25}$</td>
<td>_______</td>
<td>_______</td>
</tr>
<tr>
<td>d)</td>
<td>_______</td>
<td>1.237</td>
</tr>
<tr>
<td>e)</td>
<td>_______</td>
<td>_______</td>
</tr>
<tr>
<td>f) $\frac{2}{3}$</td>
<td>_______</td>
<td>_______</td>
</tr>
<tr>
<td>g)</td>
<td>_______</td>
<td>_______</td>
</tr>
<tr>
<td>h)</td>
<td>_______</td>
<td>_______</td>
</tr>
</tbody>
</table>
**Problem 4: You Try – Changing Forms between Decimals, Fractions, and Percents**

For each problem, you are given either a fraction, decimal, or percent. Find the other two missing forms and place them in the space provided.

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td></td>
<td>23.5%</td>
</tr>
<tr>
<td>b) (\frac{13}{20})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>c)</td>
<td>0.783</td>
<td></td>
</tr>
<tr>
<td>d)</td>
<td></td>
<td>126%</td>
</tr>
<tr>
<td>e)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>f)</td>
<td></td>
<td>200%</td>
</tr>
</tbody>
</table>
SECTION 4.3: FINDING PERCENTS GIVEN AN AMOUNT AND A WHOLE

In the last section, we wrote equivalent forms of ratios, fractions, decimals, and percents in multiple ways. In this section, we will look at application problems where we need to interpret the given information to find a ratio and write the ratio as a percent. In general, we found that

\[
\frac{\text{amount}}{\text{whole}} \times 100 = \text{percentage}
\]

We will need to use the context of the question to determine what given values are the amount and the whole and transform the result into a percent. Typically, it’s easier to determine which quantity is the whole. The whole is usually preceded by the word “of”. For example, in the test problem, 18 out of 25 points meant that 25 is the whole amount of points and 18 is the relative amount of points. We will observe this more in the problems that follow.

Problem 5: Media Example – Finding a Percent Given an Amount and a Whole

Write the corresponding scenario as a ratio of an amount multiplicatively compared to a whole. Then write the corresponding percent. Round any percents to two decimal places as needed.

a) Chanelle is driving to Washington on a 20 hour road trip. So far, she has driven for 8 hours. What percent of the hours has Chanelle already driven?

Write the corresponding ratio for this situation. Include units in your ratio.

Ratio: \[
\frac{\text{amount}}{\text{whole}} =
\]

Write the percent that corresponds to this ratio:

Write your answer as a complete sentence:

What percent of the trip remains? Explain.
b) Christian bought a $60 sweater. The tax on the sweater was $4.95 for a total cost of $64.95. What percent of the cost of the sweater was the tax?
Write the corresponding ratio and percent for this situation. Include units in your ratio.

Ratio: \( \frac{\text{amount}}{\text{whole}} = \) Percent:

Write your answer as a complete sentence:

What is the ratio and percent of the total cost including tax to the cost of the sweater? Include units in your ratio.

Ratio: \( \frac{\text{amount}}{\text{whole}} = \) Percent:

How does this percent compare to the percent of tax? What relationship do you notice?

c) Carol went shopping for a cell phone. The price was listed as $400. She had a coupon for $50 off. What percent of the original price is the coupon savings amount?
Write the corresponding ratio and percent for this situation. Include units in your ratio.

Ratio: \( \frac{\text{amount}}{\text{whole}} = \) Percent:

Write your answer as a complete sentence:

What is the ratio and percent of the reduced cost of the phone to the original cost of the phone? Include units in your ratio.

Ratio: \( \frac{\text{amount}}{\text{whole}} = \) Percent:

How does this percent compare to the percent of the coupon? What relationship do you notice?
Problem 6: You Try – Finding a Percent Given an Amount and a Whole

Write the corresponding scenario as a ratio of an amount multiplicatively compared to a whole. Then write the corresponding percent. Round any percents to two decimal places as needed.

a) Travis bought 60 cans of soda for a party. He bought 24 cans of diet cola and 36 cans of regular cola. What percent of the soda that Travis bought is diet cola? Write the corresponding ratio for this situation. Include units in your ratio.

Ratio: \( \frac{\text{amount}}{\text{whole}} = \)

Write the percent that corresponds to this ratio:

Write your answer as a complete sentence:

What percent of the soda is regular cola? Explain.

b) Faith was selling her old math book online. The book originally cost her $150. Based on her research, she can sell the book for $67.50. What percent of the original cost of the book can Faith earn back by selling her book? Write the corresponding ratio and percent for this situation. Include units in your ratio.

Ratio: \( \frac{\text{amount}}{\text{whole}} = \) Percent:

Write your answer as a complete sentence:

What percent of the original cost of the book will Faith lose by selling her book? Explain.
SECTION 4.4: FINDING AN AMOUNT GIVEN A PERCENT AND A WHOLE

In the previous sections, we were given ratios, fractions, or decimals and wrote them as percentages. In this section, we will learn how to find a percent of a quantity. A percent is always referring to a percent of something. We typically call this something the whole. For example,

1) You earned 83% of the points on a test. The whole refers to the total possible points on the test.
2) You gained 3% of your body weight last year. The whole refers to your body weight last year.
3) You are charged 7% tax on a purchase. The whole refers to the cost of your purchase.
4) The interest rate on your mortgage is 4.35%. The whole is how much you owe on your mortgage.

As you work through this section, make certain to focus on which quantities represents the whole, the amount multiplicatively compared to the whole, and the percent.

FACT: \( n\% \) means \( n \) per 100 or \( n \) for every 100.

When we find \( n\% \) of a number, we can think of cutting the whole into 100 equal pieces (each of size 1%) and then taking \( n \) copies of 1% to attain \( n \% \). Cutting into 100 pieces is equivalent to dividing by 100. Taking \( n \) copies is equivalent to multiplying by \( n \).

RULE: To find a percent of a whole (the amount),

1. Divide the percent by 100.
2. Multiply the whole by the equivalent fraction or decimal.
3. A general formula is

\[
\text{whole} \times \frac{\text{percent}}{100} = \text{amount} \quad \text{or} \quad \text{whole} \times \text{percent written as a decimal} = \text{amount}
\]
Problem 7: Worked Example – Finding an Amount Given a Percent and a Whole
State the givens and the goal and then find the indicated amounts.

a) Find 63% of 270.

Solution: Givens: Percent = 63%  Whole = 270  (The whole is after the word “of”)  
Goal: Amount

Fraction Method: \( \frac{270 \cdot 63}{100} = 170.1 \)

Decimal Method: 63% as a decimal is 0.63,  \( 270 \cdot 0.63 = 170.1 \)

Answer: 170.1

b) What is 23.4% of 18?

Solution: Givens: Percent = 23.4%  Whole = 18  (The whole is after the word “of”)  
Goal: Amount

Fraction Method: \( \frac{18 \cdot 23.4}{100} = 4.212 \)

Decimal Method: 23.4% as a decimal is 0.234,  \( 18 \cdot 0.234 = 4.212 \)

Answer: 4.212

c) 0.5% of $32,000 is what number?

Solution: Givens: Percent = 0.5%  Whole = 32,000  (The whole is after the word “of”)  
Goal: Amount

Fraction Method: \( \frac{32000 \cdot 0.5}{100} = 160 \)

Decimal Method: 0.5% as a decimal is 0.005,  \( 32,000 \cdot 0.005 = 160 \)

Answer: 160
d) Find 136% of 2.85.

**Solution:** Givens: Percent = 136% Whole = 2.85 (The whole is after the word “of”) Goal: Amount

Fraction Method: $2.85 \cdot \frac{136}{100} = 3.876$

Decimal Method: 136% as a decimal is 1.36, $2.85 \cdot 1.36 = 3.876$

**Answer:** 3.876

e) What is 0.87% of 92?

**Solution:** Givens: Percent = 0.87% Whole = 92 (The whole is after the word “of”) Goal: Amount

Fraction Method: $92 \cdot \frac{0.87}{100} = 0.8004$

Decimal Method: 0.87% as a decimal is 0.0087, $92 \cdot 0.0087 = 0.8004$

**Answer:** 0.8004

**Problem 8:** Media Example – Finding an Amount Given a Percent and a Whole Applications

Find the indicated amounts. Round your final answer to two decimal places as needed. Feel free to use your calculator for your computations. However, make sure to write down the expression that you put in the calculator.

a) Joey is taking a road trip from New York to Washington D.C. The trip is 226 miles. So far, he has driven 43% of the trip.

How many miles has Joey driven so far? Write your answer as a complete sentence.

How many miles does Joey have left to travel?

What percent of miles does Joey have left to travel?
b) Erica went shopping in Tempe and spent $213.53 on new work clothes. The sales tax rate in Tempe is 8.1%.

How much tax will Erica have to pay? Write your answer as a complete sentence.

What is the total cost of her purchase including tax?

What percent is the total cost of her purchase compared to the total cost without tax?

c) Ahmed went shopping for a tablet. The regular price was listed as $370. The store was having a 20% off sale.

How much will Ahmed save because of the sale? Write your answer as a complete sentence.

What is the reduced price of the tablet after the discount?

What percent is the reduced price of his purchase compared to the original price?
Problem 9: You Try – Finding an Amount Given a Percent and a Whole

Find the indicated amounts. Round your final answer to two decimal places as needed. Feel free to use your calculator for your computations. However, make sure to write down the expression that you put in the calculator.

a) Find 27% of 302.

b) What is 6.7% of 78?

c) Find 114% of 4.9.

d) 0.62% of 42 is what number?

e) Taylor wants to buy a new Fender guitar. The regular price is listed as $1200. The online merchant is having a sale for 35% off all purchases over $1000.

How much will Taylor save because of the sale? Write your answer as a complete sentence.

What is the reduced price of the guitar after the discount?

What percent is the reduced price of the guitar compared to the original price of the guitar?
SECTION 4.5: FINDING THE WHOLE GIVEN A PERCENT AND AN AMOUNT

In this section, we will be given a percent and the amount and will need to find the whole. As you work through this section, make certain to focus on which quantities represents the whole, the amount multiplicatively compared to the whole, and the percent.

Problem 10: Worked Example – Finding the Whole Given a Percent and an Amount

Question: 15 is 30% of what number?

Solution: First determine the givens and goal. Identifying the percent is easy since it is labeled with a percent symbol (or sometimes the word percent). In this case, 30%. Next determine the whole. The whole is after the word “of”. In this case, it says “of what number”. This means we are solving for the whole. Whatever remains (unknown or given) is the amount. In this case, the amount is 15.

Givens: 30% is the percent, 15 is the amount

Goal: The whole, or 100% is unknown.

Reasoning: When we want to find the whole, we want to find 100% of the known amount. In this example, we are given that 15 is 30% of the whole amount. So we can think of cutting the amount 15 into 30 equal pieces (each of size \( \frac{1}{100} \) or 1%) and then taking 100 copies of 1% to attain 100%. Cutting into 30 pieces is equivalent to dividing by 30. Taking 100 copies is equivalent to multiplying by 100.

So for this example,

\[
\text{The whole is } \frac{15 \times 100}{30} = \frac{15 \times 100}{30} = \frac{1500}{30} = 50
\]

1) The 30 in the denominator cuts the amount into 30 pieces of size \( \frac{1}{100} \) or 1%.

2) The 100 in the numerator takes 100 copies of these pieces of size \( \frac{1}{100} \) or 1%.

3) So multiplying the amount by \( \frac{100}{30} \) finds the whole or 100%.

RULE: To find the whole given an amount and its corresponding percent

\[
\text{whole} = \text{amount} \times \frac{100}{\text{percent}} \quad \text{or} \quad \text{whole} = \frac{\text{amount}}{\text{percent (written as decimal)}}
\]
Unit 4 – Percents and Applications

Problem 11: Worked Example – Finding the Whole Given a Percent and an Amount
Find the indicated wholes. Round your final answer to two decimal places as needed.

a) 4.32% of what number is 7.5?

Solution: Givens: Percent = 4.32%  Amount = 7.5
Goal: Whole (The whole is after the word “of”)

Fraction Method: $\frac{7.5 \times 100}{4.32} \approx 173.61$

Decimal Method: 4.32% as a decimal is 0.0432, $\frac{7.5}{0.0432} \approx 173.61$

b) 420 is 0.5% of what number?

Solution: Givens: Percent = 0.5%  Amount = 420
Goal: Whole (The whole is after the word “of”)

Fraction Method: $\frac{420 \times 100}{0.5} = 84,000$

Decimal Method: 0.5% as a decimal is 0.005, $\frac{420}{0.005} = 84,000$

c) 134.7% of what number is 2300?

Solution: Givens: Percent = 134.7%  Amount = 2300
Goal: Whole (The whole is after the word “of”)

Fraction Method: $\frac{2300 \times 100}{134.7} \approx 1707.50$

Decimal Method: 134.7% as a decimal is 1.347, $\frac{2300}{1.347} \approx 1707.50$
Problem 12: Media Example – Finding the Whole Given a Percent and an Amount Application

Find the indicated amounts. Round your final answer to two decimal places as needed. Feel free to use your calculator for your computations. However, make sure to write down the expression that you put in the calculator.

Dave has worked for the same employer for 5 years. His current salary is $73,500 which is 122.5% of his starting salary.

a) What was Dave’s starting salary?

b) If Dave received equal increases in pay every year, what was his raise per year?

Problem 13: You Try – Finding an Amount Given a Percent and a Whole

Find the indicated wholes. Round your final answer to two decimal places as needed. Feel free to use your calculator for your computations. However, make sure to write down the expression that you put in the calculator.

a) 18.7% of what number is 29.17?

b) 1915 is 23% of what number?

c) Amelia earned a scholarship and only needed to pay 47.3% of her tuition. If she paid $638.55, what was the full cost of her tuition before the scholarship?
SECTION 4.6: PERCENT INCREASE AND DECREASE

In this last section, we will learn about percent increase and percent decrease. We have already seen some problems that can be considered to fall in this category. For example,

1) A sale of 20% at a store is a percent decrease. The original price is 100%, we subtract off 20% of the original price, and the sale price is 80% of the original price.

2) The total amount of an item including 7% tax is a percent increase. The amount without tax is 100%, we add on 7% of the amount for tax, and the total price with tax is 107% of the amount without tax.

It is important to distinguish between the percent you are adding on (such as tax) or subtracting off (such as a discount) with the value after you have made these adjustments. A 50% increase means the new value is 100% + 50% = 150% of the original value or 1.5 times as large as the original value. You are not only finding 50% of the whole. You are increasing the whole by this 50%. We call 1.5 in this example the multiplicative factor since it is the number we multiply the original value by to obtain the new value. Keep this idea in mind when you solve percent increase and decrease problems as compared to problems where you are only finding a percent of a number.

Problem 14: Worked Example – Multiplicative Factors and Percent Increase and Decrease

Find the new percent that corresponds to the percent change. Write the multiplicative factor as a ratio over 100 and as a decimal.

<table>
<thead>
<tr>
<th>Percent Change</th>
<th>New Percent of Whole</th>
<th>Multiplicative Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 25% increase</td>
<td>100% + 25% = 125%</td>
<td>125/100 = 1.25</td>
</tr>
<tr>
<td>b) 13% decrease</td>
<td>100% − 13% = 87%</td>
<td>87/100 = 0.87</td>
</tr>
<tr>
<td>c) 4.25% increase</td>
<td>100% + 4.25% = 104.25%</td>
<td>104.25/100 = 1.0425</td>
</tr>
<tr>
<td>d) 32.7% decrease</td>
<td>100% − 32.7% = 67.3%</td>
<td>67.3/100 = 0.673</td>
</tr>
<tr>
<td>e) 115% increase</td>
<td>100% + 115% = 215%</td>
<td>215/100 = 2.15</td>
</tr>
<tr>
<td>f) 99% decrease</td>
<td>100% − 99% = 1%</td>
<td>1/100 = 0.01</td>
</tr>
</tbody>
</table>
Problem 15: Media Example – Percent Increase and Decrease

Determine the new amounts given the percent change. Feel free to use your calculator for your computations. However, make sure to write down the expression that you put in the calculator.

a) 150 is increased by 12%. What is the new amount?

b) 3000 is decreased by 27.5%. What is the new amount?

c) 1000 is decreased by 50%. The resulting amount is then increased by 50%. What is the new amount?

d) 600 is doubled. What is the new amount? What is the corresponding percent increase?

e) 500 is decreased by half. What is the new amount? What is the corresponding percent decrease?

Problem 16: You Try – Percent Increase and Decrease

a) Fill in the blank spaces below. Write the multiplicative factor as a ratio over 100 and a decimal.

<table>
<thead>
<tr>
<th>Percent Change</th>
<th>New Percent of Whole</th>
<th>Multiplicative Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>i. 8.75% increase</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ii. 27.4% decrease</td>
<td></td>
<td></td>
</tr>
<tr>
<td>iii. 132% increase</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
b) 37 is increased by 43%. What is the new amount?

c) 3000 is decreased by 65.4%. What is the new amount?

**Problem 17: Media Example – Percent Increase and Decrease Applications**

Find the indicated amounts. Round your final answer to two decimal places as needed. Feel free to use your calculator for your computations. However, make sure to write down the expression that you put in the calculator.

a) Julianna changed careers. Her old salary was $53,000 a year. Her new salary is 26% more per year. What is her new salary?

b) Jordan lost 17% of his body weight over the school year. If he originally weighed 247 pounds, what is his new weight?
c) The CPI Inflation Calculator measures the buying power of a dollar relative to different years. According to the Bureau of Labor Statistics, $1.00 in 1985 has the same buying power as $2.23 in 2016.

What is the multiplicative factor and percent increase from $1.00 to $2.23?

d) If Joe's salary in 1985 was $25,000 a year, how much would he need to make now just to keep up with inflation?

e) If Joe's salary is $62,000 a year in 2016, how much more is he making in addition to the inflation adjustment?

Problem 18: You Try – Percent Increase and Decrease

Find the indicated amounts. Round your final answer to two decimal places as needed. Feel free to use your calculator for your computations. However, make sure to write down the expression that you put in the calculator.

In 1970, the minimum wage was $1.60 per hour. According to the Bureau of Labor Statistics, $1.00 in 1970 has the same buying power as $6.19 in 2016.

a) What is the multiplicative factor and percent increase from $1.00 to $6.19?

b) What should the minimum wage be in 2016 adjusted for inflation to be comparable to the minimum wage in 1970?
UNIT 5 – SIMPLE AND COMPOUND INTEREST

INTRODUCTION

In this Unit, we will learn about simple and compound interest and their applications. These two types of interest are extensions of our study of percents and the foundation of financial formulas for saving and lending.

Section 5.1: Determine the interest and future amount of a one-time interest problem
Section 5.1: Determine the interest and future amount of a simple interest problem
Section 5.1: Use percents and dollar amounts to compare one-time and simple interest.
Section 5.1: Use formulas for simple interest problems solving for any letter.
Section 5.2: Determine the interest and future amount of a compound interest problem
Section 5.2: Use percents and dollar amounts to compare compound interest with different compounding periods
Section 5.2: Compare simple interest and compound interest problems
Section 5.2: Use formulas for compound interest problems solving for a future amount or principal.
Section 5.3: Use a TVM Solver to solve Compound Interest Problems
Section 5.4: Compute the Annual Percentage Yield for a savings and loan problem.
UNIT 5 – MEDIA LESSON

SECTION 5.1: SIMPLE INTEREST

Definition: Simple interest is a percent of the principal, the original amount saved or borrowed.

I. One-Time Simple Interest

Let's first look at an example of a one-time simple interest payment. By “one-time” we mean there is a flat single percentage rate and flat fee charged that corresponds to some given amount of time.

Definition: A flat single percentage rate is the percent of the original amount borrowed for some fixed amount of time. The percentage is only applied once to the original balance.

Definition: A flat single rate fee is the dollar amount that corresponds to the flat single percentage rate of the principal for the fixed amount of time.

Problem 1: Worked Example – One Time Simple Interest

David borrows $10,000 from his parents for college. He agrees to pay the money back in 3 years including 6% interest on the amount borrowed. How much interest will he pay? How much will he have to give to his parents in total after 3 years?

Solution: Let's first list the given information and our goals to organize our problem solving.

Given:
Principal: $10,000 (the original amount David borrows)
Flat single percentage rate: 6% (the percent interest charge agreed for the fixed time period)
Time: 3 years (the fixed time period is 3 years, but since it as a flat rate, it won't affect our computations)

Goals:
a) How much interest will he pay? (Flat rate fee)
   David will pay 6% of $10,000 in interest: $10,000 \cdot 0.06 = $600
   Answer: David will pay the flat rate fee of $600.

b) How much will he pay back in total?
   David will pay back original amount borrowed + flat rate fee = $10,000 + $600 = $10,600
   Answer: David will pay back $10,600 in total after 3 years.
II. Simple Interest over Time with an Annual Percentage Rate

One-time simple interest or flat percentage rates are not common in finance. Typically, a percentage rate is given that applies per year or annually. This means that every year of the loan or investment, the interest rate applies and accrues. Additionally, if the loan is for less than a year, only a corresponding fraction of the percentage applies.

Definition: An **annual percentage rate or APR** is the total interest that will be paid in one year. Definition: The **total percentage rate or Total Return** is the total interest as a percentage of the original amount.

**Problem 2: Worked Example – Simple Interest over Time with Annual Percentage Rates**

David tells his older brother John that he plans on borrowing $10,000 from his parents for college and that he will pay the money back in 3 years including 6% interest on the amount borrowed. John tells David that his parents are ripping him off! John says I will lend you the $10,000 with simple interest at a 2.5% APR for 3 years. If David accepts John’s offer, how much interest will he pay? How much will David have to give to John in total after 3 years? What is the corresponding total percentage rate for the deal John offered David? How does the deal John is offering compare to the parent’s offer?

**Solution:** Let’s first list the given information and our goals to organize our problem solving.

**Given:**
- Principal: $10,000 (the original amount David borrows)
- Annual Percentage Rate: 2.5% (the percent interest charge *per year*)
- Time: 3 years (length of loan)

**Goals:**

a) How much interest will he pay?
   - David will pay 2.5% of $10,000 in interest per year for 3 years.
   - Per year interest: $10,000 \cdot 0.025 = $250
   - Per year interest for 3 years: 3 \cdot $250 = $750
   - Answer: David will pay $750 in interest.

b) How much will he pay back in total?
   - David will pay back the original amount borrowed + interest = $10,000 + $750 = $10,750
   - Answer: David will pay back $10,750 in total after 3 years.

c) What is the corresponding total percentage rate for the deal John offered David?
   - David paid 2.5% per year for 3 years = 2.5% \cdot 3 = 7.5%
   - Answer: The corresponding total percentage rate is 7.5%.

d) Since the parent’s offer totaled 6% and $600 over 3 years, and John’s offer totaled 7.5% and $750 for 3 years, the parent’s offer is a better deal for David.
III. Patterns and Structure

One-Time Simple Interest
For one-time simple interest, we computed the interest and total repayment amount as follows.

Interest = principal \cdot \text{flat single percentage rate}

Total repayment amount = \text{principal} + \text{interest}
= \text{principal} + \text{principal} \cdot \text{flat single percentage rate}
= \text{principal} (1 + \text{flat single percentage rate})

Simple Interest over Time with an Annual Percentage Rate
To develop a formula for simple interest over time, we will create a table of values over time for Problem 2 and look at some patterns. Recall we had a principal of $10,000 and an APR of 2.5% for 3 years. We will generalize our pattern for \( t \) years, where \( t \) is an unknown number of years.

<table>
<thead>
<tr>
<th>Time in years</th>
<th>Accumulated interest</th>
<th>Accumulated Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$10,000</td>
</tr>
<tr>
<td>1</td>
<td>$10,000 \cdot 0.025(1) = $250</td>
<td>$10,000 + $250 = $10,250</td>
</tr>
<tr>
<td>2</td>
<td>$10,000 \cdot 0.025(2) = $500</td>
<td>$10,000 + $500 = $10,500</td>
</tr>
<tr>
<td>3</td>
<td>$10,000 \cdot 0.025(3) = $750</td>
<td>$10,000 + $750 = $10,750</td>
</tr>
<tr>
<td>( t )</td>
<td>$10,000 \cdot 0.025(( t ))</td>
<td>$10,000 + $10,000 \cdot 0.025(( t ))</td>
</tr>
</tbody>
</table>

Notice as time increases by 1 year, the accumulated interest and balance each increase by $250. This constant rate of change of $250 indicates that the accumulated interest and balance are \textit{linear with respect to time}. In general, this is the case for all simple interest relationships with a constant yearly increase of the original principal times the simple interest rate.

To create a general formula for \textit{any} principal value, \( P \), and \textit{any} interest rate, \( r \), we replace the given values of \( P = $10,000 \) and \( r = 0.025 \) with their corresponding letters.

<table>
<thead>
<tr>
<th>Time in years</th>
<th>Accumulated interest</th>
<th>Accumulated Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>( P \cdot r \cdot t )</td>
<td>( P + P \cdot r \cdot t )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( P(1 + r \cdot t) )</td>
</tr>
</tbody>
</table>
IV. Transforming Simple Interest Formula to Solve for P, r, or t.

We can solve the simple interest formula $A = P(1 + rt)$ for the letters $P$, $r$, or $t$. This way we will have a formula for *any letter* in a simple interest problem.

**Solve for P:** Divide both sides of the equation $A = P(1 + rt)$ by $(1 + rt)$.

\[
\frac{A}{1+rt} = \frac{P(1+rt)}{1+rt}
\]

\[
P = \frac{A}{1+rt}
\]

**Solve for r:** Divide both sides of the equation $A = P(1 + rt)$ by $P$, subtract 1, and divide by $t$.

\[
\frac{A}{P} = 1 + rt
\]

\[
A - P = rt
\]

\[
\frac{A - P}{t} = rt
\]

\[
r = \frac{A - P}{t}
\]

**Solve for t:** Follow the same steps for solving for $r$, but in the last step divide by $r$ instead of $t$.

\[
t = \frac{A - P}{r}
\]
Unit 5 – Simple and Compound Interest

V. Summary of Formulas

Simple Interest over Time with an Annual Percentage Rate

P = Principal in dollars (original amount borrowed or invested)

r = annual interest rate written as a decimal (i.e. Use 0.05 for 5%)

t = time in years

I = dollar amount of the interest (percent of principal)

A = end amount in dollars including principal and interest

q = total interest as a percentage of original amount

Formulas

\[ I = Prt \quad \text{or} \quad I = A - P \]

\[ A = P(1 + rt) \]

\[ q = rt \quad \text{or} \quad q = \frac{\text{Interest}}{\text{Principal}} \quad \text{(Note: Multiply by 100% to change to a percent)} \]

\[ P = \frac{A}{(1 + rt)} \]

\[ r = \frac{(A \div P - 1)}{t} \quad \text{(Note: Multiply by 100% to change to a percent)} \]

\[ t = \frac{(A \div P - 1)}{r} \]

Notes

1) Note that in most cases, capital letters in finance formulas refer to dollar amounts as do I, P, and A above. This can help you remember the units and meaning of the letters.

2) As mentioned earlier, one time simple interest is rarely used. If you are told that there is a simple interest rate being applied, it implies that it is over time and not a flat fee. The one time simple interest example is useful because it shows the difference between a flat percentage and a time dependent percentage, and also as a method of comparison. You will see the types of problems you will likely encounter with annual interest rates over time in the next Media Example.
Problem 3: Media Example – Simple Interest

Identify the givens and the goals in each problem and write your final answers as complete sentences.

a) Charlene invests $11,000 at 3% simple interest for 8 years. How much interest does she earn? How much is in the account at the end of the 8 year period? What percent of the original amount is the interest? Round your answer to the nearest cent.

b) Jordan needs to borrow $1000 for 5 months. If the bank charges him 6% simple interest per year. How much will he need to pay back after 5 months? How much interest will he pay?

c) You are investing in an account that earns 8% simple interest. How much would you need to deposit in an account now in order to have $5000 in the account in 10 years? What percent of the original amount is the interest? Round your answer to the nearest cent.
d) Trey wants to invest $4400 in a savings account that pays 7% simple interest. How long will it take for this investment to double in value? What percent of the original amount is the interest? Round your answer to the nearest tenth.

e) Yvonne invests $3500 in a savings account. Determine the simple interest rate required for Yvonne’s investment to double in value in 10 years. Round your answer to the nearest tenth of a percent.

**Problem 4: You Try – Simple Interest**

Identify the givens and the goals in each problem and write your final answers as complete sentences.

a) Kate invests $15,000 at 5.2% simple interest for 10 years. How much interest does she earn? How much is in the account at the end of the 10 year period? What percent of the original amount is the interest? Round your answer to the nearest cent.

b) Jim wants to invest $2300 in a savings account that pays 6% simple interest. How long will it take for this investment to double in value? What percent of the original amount is the interest? Round your answer to the nearest tenth.
With simple interest, you earn (or pay) a fixed constant amount of interest every year. For example, in Problem 2, David borrowed $10,000 at 2.5% interest for 3 years. The interest was $P \cdot r = $10,000 \cdot 0.025 = $250 per year, regardless of what year we considered. With compound interest, you earn (or pay) varying amounts of interest depending on the new balance of your account at the time of compounding.

I. Compound Interest over Time with Annual Compounding

In our first example, we will consider what David would pay if he borrowed $10,000 from a bank that charged the same interest rate as his brother John, namely, 2.5%. However, the bank will use compound interest instead of simple interest. In particular, the bank will compound interest on David's balance annually. This means that at the end of every year, the bank will compute the interest on the current balance of David's loan including prior interest over the time period of the entire loan.

Problem 5: Worked Example – Annual Compound Interest

David goes to the bank to borrow $10,000 for college. The bank offers him a loan of $10,000 at an interest rate of 2.5% compounded annually. This means that at the end of every year, the bank will add on 2.5% of his balance from the previous year.

How much will David have to give to the bank in total after 3 years? How much interest will David pay the bank in total? What is the corresponding total percentage rate for the deal the bank offered David? How does the deal the bank is offering compare to his brother John's offer?

Solution: Let's first list the given information and our goals to organize our problem solving.

Given:

Principal: $10,000 (the original amount David borrows)
Annual Percentage Rate: 2.5% (the percent interest charge per year)
Time: 3 years (length of loan)
Compounded: Annually (interest added on at the end of every year)
Goals:

a) How much will David pay back in total after 3 years?

We will keep a running tally of his account balance over the 3 years by computing his interest as simple interest of the starting balance which will include interest from the prior year.

<table>
<thead>
<tr>
<th>Time in Years</th>
<th>Starting Balance</th>
<th>Interest</th>
<th>Ending Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$10,000</td>
</tr>
<tr>
<td>1</td>
<td>$10,000</td>
<td>$10,000 \cdot 0.025 = $250</td>
<td>$10,000 + $250 = $10,250</td>
</tr>
<tr>
<td>2</td>
<td>$10,250</td>
<td>$10,250 \cdot 0.025 = $256.25</td>
<td>$10,250 + $256.25 = $10,506.25</td>
</tr>
<tr>
<td>3</td>
<td>$10,506.25</td>
<td>$10,506.25 \cdot 0.025 = $262.66</td>
<td>$10,506.25 + $262.66 = $10,762.91</td>
</tr>
</tbody>
</table>

Answer: David will pay back $10,762.91 in total after 3 years.

b) How much interest will he pay?

David will pay back: total amount paid back with interest – original amount borrowed

$10,762.91 – $10,000 = $762.91

Answer: David will pay $762.91 in interest.

c) What is the corresponding total percentage rate for the deal the bank offered David?

Total percentage rate = \( \frac{\text{interest}}{\text{principal}} \cdot 100\% = \frac{\$768.91}{\$10,000} \cdot 100\% = 7.6891\% \)

Answer: David paid 7.6891\% of the original principal in interest.

d) How does the deal the bank is offering compare to his brother John’s offer?

Answer: John’s offer totaled 7.5\% and $750 for 3 years. The bank’s offer totaled 7.6891\% and $762.91 for 3 years. So John’s simple interest offer is better than the bank’s compound interest offer even though they use the same interest rate.
Observe that for part $a$ in Problem 5, we computed each yearly ending amount to find the next amount. Let’s look at the patterns and structure for our computations to develop a general formula.

Year 0: $10,000 

Year 1: $10,000 \cdot (1 + 0.025) 
= $10,000(1 + 0.025) 
= $10,250 

Year 2: $10,250 \cdot (1 + 0.025) 
= $10,250(1 + 0.025) 
= $10,000(1 + 0.025)(1 + 0.025) 
= $10,506.25 

Year 3: $10,506.25 \cdot (1 + 0.025) 
= $10,506.25(1 + 0.025) 
= $10,250(1 + 0.025)(1 + 0.025) 
= $10,000(1 + 0.025)(1 + 0.025)(1 + 0.025) 
= $10,762.91 

Observe that, each year’s ending balance is the previous year’s balance times $(1 + r)$. So for every year, we have another factor of $(1 + r)$ in the computation. So our end amount after $t$ years is,

Annual Compounding Final Amount: $P (1 + r)^t$

So an alternative way for computing part $a$ is $P = $10,000, $r = 0.025$, and $t = 3$.

Annual Compounding Final Amount

$$= $10,000(1+0.025)^3 $$
$$= $10,000(1.076890625) $$
$$= $10,768.90625 $$
$$\approx $10,768.91 $$

Notice in the second line, we are multiplying the original principal, $10,000, by $1.076890625 = (1 + 0.025)^3$. In effect, we are finding $107.6890625\%$ of the original amount to determine the end amount. Since the original amount is $100\%$ of itself, this tells us that the corresponding total percentage rate for the end amount is $107.6890625\% - 100\% = 7.6890625\%$. This aligns with our approximated answer for part $c$, $7.6891\%$. 
II. Compound Interest and Compounding Periods

In the last problem, we saw that compounding $10,000 and 2.5% per year increased the total interest, as compared to simple interest, by $12.91 or 0.1891%. This may not seem like very much of a difference, but the compound interest only had 3 compounding periods in total. It is more likely in savings and loans that compounding will occur more frequently throughout the year. The more often compounding occurs, the more cumulative interest is charged (or paid).

We will define the letter $n$ to represent the number of compoundings that occur in one year when using compound interest.

- If the compounding is done annually (once a year), $n = 1$
- If the compounding is done semi-annually, $n = 2$
- If the compounding is done quarterly, $n = 4$
- If the compounding is done monthly, $n = 12$
- If the compounding is done daily, $n = 365$

Now let’s consider an example with quarterly compounding. We will use our results to develop a general formula for any number of compoundings.

Problem 6: Worked Example – Quarterly Compound Interest

David goes to the bank to borrow $10,000 for college. The bank offers him a loan of $10,000 at an interest rate of 2.5% compounded quarterly.

How much will David have to give to the bank in total after 3 years? How much interest will David pay the bank in total? What is the corresponding total percentage rate for the deal the bank offered David? How does this compare to the example with annual compoundings?

Given:

- Principal: $10,000 (the original amount David borrows)
- Annual Percentage Rate: 2.5% (the percent interest charge per year)
- Time: 3 years (length of loan)
- Compounded: quarterly (interest added on at the end of quarter year)
- Total number of compoundings: 12 (3 years $\times$ 4 per year $= 12$ compoundings)
- Quarterly Percentage Rate: 0.625% (divide the APR by the number of compoundings per year: $2.5\% \div 4 = 0.625\%$)
Goals:

a) How much will David have to give to the bank in total after 3 years?

We know the annual compounding formula is \( A = P (1 + r)^t \). How will this change for quarterly compounding?

Let’s temporarily ignore time and think in terms of the number of compoundings and the interest rate. We want to compound the interest 12 times with an interest rate of 0.625% (from the annual interest rate divided by 4: \( 2.5\% ÷ 4 = 0.625\% \)). This is equivalent to a loan for 12 years with an annual interest rate of 0.625%. In particular, we will use \( r = 0.625\% \) and \( t = 12 \). The principal will remain the same.

\[
A = 10,000 \left(1 + 0.00625\right)^{12} \approx \$10,776.33 \quad (\text{rounded to the nearest cent})
\]

Answer: David will give the bank \$10,776.33

b) How much interest will David pay the bank in total?

David will pay back: total amount paid back with interest – original amount borrowed

\[
\$10,776.33 – \$10,000 = \$776.33
\]

Answer: David will pay \$776.33 in interest.

c) What is the corresponding total percentage rate for the deal the bank offered David?

\[
\text{Total percentage rate} = \frac{\text{Interest}}{\text{Principal}} \times 100\% = \frac{\$776.33}{\$10,000} \times 100\% = 7.7633\% \\
\]

Answer: David paid 7.7633\% of the original principal in interest.

d) How does this compare to the example with annual compoundings?

Answer: The bank’s offer with annual compoundings totaled 7.6891\% and \$762.91 for 3 years. The bank’s offer with quarterly compoundings totaled 7.7633\% and \$776.33 for 3 years. So the annual compoundings are a better deal compared to the quarterly compoundings.
III. Summary and Formulas

Compound Interest
P = Principal in dollars (original amount borrowed or invested)
r = annual interest rate written as a decimal (i.e. Use 0.05 for 5%)
t = time in years
n = number of compoundings per year
I = dollar amount of the interest (percent of principal)
A = end amount in dollars including principal and interest
q = total interest as a percentage of the original amount

Formulas
\[ A = P \left(1 + \frac{r}{n}\right)^{nt} \]
\[ I = A - P \quad \text{or} \quad I = P \left(1 + \frac{r}{n}\right)^{nt} - P \]
\[ q = \frac{I}{P} \cdot 100\% \quad \text{(interest divided by principal as a percent)} \quad \text{or} \quad q = \left(1 + \frac{r}{n}\right)^{nt} - 1 \quad \text{(write as %)} \]

Note: We mentioned in Section 2.1 that with simple interest, the accumulated interest and balance are \textit{linear with respect to time}. With compound interest, notice that the variable \( t \) is located in the exponent of the formulas. This means that with compound interest, the accumulated interest and balance are \textit{exponential with respect to time}.

The graph to the right shows the account balance, in dollars, over time, in years, for an initial principal of $1 and an interest rate of 5%.

The red line graph represents simple interest and the blue exponential graph represents compound interest with annual compoundings.

If you follow each graph from left to right, you will see that as time increases, the difference between the graphs is also increasing. This is a good display of how compound interest is affected by time.
Problem 7: Worked Example – Solving for the Principal Given an End Amount

Ray needs $10,000 for a house down payment in 5 years. The bank offers him an investment at an interest rate of 4.8% compounded monthly. How much should Ray invest now to have $10,000 in 5 years? How much interest will Ray earn in total?

Given:
Principal: Goal (the original amount Ray invests)
Annual Percentage Rate: 4.8% (the percent interest earned per year)
Time: 5 years (length of investment)
Compounded: monthly (interest added on at the end of the month)
Total number of compoundings: 12 (5 years \(\times12\) per year = 60 compoundings)
Monthly Percentage Rate: 0.4% (divide the APR by the number of compoundings per year: 4.8% \(\div12\) = 0.4%)
(Also called periodic rate)

Goals:

a) How much should Ray invest now to have $10,000 in 5 years?

We know the general compounding formula is \(A = P \left(1 + \frac{r}{n}\right)^{nt}\). Since we want to solve for the initial investment, \(P\), we can divide both sides of this equation by the exponential term and the results is the formula below.

\[
P = \frac{A}{\left(1 + \frac{r}{n}\right)^{nt}}
\]

So using \(\frac{r}{n} = 0.004\), \(nt = 60\), and \(A = 10,000\) gives

\[
P = \frac{10,000}{(1+0.004)^{60}} = 7870.05
\]

Answer: Ray should invest $7870.05 now to have $10,000 in 5 years.

b) How much interest will Ray earn in total?

Ray will earn: total amount earned with interest – investment

\[
$10,000 - $7870.05 = $2129.95
\]

Answer: Ray will earn $2129.95 in interest.
Problem 8: Media Example – Compound Interest

Identify the givens and the goals in each problem and write your final answers as complete sentences.

a) Riley invests $7,000 at 3.4% interest compounded monthly for 8 years. How much is in the account at the end of the 8 year period? How much interest does he earn? What percent of the original amount is the interest? Round your answer to the nearest cent.

b) Melissa is investing in an account that earns 8% interest compounded quarterly. How much will she need to deposit in an account now in order to have $5000 in the account in 10 years? What percent of the original amount is the interest? Round your answer to the nearest cent.

Problem 9: You Try – Compound Interest

Identify the givens and the goals in each problem and write your final answers as complete sentences.

a) Aranza invests $23,000 at 12.9% interest compounded quarterly for 10 years. How much is in the account at the end of the 10 year period? How much interest does she earn? What percent of the original amount is the interest? Round your answer to the nearest cent.

b) Jenell is investing in an account that earns 6% interest compounded quarterly. How much will she need to deposit in an account now in order to have $12000 in the account in 10 years? What percent of the original amount is the interest? Round your answer to the nearest cent.
SECTION 5.3: USING A TVM SOLVER FOR COMPOUND INTEREST

The computations for solving for the rate or time in the compound interest formula can be a bit tedious. We can use an Online TVM calculator to solve for any value in the compound interest formula provided we are given all of the others. TVM stand for Time, Value, and Money. This gives us the ability to solve for r or t without using more complex mathematics. If you have a Ti graphing calculator, you likely have a TVM solver that you can use there as well.

The TVM calculator we will use is shown below. Read the meaning of each entry box below carefully.

PV: Present Value – initial amount in the account (P in our formula)
PMT: Payment – regular payment amount (Does not apply to compound interest. Set to 0.)
FV: Future Value – end amount of investment (A in our formula)
N: Total number of payments or compoundings (N = n · t in our formula)
I%: Annual interest rate (r in our formula)
P/Y or C/Y: How many payments or compoundings occur in one year

Solve: Click the value you want to solve for after entering the other information.

Important Note: All dollar values paid from the investor to the bank must be negative. All dollar values paid from the bank to the investor must be positive. Sometimes, it won't make a difference in the TVM calculator's answer except the sign of a result, but sometimes it will give you a wrong result. So make sure to follow this convention.

Investor pays $ to bank – negative $  
Investor receives $ from bank – positive $
Problem 10: Worked Example – Compound Interest – Using the TVM Solver

Identify the givens and the goals in each problem and write your final answers as complete sentences.

a) Penny wants to invest $12,000 and double her money in 15 years. If the interest is compounded monthly, what interest rate will she need to attain this goal?

Solution: First make a list of the givens and the goal for all the values in the TVM Solver.

- **PV:** $-12,000 (invest/pay 12,000)
- **PMT:** 0 (no payments made)
- **FV:** $24,000 (want 2 × 12,000 = 24,000)
- **N:** 180 (15 years · 12 compoundings per year)
- **I%:** Goal
- **P/Y** and **C/Y:** 12 (monthly compoundings)

Enter your values in the calculator as shown below and leave the “I%” box blank. After all of your values are entered, click “I% Solve” and the result will appear in the “I%” box as shown in the image below.

![](image)

Answer: Penny will need an interest rate of 4.63% to double her investment in 15 years.
b) Jake is borrowing $15,000 for home repairs at an interest rate of 7.26% compounded quarterly. He pays back $20,000 at the end of the loan period. How many years was his loan?

Solution: First make a list of the givens and the goal for all the values in the TVM Solver.

- PV: 15,000 (borrows 15,000)
- PMT: 0 (no payments made)
- I%: 7.26
- FV: −20,000 (pays back)
- P/Y and C/Y: 4 (quarterly compoundings)

Enter your values in the calculator as shown below and leave the “N” box blank. After all of your values are entered, click “N Solve” and the result will appear in the “N” box as shown in the image below.

The N value is 16. Note that this is 16 compounding periods which are each one quarter of a year. So the number of years is $\frac{16}{4} = 4$ years.

Answer: Jake’s loan was for 4 years.

**Problem 11:** You Try – Compound Interest – Using the TVM Solver

Sally invested $22,000 at an interest rate of 7.5% compounded monthly. Her account now has $40,000. How many years has her money been accruing interest?
We have seen that determining the value of an investment or loan isn’t as straightforward as looking at the percentage rate or APR. We have already computed one method of determining the total value for savings and loans, namely, the total percentage rate or total return. Another method similar to the total return is called the Annual Percentage Yield or APY.

Definition: The Annual Percentage Yield is the percent of the original amount the loan or investment will accumulate in interest in one year.

The APY is useful because it fixes time at one year so you can see the effect of different rates and compounding periods over the same period of time. Note that this doesn’t mean that one loan is better than another over the whole time period or varying time periods. The APY just gives us a benchmark for determine the interest over one year.

One way we computed the total percentage rate or total return is shown on the left. The APY uses the same general formula as the total return except time is fixed to one year. So in the formula, if replace \( t \) with 1, we get the following APY formula on the right.

\[
q = \left(1 + \frac{r}{n}\right)^n - 1
\]

\[
\text{APY} = \left(1 + \frac{r}{n}\right)^n - 1 \quad \text{(written as a percent)}
\]

Note that the APY is only dependent on two variables: the annual interest rate, \( r \), and the number of compoundings per year, \( n \).

Problem 12: Worked Example – Annual Percentage Yield

Rebecca wants to borrow $10,000. She sees the APR is 5% compounded daily. What is the APY for the loan? Write your answer as a percent rounded to three decimal places.

Solution: How much Rebecca wants to borrow isn't necessary to find the APY. We only need to use the information \( r = 0.05 \) and \( n = 365 \).

\[
\text{APY} = \left(1 + \frac{0.05}{365}\right)^{365} - 1 \approx 0.05127 \times 100\% = 5.127\%
\]

Problem 13: You Try – Annual Percentage Yield

Compute the APY for a 5% APR compounded quarterly. Write your answer as a percent rounded to three decimal places.
UNIT 6 – ANNUITIES AND LOANS

INTRODUCTION

In this Unit, we will learn about annuities and loans that use multiple payments or deposits and compound interest. These types of savings and loans are more practical than regular compound interest where we assumed a single lump sum was invested (or borrowed) at the beginning, and a single lump sum was withdrawn (or paid back) at the end.

Section 6.1: Develop the formula for a savings annuity

Section 6.1: Determine the interest and future amount of a savings annuity

Section 6.2: Use a TVM solver to find any missing value in a savings annuity problem

Section 6.3: Determine the interest and initial amount of a payout annuity

Section 6.3: Use a TVM solver to find any missing value in a payout annuity problem

Section 6.4: Use a TVM solver to find any missing value in a conventional loan

Section 6.5: Determine the number of payments and interest paid when using the minimum payment due on a credit card

Section 6.5: Determine the number of payments and interest paid when using equal fixed payments on a credit card

Section 6.5: Compare the number of payments and interest paid between the minimum payment due and fixed equal payments.

Section 6.6: Find a balance at various increments throughout a loan and determine the principal and interest paid.
UNIT 6 – MEDIA LESSON

SECTION 6.1: SAVINGS ANNUITIES

In Unit 5, we learned about simple and compound interest. One assumption we made for both types of interest was that there was a starting time and an ending time for the investment (or loan), and that a single amount would be deposited (or lent) at the beginning, and a single amount (with interest) would be withdrawn (or paid back) at the end.

These types of loans and investments are useful if you have a large sum of money to deposit or plan on having a large sum of money to pay back a loan in the future. However, for most people it is more practical to make smaller deposits or payments over time.

I. Savings Annuity Introduction and Requirements

We will first consider a savings annuity. A common use of a savings annuity is a retirement account such as a 401K, IRA, or a retirement agreement through an employer. We will look at an example of a savings annuity under this context and see how the money grows.

We will first state a few requirements and assumptions in our study of annuities to simplify the process and understand the patterns and structure better.

1) In general, an annuity requires equal and regular deposits or payments over time that earn compound interest. Meaning there is a fixed amount of money you will deposit at regular time intervals: a fixed amount, such as $100 and a regular time interval such as monthly. These values do not change over the life of the annuity.

2) We will assume all annuities are ordinary annuities. This means that the payment is made at the end of the time period. This will help us stick to a single formula. Another type of annuity is called an annuity due where the payments are made at the beginning of the time period. It's not very different so we will stick to ordinary annuities for consistency and simplicity.

3) We will assume all annuities are simple annuities. This means that the compounding period is the same as the period between payments. So if you make a payment once a month, your interest is compounded monthly.

4) All of the money you deposit stays in the account for the life of the annuity. You cannot take any money out of the account along the way.
II. A Savings Annuity Example

Problem 1: Worked Example – A Very Short Savings Annuity

We will first look at an example of a short term annuity so we can observe patterns and determine the general structure of an annuity.

Regina invests in a short term annuity. She makes equal payments of $100 per month for four months at an annual rate of 12%. Determine the value of her individual payments over time and find the ending value of her annuity.

Solution: Let’s first list the given information and our goals to organize our problem solving.

Given:

- Payment: $100 (the monthly deposit)
- Compoundings per year: 12 (same as number of payments per year)
- Annual rate: 12% (the percent of interest per year)
- Periodic interest rate: 1% (annual rate divided by compoundings per year)
- Time: \( \frac{4}{12} \) of a year (number of years)
- Total Compoundings: 4 (number of years times compoundings per year)

Goals:

1) Determine the value of her individual payments over time.

The table below shows how each individual payment accrues interest over the 4 months. Read this table by following each payment column downwards over time. Look for patterns and then read the analysis after the table.

<table>
<thead>
<tr>
<th></th>
<th>Payment 1</th>
<th>Payment 2</th>
<th>Payment 3</th>
<th>Payment 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Month 1</td>
<td>$100</td>
<td>$0</td>
<td>$0</td>
<td>$0</td>
</tr>
<tr>
<td>Month 2</td>
<td>$100 (1 + 0.01)</td>
<td>$100</td>
<td>$0</td>
<td>$0</td>
</tr>
<tr>
<td>Month 3</td>
<td>$100 (1 + 0.01)^2</td>
<td>$100 (1 + 0.01)</td>
<td>$100</td>
<td>$0</td>
</tr>
<tr>
<td>Month 4</td>
<td>$100 (1 + 0.01)^3</td>
<td>$100 (1 + 0.01)^2</td>
<td>$100 (1 + 0.01)</td>
<td>$100</td>
</tr>
<tr>
<td>End amount</td>
<td>$103.03</td>
<td>$102.01</td>
<td>$101.00</td>
<td>$100</td>
</tr>
</tbody>
</table>
Unit 6 – Annuities and Loans

First let’s consider the individual payment, Payment 1, and its value over time.

a) Payment 1 is made at the end of the month 1 so it doesn’t earn any interest for the month it is made. Every month after this, it earns compound interest.

b) In month 2, Payment 1 has a value of $100 \times (1 + 0.01). Notice that this is the compound interest formula for the end amount after one compounding at 1%.

c) In month 3, payment 1 will be compounded twice. So its value in month 3 is $100 \times (1 + 0.01)^2$ which is the compound interest formula for the end amount after 2 compoundings at 1%.

d) In month 4, payment 1 will be compounded three times. So its value in month 4 is $100 \times (1 + 0.01)^3$ which is the compound interest formula for the end amount after 3 compoundings at 1%.

Now each payment will follow this same pattern, but the number of compoundings (the exponent) will be different based on the month it was deposited.

2) Find the ending value of the annuity.

In the bottom row labeled End Amount, we can see the value of each of the 4 payments at end of the annuity. To find the total end amount of the annuity after 4 months, we would find the sum of these values.

End amount = $100 \times (1 + 0.01)^3 + $100 \times (1 + 0.01)^2 + $100 \times (1 + 0.01) + $100$

= $103.03 + 102.01 + 101 + 100 = 406.04$

Answer: The end value of the annuity is $406.04.

III. Developing a Formula

Computing this for many compoundings would be incredibly tedious! Fortunately, this is a type of mathematical expression called a Geometric Series and there is a formula to turn this sum into a more simplified expression. We won’t discuss this transformation here, but feel free to follow this Geometric Series Link to learn more about it.

The simplified form below represents the sum in an alternative form using a geometric series formula. You should verify using your calculator that the two expressions below are in fact equal and both compute to $406.04.$
Sum Form: $100 + $100 (1 + 0.01) + $100 (1 + 0.01)^2 + $100 (1 + 0.01)^3 = $406.04

Simplified Form: \[
\frac{100((1+0.01)^4-1)}{0.01} = 406.04
\]

Note the values in the simplified form that relate to our original problem;

Payment = $100
Periodic interest rate = 1% or 0.01
Total number of compoundings = 4

Using these values, we can now define a general formula for a savings annuity.

Savings Annuity

PMT = fixed equal payment that you deposit at regular time intervals
r = annual interest rate written as a decimal (i.e. Use 0.05 for 5%)
t = time in years
n = the number of compoundings and equal payments in one year
A = end amount in dollars including payments and interest
R = periodic interest rate: \[ \frac{r}{n} \]
N = total number of compoundings: \[ n \cdot t \]
I = total interest earned on the annuity

Formulas

\[
A = PMT \cdot \left(\frac{(1 + R)^N - 1}{R}\right)
\]

\[
I = A - N \cdot PMT \quad \text{Interest} = \text{End Amount} - \text{total # of payments} \times \text{payment amount}
\]
Problem 2: Worked Example – Savings Annuity

Sally is saving for retirement by making regular monthly payments into an IRA, a retirement account annuity. She deposits $150 per month at an interest rate 7.2% for 25 years. How much will Sally have in her account at the end of 25 years? How much interest will she earn?

Given: PMT = $150 per month  
\( r = 0.072 \)  
\( t = 25 \)  
\( n = 12 \)

Compute based on the given values:
\[ R = \frac{r}{n} = \frac{0.072}{12} = 0.006 \]  
(Note: If this decimal repeats after division, leave it in fraction form.)

\[ N = n \times t = 12 \times 25 = 300 \]

Goals:

a) How much will Sally have in her account at the end of 25 years?

\[ A = 150 \times \left( \frac{(1+0.006)^{300} - 1}{0.006} \right) = $125,429.92 \]

Answer: Sally will have $125,429.92 in her account after 25 years.

b) How much interest she will earn?

Sally made 300 payments of $150. So she deposited 300 \( \times \) $150 = $45,000

The difference between her payment total and the end value of her investment is her interest.

\( I = $125,429.92 - $45,000 = $80,423.92 \)

Answer: Sally earned $80,423.92 in interest.

Problem 3: You Try – Savings Annuity

Jamison is saving for retirement by making regular monthly payments into an IRA. He deposits $100 per month at an interest rate 4.8% for 20 years. How much will Jamison have in his account at the end of 20 years? How much interest will he earn?
SECTION 6.2: USING A TVM SOLVER

The computations for annuities can be a bit tedious. You cannot round any values beforehand or you will have a round off error. The previous example used nice numbers to make the computation easier. However, now that we have developed the general idea of an annuity, we can use software to perform the computations.

We will use an Online TVM calculator for our calculations. TVM stand for Time, Value, and Money. We can use this calculator to solve for any value in the annuity formula provided we are given all of the others. This gives us the ability to solve for R or N without using more complex mathematics. If you have a TI graphing calculator, you likely have a TVM solver that you can use there as well.

The TVM calculator we will use is shown below. Read the meaning of each entry box below.

| PV: Present Value – initial amount in the account |
| PMT: Payment – regular payment amount |
| FV: Future Value – end amount of annuity |
| N: Total number of payments or compoundings |
| I%: Annual interest rate |
| P/Y or C/Y: Number of payments or compoundings that occur in one year |

Buttons on the bottom: Click the one you want after entering the other information

**Important Note:** All dollar values paid from the investor to the bank must be negative. All dollar values paid from the bank to the investor must be positive. Sometimes, it won’t make a difference in the TVM calculator’s answer except the sign of a result, but sometimes it will give you a wrong result. So make sure to follow this convention.

*Investor pays* $ to bank – *negative* $    
*Investor receives* $ from bank – *positive* $
Problem 4: Worked Example – Savings Annuity with TVM Calculator – End Amount

We will solve Problem 2 using the TVM calculator.

Sally is saving for retirement by making regular monthly payments into an IRA, a retirement account annuity. She deposits $150 per month at an interest rate 7.2% for 25 years. How much will Sally have in her account at the end of 25 years? How much interest will she earn?

a) How much will Sally have in her account at the end of 25 years?

1) Setup for TVM calculator:
   PV: $0 (account starts at $0)
   PMT: −$150 (investor pays $150 a month)
   FV: Blank (what we want to calculate)
   Rate: 7.2% (annual interest rate)
   Periods: 300 (from 12 ∙ 25 = 300)
   Drop Downs: 12 (monthly payments)

2) Click on FV Solve to have the calculator solve for the future value or end amount.

After you click on the FV button, the future value or end amount will appear in the box you had left blank.

Observe that $125,423.92 is the same answer we found using the formula. This value is positive because amounts the investor receives are positive with this calculator.

Answer: Sally will have $125,423.92 in her account after 25 years.

b) How much interest will Sally earn?

Interest = FV – N × PMT = $125,423.92 – 300 × $150 = $125,423.92 – $45,000 = $80,423.92

Answer: Sally earned $80,423.92 in interest.
**Problem 5: Worked Example – Savings Annuity with TVM Calculator – Payment**

Danielle is opening a retirement savings account that offers an interest rate of 6.5%. She plans on retiring in 23 years. How much should her monthly payments be if she wants to have $300,000 when she retires?

**GOAL:** How much should Danielle’s monthly payments be?

1) **Setup for TVM calculator:**

   - **PV:** $0 (account starts at $0)
   - **PMT:** Blank (what we want to calculate)
   - **FV:** $300,000 (future value to investor)
   - **Rate:** 6.5% (annual interest rate)
   - **Periods:** 276 (from $12 \cdot 23 = 276$)
   - **Drop Down:** 12 (monthly payments)

2) **Click on PMT to have the calculator solve for the payment.**

   After you click on the PMT button, the payment will appear in the box you had left blank.

   Observe that the payment value is listed as −$472.19. This value is negative because amounts the investor pays to the bank are negative with this calculator.

**Answer:** Danielle will need to make monthly payments of $472.19 to have $300,000 at the time of retirement.
Problem 6: Media Example – Savings Annuity with TVM Calculator

a) Find the end value of a savings annuity at an annual rate of 4.7% for 20 years and a weekly payment of $25.

b) Find the monthly payment needed to have an end value of $100,000 after 12 years at an annual rate of 9.3%.

Problem 7: You Try – Savings Annuity with TVM Calculator

a) Tom invests in a savings annuity with an interest rate of 8.8% compounded monthly. If Tom makes payments of $60 for 22 years, how much will he have in his account at the end of 22 years?

b) Maggie is planning to save for retirement by making regular monthly payments into a 401K earning 11.9% interest. She wants to have a total of $240,000 at the end of 20 years. How much should Maggie's monthly payments be?
SECTION 6.3: PAYOUT ANNUITY

In the last section, we learned about savings annuities. With a savings annuity, you make regular payments over time. At the end of the annuity time period, you have an end amount that includes all of your payments and interest.

I. Introduction to Payout Annuities

With a **payout annuity**, you start with money in your account, and you withdraw regular payments over time. Any remaining money after your series of withdrawals continues to earn interest. At the end of the annuity time period, you have withdrawn your initial investment and interest and your account balance is $0.

Payout annuities are useful once a person has retired. When you have retired, you may want a certain amount of money a month to pay your general expenses. You probably don’t need the full amount of your IRA retirement account all at once. This way, you can still earn interest on the amount you don’t need right away.

II. Payout Annuity Formula

The payout annuity formula is similar to the savings annuity formula. Without going into too much depth, you can think of it as reversing time on a savings annuity and changing payments to withdrawals. This makes the payout formula reverse the order of subtraction in the numerator of the savings formula, and negate the total number of compoundings which is time dependent.

**Payout Annuity**

- **PMT** = fixed equal payment that you withdraw at regular time intervals (considered positive)
- **r** = annual interest rate written as a decimal (i.e. Use 0.05 for 5%)
- **t** = time in years
- **n** = the number of compounding and equal withdrawals in one year
- **P** = the initial amount in the account (considered negative)
- **R** = periodic interest rate: \( \frac{r}{n} \)
- **N** = total number of compoundings: \( n \cdot t \)

**Formula**

\[
P = PMT \cdot \frac{1 - (1 + R)^{-N}}{R}
\]
Problem 8: Worked Example – Payout Annuity – Initial Amount

When you retire, you want to be able to take $1500 a month out of a payout annuity for your expenses. If you are planning on a 20 year retirement, and the annuity earns 8.4% annual interest, how much will you need to put in your payout annuity at the beginning of your retirement?

Givens:
PMT = $1500
r = 0.084
t = 20
n = 12
R = 0.007
N = 240

Goal: Solve for the original deposit, P.

\[ P = 1500 \cdot \frac{1 - (1 + 0.007)^{-240}}{0.007} = -174,114.01 \]

TVM Calculator Settings:

Start:

<table>
<thead>
<tr>
<th>PV:</th>
<th>PMT:</th>
<th>FV:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1500</td>
<td>0</td>
</tr>
</tbody>
</table>

Click on PV Solve:

<table>
<thead>
<tr>
<th>PV:</th>
<th>PMT:</th>
<th>FV:</th>
</tr>
</thead>
<tbody>
<tr>
<td>-174,114.01</td>
<td>1500</td>
<td>0</td>
</tr>
</tbody>
</table>

Note: The result is negative because the investor needs to pay this amount to the bank. The payments are positive because the investor receives this money as a withdrawal monthly.

Answer: You need to deposit $174,114.01 at the beginning of retirement to withdraw payments of $1500 a month for 20 years.
Problem 9: Worked Example – Payout Annuity – Payout Amount

Jeff has $250,000 from a savings annuity to invest in a payout annuity for retirement with a 6.75% annual interest rate. He wants the money to be paid out annually over 20 years. How much will he be given as a payout per year? How much will he earn in interest?

Solution:

a) How much will he be given as a payout per year?

Initial TVM Solver Settings:

| PV: | -250000 |
| PMT: | 0 |
| FV: | 0 |
| N: | 20 |
| I%: | 6.75 |

Click on PMT Solve:

| PV: | -250000 |
| PMT: | 23141.67 |
| FV: | 0 |
| N: | 20 |
| I%: | 6.75 |

Answer: Jeff will receive a payout of $23,141.67 per year for 20 years.

b) How much will he earn in interest?

Jeff will receive a total of 20 × $23,141.67 = $462,833.40

The interest earned is the difference between his initial investment and the total amount he received in payouts.

$462,833.40 - $250,000 = $212,833.40

Answer: Jeff will earn $212,833.40 in interest.
Problem 10: Media Example – Payout Annuity with TVM Calculator

a) Find the initial amount that needs to be invested in a payout annuity with an annual rate of 4.7% for 20 years to receive a monthly payout of $1250.

b) Find the monthly payment you receive for a payout annuity over 10 years at an annual rate of 9.3% with an initial investment of $100,000.

Problem 11: You Try – Payout Annuity with TVM Calculator

a) Lara inherits some money and wants to deposit a portion in a payout annuity to receive equal payments over time while earning interest. She wants monthly payouts of $2,900 for 20 years. If the interest rate is 10.3% how much should he invest now? How much interest will have earned in total?

b) You are retiring and investing $290,000 in a payout annuity for your monthly expenses. If you are planning on a 16 year retirement, and the payout annuity earns 11.5% annual interest, how much will your monthly payments be? How much interest will you earn?
SECTION 6.4: MORTGAGES AND CAR LOANS

In the last section, we discussed payout annuities. With a payout annuity, you deposited a lump sum into the bank, and then made equal regular withdrawals over time until your balance was $0. When you take out a home mortgage or car loan, you follow this same process in reverse. You receive (withdraw) a lump sum from the bank to pay for your house or car, and then you make equal regular payments (deposit) over time until your balance is $0. These types of loans are called conventional, installment, or amortized loans.

We can use the same formula as we used for payout annuities, but instead of depositing a lump sum at the beginning, you withdraw a lump sum at the beginning. And instead of withdrawing payments you deposit payments. This will change the meaning of our variables and the sign of two of our values in the TVM solver. Otherwise, it’s the same process.

Conventional Loans

P = the initial amount you withdraw (considered positive)
PMT = fixed equal payment that you pay at regular time intervals (considered negative)
FV = future value (end amount)
r = annual interest rate written as a decimal (i.e. Use 0.05 for 5%)
t = time in years
n = the number of compounding and equal payments in one year
R = periodic interest rate: \( \frac{r}{n} \)
N = total number of compoundings: \( n \cdot t \)

Formula

\[
P = PMT \cdot \frac{1-(1+R)^{-N}}{R}
\]
Problem 12: Worked Example – Car Loan – How much can you borrow?

You review your personal budget and you can afford car payments of $350 per month. If a bank offers you a loan at 6.78% for 5 years, how much can you borrow for your car? How much will you pay in total for your loan including interest?

Givens:

PMT = $350
r = 0.0678
t = 5
n = 12
R = \frac{0.0678}{12}
N = 60

Goal: a) Solve for how much you can borrow, P.

\[
P = 350 \cdot \frac{1 - (1 + \frac{0.0678}{12})^{-60}}{\frac{0.0678}{12}} = $17,768.69
\]

TVM Calculator Settings:

Start:

Click on PV Solve button:

Answer: You can borrow $17,768.89 to purchase a car.

b) How much will you pay in total for your loan including interest?

60 payments of $350 = $21,000

Answer: You will pay $21,000 in total for your loan.
Problem 13: Worked Example – Mortgage – How much is your payment?

You found the home of your dreams! After the down payment, you need to borrow $210,000. If a bank offers you a loan at 5.72% for 30 years, how much will your monthly payment be? How much will you pay in total for your loan including interest?

Givens:

\[ \begin{align*} 
& PV = \$210,000 \\
& r = 0.0572 \\
& t = 30 \\
& n = 12 \\
& R = \frac{0.0572}{12} \\
& N = 360 
\end{align*} \]

(a) Goal: How much will your monthly payment be?

TVM Calculator Settings:

Start:

Click on PMT Solve button:

Answer: Your monthly payment will be $1221.50

(b) How much will you pay in total for your loan including interest?

360 payments of $1221.50 = $439,740

Answer: You will pay $439,740 in total for your loan.
Problem 14: Media Example – Conventional Loans

a) You make car payments of $421.76 per month. If the APR is 7.23% for 5 years, how much did you borrow for your car? How much will you pay in total for your loan including interest?

b) You borrow $340,000 to purchase a home. The interest rate is 6.5%, and the loan is for 30 years. How much will your monthly payments be?

Problem 15: You Try – Conventional Loans

a) You review your personal budget and you can afford car payments of $340 per month. If a bank offers you a loan at 7.8% for 4 years, how much can you borrow for your car? How much will you pay in interest?

b) You are taking out a mortgage to buy a home. After the down payment, you need to borrow $290,000. If a bank offers you a loan at 6.6% for 25 years, how much will your monthly payment be? How much will you pay in interest?
SECTION 6.5: CREDIT CARDS

We have seen the power of compound interest over time. In the previous sections we saw that even when you are making equal regular payments, you will pay a lot of interest if you borrow money. The banks know this and do their best to maximize their earnings.

I. Credit Cards: Minimum Payment Due

With credit cards, you do not make equal regular payments. The credit card company gives you a minimum payment due that is based on your current balance. It varies by company and card, but the minimum payment usually covers the interest you owe that month and a small percentage of your principal. This way your debt will decrease, but not by very much each month. They also usually compound your interest daily even though you make payments monthly. This way they can charge you even more interest.

In this section, we will assume you are trying to pay off your credit card debt and are not making any new charges. We will use an online credit card payment tool to determine how long it will take you to pay off your credit card if you only make the minimum payment due every month. Then we will use what we learned in previous sections about equal regular payments to see the cost savings over time of making payments larger than the minimum due.

Problem 16: Worked Example – Credit Cards – Minimum Payment Due

John has a credit card that has a balance of $3000 charging a 17% APR (credit cards usually have higher interest rates too). He decides to pay it off and not make any more charges. The credit card company computes the minimum payment by adding the interest due + 1% of the principal of the account.

If John makes only the minimum payment every month, how long will it take him to pay off the credit card? How much will he pay in interest? What percent of the original $3000 is the total interest paid?

Input: I entered the information given in the problem at this Minimum Payment Due website. Here is how I filled in the values.

Current Balance: $3000

Interest Rate: 17%

What are the components of your minimum payment calculation?

Percentage of balance: Interest + 1% (dropdown menu)

Minimum Dollar Amount: $25 (cards usually have a minimum charge if the above value is less)
Output: By clicking on “calculate” I was given the following result:

“It will take you 169 months to pay off your debt, if you make minimum monthly payments on a balance of $3000.00 with a 17% APR. In that time, you will pay $3318.15 in interest charges

We recommend that you pay more than the minimum payment whenever possible. If you make only the minimum payment each month, it will take you longer and cost you more to clear your balance.”

Answers:

a) It will take John 169 months or 14 years and 1 month to pay off his credit card in full.

b) John will pay $3318.15 in interest. (Note: This is just the interest! He also paid back $3000)

c) \[
\frac{3318.15}{3000} \times 100\% = 110.65\%
\]

John paid a total of 110.65% interest on the original balance on his credit card.

II. Credit Cards: Further Analysis of Minimum Payment Due

You will also see a tab called “Table” on this website. It tells you; the amount of each payment, how much of each payment goes to principal and interest, your remaining balance, and the total interest paid. The output for the first three payments for Problem 12 are displayed below.

<table>
<thead>
<tr>
<th>Minimum Payment</th>
<th>Principal Paid</th>
<th>Interest Paid</th>
<th>Remaining Balance</th>
<th>Total Interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>72.5</td>
<td>30</td>
<td>42.5</td>
<td>2970</td>
<td>42.5</td>
</tr>
<tr>
<td>71.78</td>
<td>29.7</td>
<td>42.08</td>
<td>2940.3</td>
<td>84.85</td>
</tr>
<tr>
<td>71.05</td>
<td>29.4</td>
<td>41.65</td>
<td>2910.9</td>
<td>126.23</td>
</tr>
</tbody>
</table>

Since there are 169 payments, it is not feasible to get a sense of the patterns without a graph. There are two graphs for us to analyze below based on the entire 169 month time interval.
Graph 1: This graph shows the remaining balance and total interest paid over 169 months’ time. Notice at payment 107, the remaining balance is decreasing more rapidly and the total interest is increasing less rapidly. This is when the minimum payment hits the constant value of $25 for the remaining payments. This shows when you are paying more than the 1% principal + interest, you are charged less interest and the debt gets paid off more rapidly.

![Graph 1: Credit Card Repayment](image1.png)

Graph 2: This graph shows the minimum payment, principal paid, and interest paid over 169 months’ time. Again notice the change in the graphs at 107 payments. This is when the minimum payment due is $25 per month.

![Graph 2: Credit Card Repayment](image2.png)

Based on these graphs and our knowledge of compound interest and amortized loans with equal regular payments, we can investigate making equal regular repayments and seeing the cost savings.
Note that the first payment was a minimum payment of $72.50. So we'll use this as a monthly payment in the TVM solver and compare the results.

**Problem 17: Worked Example – Credit Cards – Making Equal Regular Payments**

John has a credit card that has a balance of $3000 charging a 17% APR. He decides to pay it off and not make any more charges. After seeing the results of paying only the minimum payment due, he decides to commit to paying $72.50 a month until the credit card is paid off.

If John makes a payment of $72.50 every month, how long will it take him to pay off the credit card? How much will he pay in interest? What percent of the original $3000 is the total interest paid? How do these values compare to making the minimum payment due?

**TVM Calculator Settings:**

a) Goal: How long will it take him to pay off the credit card?

Note: This is an approximation based on the credit card company compounding interest monthly. They usually compound daily even though you make monthly payments. This will give us a decent underestimate.

Start: Click on N Solve button:

We will round up to 63 periods which is 5 years and 3 months.

Answer: John will pay off his credit card in 5 years and 3 months.

b) How much will he pay in interest?

Total Amount Paid: 63 payments \( \times \) $72.50 per payment = $4567.50

Interest Paid: $4567.50 – $3000 = $1567.50

c) What percent of the original $3000 is the total interest paid?

\[
\frac{1567.50}{3000} \times 100\% = 52.5\%
\]
d) How do these values compare to making the minimum payment due?

Let's make a chart for comparison:

<table>
<thead>
<tr>
<th></th>
<th>Time until paid off</th>
<th>Total interest in dollars</th>
<th>Total interest as a % of the original $3000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum Payment</td>
<td>14 years and 1 month</td>
<td>$3318.15</td>
<td>110.65%</td>
</tr>
<tr>
<td>$72.50 per month</td>
<td>5 years and 3 months</td>
<td>$1567.50</td>
<td>52.50%</td>
</tr>
<tr>
<td>Difference (row 1 – row 2)</td>
<td>8 years and 10 months</td>
<td>$1750.65</td>
<td>58.15%</td>
</tr>
</tbody>
</table>

**Problem 18: Media Example – Credit Card Payments**

Sylvia wants to pay off a credit card loan of $4500 charging a 10.95% APR. She decides to pay it off and not make any more charges. The credit card company computes the minimum payment by adding the interest due + 1% of the principal of the account.

a) If Sylvia makes only the minimum payment every month, how long will it take her to pay off the credit card? How much will she pay in interest? What percent of the original $4500 is the total interest paid? (Use this [Credit Card Payment Web Calculator](#))

b) If Sylvia makes a payment of $86.06 every month, how long will it take her to pay off the credit card? How much will she pay in interest? What percent of the original $4500 is the total interest paid? How do these values compare to making the minimum payment due? (Use the [TVM Solver](#) and round to two decimals assuming a partial payment)
Problem 19: You Try – Credit Card Payments

John has a credit card that has a balance of $5000 charging a 9% APR. He decides to pay it off and not make any more charges.

The credit card company computes the minimum payment by adding the interest due + 1% of the principal of the account with a minimum dollar amount of $25.

(a) If John makes only the minimum payment every month, how long will it take him to pay off the credit card? How much will he pay in total in interest? (Use this Credit Card Payment Web Calculator)

(b) John noticed that his first and largest minimum payment due was $87.50. After seeing the results of paying only the minimum payment due, he decides to commit to paying $87.50 a month until the credit card is paid off.

If John makes a payment of $87.50 every month, how long will it take him to pay off the credit card? How much will he pay in total in interest? (Use the TVM Solver and round to two decimals assuming a partial payment)

(c) How much will John save paying $87.50 per month compared to paying the minimum payment due every month?
SECTION 6.6: PRINCIPAL AND INTEREST

We saw in the Credit Card Example that when you increase your payment every month, the extra amount pays off more principal and shortened the length and interest paid on your loan. In this section, we will look at the balance at various stages of repayment and compare the interest paid and principal paid.

Problem 20: Worked Example – Amount of Principal and Interest per Payment.

You borrow $22,000 for a car loan and make payments of $510 a month for 5 years (60 months) at an interest rate of 13.83%.

a) Use the TVM Calculator to compute your remaining balance after every year.

Solution: To find your balance at the end of each year, Use that PV = $22,000, PMT = -$510, P/Y=12, r =13.83, and enter the # of payments based on the number of years you have been making payments on the loan, N = 12 \cdot number of years you have made payments.

Below is the TVM set up for the first year's ending balance, the opposite of the FV. For the next years, you will only change N and enter N = 24, N = 36, N = 48, and N = 60

Year 1 Ending Balance: $18,719.84
Year 2 Ending Balance: $14,956.14
Year 3 Ending Balance: $10,637.63
Year 4 Ending Balance: $5,682.52
Year 5 Ending Balance: −$3.03

(Note: Your final ending balance should be $0, but there will likely be a little round off error on your last year. The FV was +$3.03 which means you overpaid by $3.03. Take the opposite result −$3.03 like you did for the other FV's. In practice, the lender would slightly reduce or increase your final payment(s). We'll use the exact model.)
b) Given that every year, your payments total $6120, determine how much you paid in interest and principal each year.

Solution:

1) Make a chart representing the 4 quantities per year: starting balance, ending balance, principal paid, and interest paid.

2) Fill in the ending balances from part a. Note the ending balance of Year 1 equals the starting balance for Year 2 and that this holds for any two consecutive years. Fill in these amounts.

3) The principal paid in a year is the difference between the starting and ending balance for each year. So find: **Principal Paid = Starting Balance – Ending Balance** for each year.

4) Interest paid in a year is the difference between the total yearly payment, $6120 (12 \cdot $510) and the Principal Paid. So find: **Interest Paid = Total Paid per Year – Principal Paid in Year.**

<table>
<thead>
<tr>
<th>Starting Balance</th>
<th>Ending Balance</th>
<th>Principal Paid</th>
<th>Interest Paid</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year 1</td>
<td>$22,000</td>
<td>$18,719.84</td>
<td>$3280.16</td>
</tr>
<tr>
<td>Year 2</td>
<td>$18,719.84</td>
<td>$14,956.14</td>
<td>$3763.70</td>
</tr>
<tr>
<td>Year 3</td>
<td>$14,956.14</td>
<td>$10,637.63</td>
<td>$4318.51</td>
</tr>
<tr>
<td>Year 4</td>
<td>$10,637.63</td>
<td>$5,682.52</td>
<td>$4955.11</td>
</tr>
<tr>
<td>Year 5</td>
<td>$5,682.52</td>
<td>−$3.03</td>
<td>$5685.55</td>
</tr>
</tbody>
</table>

c) What trends in principal paid per year and interest paid per year do you notice as the number of payments increases?

Solution:

As the number of payments increases, the principal paid per year increases.

As the number of payments increases, the interest paid per year decreases.
UNIT 7 – ORGANIZING AND INTERPRETING DATA

INTRODUCTION

In this lesson, we will learn about a branch of mathematics that deals with collecting, organizing, and interpreting data. This branch of mathematics is called statistics. In addition, the word statistics is often used to denote the data and information that are being collected and interpreted.

Section 7.1: Interpret data from a Table

Section 7.2: Interpret data from a Pie Chart

Section 7.3: Interpret data from a Bar Chart

Section 7.4: Interpret data from a Histogram

Section 7.5: Interpret data from a Line Graph

Section 7.6: Interpret data from a Scatterplot and Regression Line
Our world is filled with information. We need to have the ability to organize, analyze, and make sense of this vast amount of information to make informed and well-reasoned decisions. When we are given a set of data points, particularly if that set is very large, we want to get a feel for the data. A first step is often to analyze or create tables and graphs to help us visualize and organize the information.

In the first three sections, we will look at categorical data. Data are categorical if they can be broken into categories that are not numerical in value. For example, if you ask a class of first graders their favorite color, their answers may include red, blue, green, etc. These values are not numerical. We cannot add together red + green. However, we can collect the frequency or percentages (or relative frequency) of these categories in the responses.

Definitions:

The frequency of a category is the number that counts how many data values are in the specific category. For example, if you ask a class of first graders their favorite color, we might find that 10 students responded blue. The number 10 is the frequency of students who responded blue.

The percentage or relative frequency of a category is the frequency of the category divided by the total number of data points (or sum of the frequencies of all the categories) written as a percent. These percentages are useful for comparing the frequencies in the different categories relative to the whole. For example, if you ask a class of first graders their favorite color and 10 responded blue, we might want to ask the question, “out of how many” to determine if 10 is large amount relative to the total number of students in the class. If there were 25 students in total and 10 responded blue, 40% of the students responded blue. So 40% is the relative frequency of students who responded blue. In contrast, if there were 200 students in total and 10 responded blue, only 5% of the students responded blue. So relative frequencies provide us with a numerical way to compare categories relative to the whole data set.

When analyzing a data table, chart or graph, make sure to look for information on what categories are represented. A legend is a key next to data display that describes the meaning of the colors or categories in the chart. You should also look for headers or labels that correspond to the title of the graph or define the horizontal or vertical axes of a graph.
SECTION 7.1: INTERPRETING DATA FROM TABLES

Problem 1: Media Example – Interpreting Data from a Table

A table presents information in rows and columns as shown in this example.

I. Birth Rates and Populations around the World in 2011

<table>
<thead>
<tr>
<th>Country</th>
<th>Birth Rate (per 1000 people per year)</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>French Polynesia</td>
<td>15.53</td>
<td>294,935</td>
</tr>
<tr>
<td>Brazil</td>
<td>17.79</td>
<td>203,429,800</td>
</tr>
<tr>
<td>Australia</td>
<td>12.33</td>
<td>21,766,710</td>
</tr>
<tr>
<td>Sudan</td>
<td>36.12</td>
<td>45,047,500</td>
</tr>
<tr>
<td>Russia</td>
<td>11.05</td>
<td>138,739,900</td>
</tr>
<tr>
<td>India</td>
<td>20.97</td>
<td>1,189,173,000</td>
</tr>
<tr>
<td>French Polynesia</td>
<td>15.53</td>
<td>294,935</td>
</tr>
</tbody>
</table>

Source: http://www.indexmundi.com/g/

a) Which country has a birth rate of 17.79 per 1000 people per year?

b) Which country has the smallest birth rate?

c) Which country has the largest population?

d) Based on the birth rate and population of French Polynesia, approximately how many births were there in 2011?
Problem 2: Media Example – Interpreting Data from a Table

A researcher collected data on the marital status and highest education level of 300 subjects. The results are in the chart below.

<table>
<thead>
<tr>
<th></th>
<th>High School</th>
<th>Some College</th>
<th>Bachelor's Degree</th>
<th>Master's Degree</th>
<th>PhD or higher</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single</td>
<td>19</td>
<td>35</td>
<td>20</td>
<td>10</td>
<td>6</td>
<td>90</td>
</tr>
<tr>
<td>Married</td>
<td>11</td>
<td>37</td>
<td>47</td>
<td>34</td>
<td>21</td>
<td>150</td>
</tr>
<tr>
<td>Divorced</td>
<td>5</td>
<td>10</td>
<td>9</td>
<td>3</td>
<td>3</td>
<td>30</td>
</tr>
<tr>
<td>Widowed</td>
<td>3</td>
<td>9</td>
<td>8</td>
<td>7</td>
<td>3</td>
<td>30</td>
</tr>
<tr>
<td>Total</td>
<td>38</td>
<td>91</td>
<td>84</td>
<td>54</td>
<td>33</td>
<td>300</td>
</tr>
</tbody>
</table>

a) How many people in the study are married?

b) How many people in the study have a highest education level of Bachelor’s Degree?

c) What percent of those with a Bachelor’s Degree are married?

d) What percent of those who are married have a Bachelor’s Degree?

e) What percent of those who are divorced have some college?

Problem 3: You Try – Interpreting Data from a Table

Use the data from Problem 2 to answer the following questions

a) How many people in the study have a highest education level of Some College?

b) What percent of those with Some College are Single?

c) What percent of those who are Single have a highest education level of Some College?

d) What percent of all people in the study are Divorced?
SECTION 7.2: INTERPRETING DATA FROM PIE CHARTS

Problem 4: Media Example – Interpreting Data from a Pie Chart

A Pie Chart (also called a Circle Graph) is used to show how a whole amount is broken up into parts that add up to the whole. If written as percents of the whole, and the sum of the percents is 100%.

I. The pie chart below represents what percentage, on average, an American spends their income on different categories.

![Pie Chart](Image)


a) What percent of their income does the average American spend on healthcare?

b) For the average person, what is the single biggest category of expense?

c) Suppose your monthly salary is $2200. How much should you be spending on Food?
Problem 5: You Try – Interpreting Data from a Table or Pie Chart

I. In 2009, the source Index Mundi gives the data below for the number of people in a country with access to the internet.

<table>
<thead>
<tr>
<th>Country</th>
<th># with Access to the Internet</th>
<th>Total Population</th>
<th>% with Access to the Internet</th>
</tr>
</thead>
<tbody>
<tr>
<td>El Salvador</td>
<td>746,000</td>
<td>7,185,218</td>
<td>10.38%</td>
</tr>
<tr>
<td>Greece</td>
<td>4,971,000</td>
<td>10,737,430</td>
<td>46.30%</td>
</tr>
<tr>
<td>Iraq</td>
<td>325,900</td>
<td>28,945,660</td>
<td>1.13%</td>
</tr>
<tr>
<td>Russia</td>
<td>40,853,000</td>
<td>140,041,200</td>
<td>29.17%</td>
</tr>
</tbody>
</table>

a) Which country has the largest number of people with access to the internet?

b) Which country has the largest percentage of the population with access to the internet?

II. In 2009, the Bureau of Labor Statistics reported a surge in volunteerism. At this time, there were a reported 63,361 volunteers in the U.S. The pie chart below shows the different categories in which these people volunteered.

a) What category has the largest percent of volunteers?

b) Find the number of people who volunteered in an Educational capacity. Round your answer to the nearest whole number.
SECTION 7.3: INTERPRETING DATA FROM BAR GRAPHS

Problem 6: Media Example – Interpreting Data from a Bar Graph

A bar graph represents categorical data by using bars that correspond to either the frequency or percentage in each category. The bars are either horizontal or vertical. Our example below uses vertical bars.

I. The bar graph below represents the frequency of eye color in a class of students.

- a) How many students have hazel eyes?
- b) What eye color is most frequent in the class?
- c) What eye color is least frequent in the class?
- d) How many students are in the class?
- e) What percentage of the class has blue eyes? Round to the nearest hundredth.
Problem 7: You Try – Interpreting Data from a Bar Graph

The blog ISeeCars collected data on 12 million used car sales by state in 2016. They determined the most frequent car color sold in each state. The bar chart below shows the number of states that had each color as their largest used car sales based on their data. (Yellow is the most popular color in used car sales in AZ. Who knew?)

![Bar Chart](image)

a) What color(s) is reported as the most popular color of used car sales in the most number of states?

b) What color(s) is reported as the most popular color of used car sales in the least number of states?

c) What percentage of states have gold as the color of the most frequently sold used car in 2016?
SECTION 7.4: INTERPRETING DATA FROM HISTOGRAMS

Histograms are very similar to bar charts except that the data on the horizontal axis are numerical and ordered, and the bars do not have spaces between them. We have the bars touch since the horizontal axis serves as a number line and we want to represent all the numbers in the range of the data. In the data below, the horizontal axis represents women's heights to the nearest inch. It is likely the women aren't the exact value represented, but we break them into these integer values to understand the data better.

Problem 8: Media Example – Histograms

Sonya surveyed 66 women at her college and asked them their height rounded to the nearest inch. The histogram below shows the number of women who indicated each height in inches.

a) How many women were 63 inches tall?

b) How many women were at least 67 inches tall?

c) What percent of women were less than 62 inches tall? (Round the percent to two decimal places as needed.)
Problem 9: Media Example – Histograms and Bin Width

In the last example, there were 12 bars or bins with width 1. Meaning, each bar represented one height value where the women’s heights were rounded to the nearest inch. When making a histogram, if we have too many bins, it would be hard to make sense of the data. For example, if we measured the heights in the last example to the nearest tenth of an inch, there would be 120 bars making it more difficult to make sense of the data. We also do not want too few bins. If we only had two bins for those shorter or taller than 65 inches, we would lose a lot of information.

When we combine value into intervals, we are aggregating the data. We lose some information about the specific values in each bin or interval, but it is easier to understand the trends in the data. Typically, as a rule of thumb, we would like to have approximately 4 – 12 equally spaced bars.

In the data below, we grouped ages into four equally sized intervals. For example, 16 – 20 represents anyone of age at least 16, up to 20. Although the next bin starts at 21, the 16 – 20 bin includes anyone greater than exactly 20 years old and less than 21 years old. If we listed each age from 16 to 35, there would be 15 bins or bars which would make it more difficult to get a grasp on the age groups.

The coordinator of a dart tournament recorded the ages of its participants and used the results to construct the histogram shown above.

a) Find the total number of participants in the tournament.

b) Find the number of participants 25 years of age or younger.

c) Find the percent of total participants that are 25 years of age or younger. (Round the percent to two decimal places as needed.)
Problem 10: Media Example – Histograms with Continuous Data

In the data below, we grouped the hemoglobin measurements into ten equally sized intervals. Notice the first bar is labeled 11.5 on the left and 12 on the right. This means the first bar contains all the hemoglobin values greater than or equal to 11.5 and strictly less than 12. The second bar includes all the values greater than or equal to 12 and strictly less than 12.5, etc… Also observe that the vertical axis represents the percent of all the subjects that fall in each interval not the count or frequency.

Kennedy collected data on the amount of hemoglobin in gallons per deciliter in the blood of 409 subjects. The histogram above shows the percent of subjects whose hemoglobin fell in the indicated ranges.

a) What percent of subjects had a hemoglobin measurement of at least 14 and less than 14.5?

b) How many subjects had a hemoglobin measurement of 15 or greater? (Round to the nearest person)

c) A hemoglobin measurement of less than 12 may indicate anemia and requires further testing. How many subjects fall in this range? (Round to the nearest person)
Problem 11: You Try – Interpreting Histograms

The Desert Botanical Garden in Phoenix gathered data on the heights of 921 Saguaro cacti. The histogram below approximates their results.

a) How many of the Saguaro Cacti's heights are at least 22 feet, but less than 26 feet?

b) What height corresponds to the most frequent bin grouping?

c) How many of the Saguaro Cacti are less than 10 feet in height?

d) What percent of the Saguaro Cacti are less than 10 feet in height?
SECTION 7.5: INTERPRETING DATA FROM LINE GRAPHS

Thus far, we have dealt with data with one variable or univariate data. In the last few examples, we looked at height of cacti, hemoglobin measurements, ages, height of women, and their corresponding frequencies or percentages. In the next two sections we will look at bivariate data or data involving two variables. We are interested in how and if two variables are correlated or change in relation to each other.

First we will look at line graphs. Line graphs contain multiple data points where each point represents two variable quantities. For example, the time of day and the corresponding temperature; 3 pm and 112°F. We collect and plot multiple points like this and connect consecutive points with a line segment.

A line graph may be used to represent ordered categorical data or numerical data. Notice the word “ordered” is before the word category. In the first example, the categories are the days of the week. You cannot add Monday + Tuesday so they are categories, but they occur in an order. With respect to time, Monday is before Tuesday, then Wednesday, etc... This is not the case for non-ordered categories such as type of clothing; pants, shirts, sweaters, shoes.

When the categories have an order, it makes sense to put them in this order. This way we can also interpret how the frequency or corresponding variable changes as the order changes. It wouldn't make sense to use a line graph if the categories are not ordered. We could rearrange the categories in any order and it would change the graph. So if there is not an order to the categories, we'd likely use a bar graph or pie chart.

To determine if two variables are correlated, observe the graph from left to right and look for patterns.

1. If the second variable increases as the first variable increases, we say the variables are positively correlated.

2. If the second variable decreases as the first variable increases, we say the variables are negatively correlated.

3. There can be intervals where the graphs are positively correlated and intervals where the graphs are negatively correlated. If these are patterns and not just random jumps in the graph, we will say the variables are correlated without indicating direction.

4. If there are no discernible patterns, we will say the variables are not correlated.
Problem 12: Media Example – Interpreting Data from a Line Graph

U.S. Climate Data reports the average high temperatures by month in Tempe, Arizona, as shown in the line graph below.

a) What is the average high temperature in Tempe AZ in April?

b) Which month has the greatest average high temperature?

c) Which month has the lowest average high temperature?

d) Which month(s) have an average high temperature of 78°F?

e) What is the largest temperature decrease between two consecutive months?

f) What is the smallest temperature increase between two consecutive months?

g) Do the data appear to be correlated? Explain.
Problem 13: You Try – Interpreting Data from a Line Graph

Anson kept track of the number of gallons of gas his car used and the number of miles he traveled. The data are in the line graph below.

a) How many miles did Anson travel on 6 gallons of gas?

b) What is the difference between the number of miles traveled for 8 and 9 gallons of gas?

c) How many miles per gallon did Anson’s car get between 8 and 9 gallons of gas?

d) How many miles per gallon did Anson’s car get between 4 and 6 gallons of gas?

e) Do the number of gallons of gas used and the number of miles traveled appear to be correlated? Explain.
SECTION 7.6: SCATTERPLOTS AND LINEAR REGRESSION

In the last example, the number of miles traveled and the number of gallons of gas used were positively correlated. The data without labels or connecting lines are shown below. This is called a scatterplot.

Observing the graph from left to right, as the number of gallons of gas used increases, the number of miles driven increases. The data do not fall exactly on a single line, but you can see that they are close to doing so. We say these data are approximately linear.

In Problem 13 parts c and d, you computed what is called the average rate of change between a pair of points. This is the slope of the line segments between these points. The average rate of change between 8 and 9 gallons was 23 miles per gallon. The average rate of change between 4 and 6 gallons was 29 miles per gallon.

The chart below shows all the average rates of change between consecutive data points.

<table>
<thead>
<tr>
<th>Points (gallons, miles)</th>
<th>Computation</th>
<th>Average Rate of Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2, 47) and (4, 156)</td>
<td>156 – 47 = \frac{109}{2} = 54.5</td>
<td>54.5 miles per gallon</td>
</tr>
<tr>
<td>(4, 156) and (6, 214)</td>
<td>214 – 156 = \frac{58}{2} = 29</td>
<td>29 miles per gallon</td>
</tr>
<tr>
<td>(6, 214) and (8, 307)</td>
<td>307 – 214 = \frac{93}{2} = 46.5</td>
<td>46.5 miles per gallon</td>
</tr>
<tr>
<td>(8, 307) and (9, 330)</td>
<td>330 – 307 = \frac{23}{1} = 23</td>
<td>23 miles per gallon</td>
</tr>
<tr>
<td>(9, 330) and (11, 424)</td>
<td>424 – 330 = \frac{94}{2} = 47</td>
<td>47 miles per gallon</td>
</tr>
</tbody>
</table>
There is variability in the miles per gallon over the time intervals. This could be due to highway versus city miles, traffic, air condition use, and many other factors of which we are not aware. Regardless, we would like a line that describes the data considering all of its points. The slope of this line will give us an idea of the rate of change for all the points. To do this we will find what is called the **line of best fit model** using **least squares linear regression**. Least squares linear regression finds the line that minimizes the sum of the squared vertical distances between the line and the data points. We will use computer applications to find this line.

The red line below is the linear regression model for the data from Problem 11.

{image}

The equation given in the right lower hand corner is the equation for the least squares regression line. The variable x represents the variable on the horizontal axis and the variable y represents the variable of the vertical axis. We will rename these variables to correspond to our context below.

Let \( g = \) the number of gallons of gas used

Let \( m = \) the number of miles driven.

\[ m = 40.46g - 23.385 \]

Interpreting the line: The slope is the number multiplied by \( g \). We can interpret this as 40.46 miles per gallon.

The vertical intercept is (0, -23.385). This can be interpreted as, if you have 0 gallons of gas you can travel -23.385 miles. Which of course makes no sense!

A famous Statistician, George Box stated, "**All models are wrong, but some are useful.**" In statistics, you should never expect a perfect answer. You are finding a best approximation. The negative miles value for the vertical intercept is likely due to the variability we mentioned above; highway versus city miles, traffic, air condition use, and many other factors of which we are not aware. So our model is wrong, but it is a good approximation and it is useful.
Problem 14: Media Example – Interpreting a Linear Regression Equation

The data below approximates NASA's data on the change in Antarctica's ice mass measured in Gigatonnes relative to 2002 (shown by the blue dots) and a linear regression equation (shown by the red line). Note: A metric tonne is approximately 2200 pounds. A Gigatonne is one billion metric tonnes.

The regression equation approximates the change in the amount of ice since 2002 in Antarctica by the following variables and formula.

\[ C = -127 \cdot t + 34 \]

Use this equation to approximate the change in the amount of ice in Antarctica in the year 2018 by completing the sentence below.

The regression equation estimates that in the year 2018 the change in the amount of ice since the year 2002 will be ________ Gigatonnes.
Problem 15: You Try – Interpreting a Linear Regression Equation

For your You Try problem, I would like you see what we call a *spurious correlation*. This is when two variables appear correlated, but they are not really directly related to one another. In the example below, both of the variables are increasing over time. When you plot the corresponding values by year, they are changing together though they are completely unrelated except by time.

This example serves as an important warning:

*Correlation does not imply causation*

Meaning, just because two variables are correlated it does not mean one is causing the other. Causation claims can only be justified when we run a properly designed experiment. An experiment cannot be done with the variables below (at least not ethically), but our common sense tells us this is not a useful correlation and definitely *not* a causal relationship.

![A Spurious Correlation](http://www.tylervigen.com/spurious-correlations)

Data Source: [http://www.tylervigen.com/spurious-correlations](http://www.tylervigen.com/spurious-correlations)

a) Based on the regression line, as the number of people who died by becoming tangled in their bedsheets increases, the per capita cheese consumption in pounds ______________.
b) How are the number of people who died by becoming tangled in their bedsheets and the per capita cheese consumption in pounds correlated?

c) The slope of the regression equation is approximately 0.0079. Interpret the slope by completing the sentence below.

As the _______________ increases by 1,

the _______________ increases by ___________ pounds.

d) The regression equation approximates the per capita cheese consumption in pounds by the following variables and formula.

\[ C = \text{the per capita cheese consumption in pounds} \]
\[ p = \text{the number of people who died by becoming tangled in their bedsheets} \]
\[ C = 0.0079 \cdot p + 26.853 \]

Use this equation to approximate the per capita cheese consumption in pounds when 403 people died by becoming tangled in their bedsheets and then complete the sentence below.

The regression equation estimates that when 403 people die by becoming tangled in their bedsheets, the per capita cheese consumption is ___________ pounds.
UNIT 8 – MEASURES OF CENTRAL TENDENCY AND DISPERSION

INTRODUCTION

In this unit, we will learn about the measures of central tendency or averages. Specifically, we will learn about the mean, median and mode and how they relate to a histogram or distribution. Then we will discuss three measures of dispersion called the range, standard deviation, and quartiles and analyze boxplots as a tool for analyzing a distribution.

Section 8.1: Compute the mean, median, and mode for small data sets

Section 8.1: Compute the mean, median, and mode from a frequency table

Section 8.2: Identify common distribution shapes

Section 8.2: Identify the approximate and exact location of the mean, median and mode for common distribution shapes given histograms and frequency tables

Section: 8.3: Determine the standard deviation of a data set

Section 8.3: Interpret information about a distribution using the standard deviation

Section 8.4: Determine the 5-number summary for small data sets

Section 8.4: Determine the 5-number summary from a box plot

Section 8.4: Interpret information about a distribution from a box plot
UNIT 8 – MEDIA LESSON

In the previous section, we learned about different ways to display data so we could visualize relationships and patterns. In this unit, we are going to compute quantities that help us understand other attributes of data sets such as average (center) and spread (dispersion).

SECTION 8.1: COMPUTING THE MEAN, MEDIAN, AND MODE

Most people use the word *average* in their everyday speech. We may talk about someone of average height, your course average in a class, or a *typical* example of something such as an average Quick Trip store or an average sit-com TV show.

In mathematics, when people hear the word average, they are usually referring to the *arithmetic mean*. There are other types of means, but since the arithmetic mean is used the most, we usually refer to it as just the *mean*. There are two other averages or measures of center that are also common called the *median* and the *mode*. The list below states the procedures for finding each of these measures of center.

Determining measures of center:

1) **Mean:** Add up all of the data values and divide by the number of data values

2) **Median:** Order the data from least to greatest. If there are an odd number of data values, the median is the data value in the middle. If there are an even number of data values in the list, the median is the mean of the two middle data values.

3) **Mode:** The most frequent value or category represented in a data set.

*Note:* Both the mean and median involve computations on the data set so we can only find the mean and mode for numerical data and not for categorical data. For example, if we asked people their favorite type of ice cream, we couldn’t add up the ice cream or put the ice cream in an ordered list. However, the description of the mode includes the most frequent category. So we can find the type of ice cream that the most people gave as a response. So the mode is the only measure of center used for categorical data.
Problem 1: Worked Example – Finding the Mode of Categorical Data

I. The bar graph below represents the frequency of eye color in a class of students. Find the mode of the data set.

![Frequency of Eye Color](image)

Solution: We actually answered this question in the last section, however it was phrased, “What eye color is most frequent in the class?” This is equivalent to asking what the mode is. Since the data are in a bar chart, we can observe that the highest bar is for the brown eye category.

Answer: The mode is brown.

II. Data may also be presented in a list or a frequency table as shown below. If you are given a list of a lot of values, you may want to create a tally chart and frequency table to better organize the data and find the mode.

Stacey is in fashion design and is interested in the most popular shirt colors of the season. She collects data and creates the table below. What is the mode color in the data set?

<table>
<thead>
<tr>
<th>Color</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pink</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Blue</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Green</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gray</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Solution: Observe that the largest number in the frequency column is 9 and that this corresponds to blue. It is common to mistake the number 6 as the mode since it is the most frequent of the frequencies. Remember that your answer should be one of the original data values and not a frequency value.

Answer: The mode is blue.
Problem 2: Worked Example – Finding the Mean, Median, and Mode of Numerical Data

Sydney ran an experiment to determine the optimal conditions for growing artichokes. The data below displays the weights of the artichokes she grew (measured in grams). Find the mean, median, and mode of the weight of the artichokes.

Weight of Artichokes in grams:
68.3, 47.2, 53.2, 72.5, 55.4, 36.8, 72.5, 72.5, 41.9, 56.8

Solution:

a) To find the mean weight of the artichokes, we add up all of the weights and divide by the number of data values

\[
mean = \frac{68.3 + 47.2 + 53.2 + 72.5 + 55.4 + 36.8 + 72.5 + 72.5 + 41.9 + 56.8}{10} = \frac{577.1}{10} = 57.71
\]

b) To find the median weight of the artichokes, we order the data from least to greatest and find the number in the middle of the ordered list.

Given Data: 68.3, 47.2, 53.2, 72.5, 55.4, 36.8, 72.5, 72.5, 41.9, 56.8

The table below displays the data ordered from least to greatest. The first row indicates the placement of the data value in the ordered list.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>36.8</td>
<td>41.9</td>
<td>47.2</td>
<td>53.2</td>
<td>55.4</td>
<td>56.8</td>
<td>68.3</td>
<td>72.5</td>
<td>72.5</td>
<td>72.5</td>
</tr>
</tbody>
</table>

There are two values in the middle of the data set highlighted above. When there is no singular data value in the middle, we find the mean of the two middle values.

\[
median = \frac{55.4 + 56.8}{2} = 56.1
\]

Observe that the median value is not in the data set. This is a possibility.

c) To find the mode weight of the artichokes, we find the most frequent weight represented in a data set. Observe that the only repeated value is 72.5 grams and its frequency is 3. So 72.5 is the mode.

Answer: The three measures of center are as follows:

\[
\text{mean} = 57.71 \text{g} \quad \text{median} = 56.1 \text{g} \quad \text{mode} = 72.5 \text{g}
\]
Problem 3: Media Example – Finding the Mean, Median, and Mode from a Frequency Table

Melissa surveyed the girls in her 4th grade class and asked them their height rounded to the nearest inch. The frequency table below shows the number of girls who indicated each height in inches. Determine the mean, median, and mode height of the girls.

<table>
<thead>
<tr>
<th>Height (in inches)</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>52</td>
<td>1</td>
</tr>
<tr>
<td>56</td>
<td>2</td>
</tr>
<tr>
<td>57</td>
<td>3</td>
</tr>
<tr>
<td>58</td>
<td>5</td>
</tr>
<tr>
<td>59</td>
<td>3</td>
</tr>
<tr>
<td>60</td>
<td>3</td>
</tr>
<tr>
<td>62</td>
<td>1</td>
</tr>
<tr>
<td>63</td>
<td>1</td>
</tr>
<tr>
<td>64</td>
<td>1</td>
</tr>
</tbody>
</table>

Problem 4: You Try – Finding the Mean, Median, and Mode

a) At a family gathering, Tom asked his relatives in which state they were born. The frequency table below shows his data. Find the mode of the categorical data.

<table>
<thead>
<tr>
<th>Color</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arizona</td>
<td>7</td>
</tr>
<tr>
<td>New York</td>
<td>3</td>
</tr>
<tr>
<td>California</td>
<td>9</td>
</tr>
<tr>
<td>Nevada</td>
<td>7</td>
</tr>
<tr>
<td>Utah</td>
<td>5</td>
</tr>
</tbody>
</table>

b) A group of adults were asked how many children were in their family. The data below show their responses. Find the mean, median, and mode of the number of children in the adult's families.

Number of Children: 1, 3, 4, 2, 2, 1, 0, 2, 3, 5, 4, 0
c) Marshall surveyed his classmates by asking them how many pairs of shoes they owned. His results are in the frequency table below. Find the mean, median, and mode of the number of pairs of shoes per person.

<table>
<thead>
<tr>
<th>Number of Pairs of Shoes per person</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
</tr>
</tbody>
</table>

SECTION 8.2: DISTRIBUTION SHAPE AND MEASURES OF CENTER

In this section, we are going to look at a few common distribution shapes for histograms. Then we will investigate how distribution shape and the different measures of center are related.

I. Common Distribution Shapes: Six common distribution shapes are shown below with brief descriptions of their properties. In general, distributions do not look exactly like these because of variability in data, but they will have this general shape.

A. Mirror image with respect to center

![Symmetric Distribution](image)

B. Two bars with highest frequency

![Bimodal Distribution](image)
II. Symmetric Distributions and Measures of Center

We will begin with symmetric distributions. We will look at distributions that are exactly symmetric. However, some distributions are approximately symmetric and their measures of center will be close to these results.

First, imagine you took 8 quizzes in class and got exactly 90 points on each of them. What would the mean, median, and mode be?

Quiz Scores: 90, 90, 90, 90, 90, 90, 90, 90

Mean: \[ \frac{90 + 90 + 90 + 90 + 90 + 90 + 90 + 90}{8} = \frac{8 \cdot 90}{8} = 90 \]

Median: Since all of your scores are 90, the middle score when arranged in order (eight 90's in a row) is the median or 90.

Mode: Since 90 is your only score, we could say it is the most frequent score and therefore 90.
So in the case where all of the data points are 90, so are the mean, median, and mode. Take a moment to convince yourself this would be the case for any set of data where all of the values are identical.

Now we are going to use this idea to look at distributions of different shapes. We will start by assuming we have eight data points that are 0 for ease of computation. The dot plot below is like a histogram, but we will use dots instead of bars so we can imagine moving some of the data points. So the dot plot below is for the data set; 0, 0, 0, 0, 0, 0, 0, 0.

Since all of the data values are 0, we know the mean, median, and mode are 0. In particular, we also know the sum of the data points is 0. We are going to rearrange the data points to look at different distribution shapes while maintaining a mean of 0. If we rearrange the data points so the sum of the data points remains 0, the mean will remain 0. We will create three such distributions.

**Dotplot 1:** The dot plots below show how the top four dots were moved to the left or right one unit. The resulting distribution is on the right. This way we added 2 and subtracted 2 from the sum of the data, maintaining a mean of 0.
Dotplot 2: The dot plots below start with the previous result and move two more dots from the 0 column. One dot was moved two to the left and the other two to the right. The resulting distribution is on the right. This way we added 2 and subtracted 2 from the sum of the data, maintaining a mean of 0.

![Dotplot 2](image)

Dotplot B

Sum: $1 \cdot -2 + 2 \cdot -1 + 2 \cdot 0 + 2 \cdot 1 + 1 \cdot 2 = 0$

Dotplot 3: Here we moved the last remaining two 0 dots to 3 and -3, maintain the sum and mean of 0.

![Dotplot 3](image)

Dotplot C

Sum: $1 \cdot -3 + 1 \cdot -2 + 2 \cdot -1 + 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 = 0$

IV: The three dotplots we constructed are below. We constructed them so they all have a mean of 0. Now we will look at the dotplots medians.

![Dotplot A](image)

![Dotplot B](image)

![Dotplot C](image)
Since we find the median by creating an ordered list and determining the middle number, we can label the dots from left to right starting with 1 to find the middle value of the dotplots. Since there are 8 data values, the median is the mean of the 4\textsuperscript{th} and 5\textsuperscript{th} values.

There is a blue line between the 4\textsuperscript{th} and 5\textsuperscript{th} data points in each dotplot indicating the location of the median. In each case, the median is 0.

V. The mode is the easiest measure of center to determine from a dotplot. It is the data value with the most dots. Dotplot A has one mode of 0. We call this \textit{unimodal}. Dotplot B has three modes; −1, 0, and 1. We call this \textit{trimodal}. Dotplot C has two modes; −1, and 1. We call this \textit{bimodal}.

\textbf{Result:} The three dotplots we created have a special property. They are \textit{symmetric distributions}. This means that we can draw a vertical line in the center of the distribution, and either side is the mirror image of the other as shown below.

\textbf{Fact:} Symmetric distribution’s mean and median fall on their line of symmetry. If the distribution is unimodal, the mode equals the mean and median.
III. Skewed Distributions and Measures of Center

We'll begin again with a dotplot with a mean, median, and mode of 0 where all the data points are 0. We will transform it into a right skewed distribution by moving data points to the right. This will not maintain the mean at 0 like the symmetric distribution did, but we will keep track of our changes and determine how the measures of center change.

Dotplot A on the left shows how we will move 3 data points from 0 to 1, and 3 more data points to 2, 3, and 4 respectively to attain Dotplot B. The sum under Dotplot B shows how the sum of the data points changed from 0 to 12.

Observe that Dotplot B now has the general shape of a right skewed distribution.
Now let’s determine the mean, median, and mode to see how they changed from 0 by performing the right skew.

**Mean:** The sum of the data points of the transformed data set is

\[ \text{Sum: } 3 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 + 4 \cdot 1 = 12 \]

Since there are 8 data points, the mean is

\[ \frac{3 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 + 4 \cdot 1}{8} = \frac{12}{8} = 1.5 \]

**Median:** Since we find the median by creating an ordered list and determining the middle number, we can label the dots from left to right starting with 1 to find the middle value of the dotplots. Since there are 8 data values, the median is the mean of the 4\(^{th}\) and 5\(^{th}\) values at 1.

Notice the “right skewing” has a larger effect on the mean than the median. Points 6, 7, and 8 could be 100 and the median would still be 1. However, the mean would be \( \frac{303}{8} = 37.875 \)

**Mode:** The mode is the data value with the most dots. In this case our data value of 1 has the largest frequency of 3. So 1 is the mode.

**Result:** With this right skewed distribution, the mean is 1.5, the median is 1, and the mode is 1.

**General Result:** When we have a right skewed distribution (with one mode)

\[ \text{mode} \leq \text{median} \leq \text{mean} \]

The diagram below is a typical result of the locations of center for right skewed distributions.
Although we did not look at an example of a left skewed distribution, observe that the same effect on the mean, median, and mode will occur except they will tend towards the left as shown in the image below.

**General Result:** When we have a left skewed distribution (with one mode)

\[ \text{mode} \geq \text{median} \geq \text{mean} \]

**Summary:** Given the following unimodal distributions, the following relationships between the measures of center holds.

1) For left-skewed distributions: \( \text{mode} \geq \text{median} \geq \text{mean} \)

2) For symmetric distributions: \( \text{mode} = \text{median} = \text{mean} \)

3) For approximately symmetric distributions: \( \text{mode} \approx \text{median} \approx \text{mean} \)

4) For right-skewed distributions: \( \text{mode} \leq \text{median} \leq \text{mean} \)
Problem 5: Media Example – Estimating the Mean, Median, and Mode from a Histogram

a) A teacher gave his class of 42 students a quiz worth 10 points. The histogram below shows the distribution of the students’ scores. The distribution is approximately symmetric.

i. Estimate the mean, median, and mode based on the shape of the distribution.

ii. Compute the actual mean, median, and mode of the distribution. Round to two decimal places as needed.

b) A teacher gave his class of 37 students a pop quiz worth 10 points. The histogram below shows the distribution of the students’ scores.

i. Estimate the mean, median, and mode based on the shape of the distribution.

ii. Compute the actual mean, median, and mode of the distribution. Round to two decimal places as needed.
Problem 6: You Try – Estimating the Mean, Median, and Mode from a Histogram

A teacher gave his class of 41 students an easy quiz before a holiday worth 10 points. The histogram below shows the distribution of the students’ scores.

i. Estimate the mean, median, and mode based on the shape of the distribution.

ii. Compute the actual mean, median, and mode of the distribution. Round to two decimal places as needed.
SECTION 8.3: VARIABILITY AND MEASURES OF DISPERSION

The three measures of center each inform us about the center of a distribution with a single number. When we looked at the different types of distributions, we saw that these measures vary in relation to each other based on the shape of the distribution. Knowing a measure of center or all three is useful, but there is more information that can be computed.

Measures of dispersion or variability are quantities that tell us how spread out the data are. Consider the three histograms below. All of the graphs are exactly symmetric. Therefore, their three measures of center (mean, median, and mode) are all equal. In particular, these values are 50 for all three graphs. However, the graphs are different.

In this section, we will look at two measures to quantify how spread out the data are using what we call the range and the standard deviation.

I. Range: The range of a data set is the difference between its minimum and maximum data values.

Problem 7: Worked Example – The Range of a Data Set

Find the range of the data sets displayed in Graphs A, B, and C.

Solution: To determine the range, find the minimum and maximum data values and subtract to find their difference.

Graph A: The minimum value in Graph A is 47. The maximum value in Graph A is 53.
Answer: Range of Graph A: 53 – 47 = 6

Graph B: The minimum value in Graph B is 46. The maximum value in Graph B is 54.
Answer: Range of Graph B: 54 – 46 = 8

Graph C: The minimum value in Graph C is 44. The maximum value in Graph C is 56.
Answer: Range of Graph C: 56 – 44 = 12
II: Population Standard Deviation: In the next example, we will compute a population standard deviation. The standard deviation is a measure of how far the data values are from the mean. The word deviation refers to the difference of the individual data points from the mean value.

We will use the population standard deviation in this example as opposed to the sample standard deviation. This means that we are assuming the data set describes all the values in which we are interested as opposed to making an inference about some larger population. Specifically, our data sets will describe three plots of water spinach plants. We did not obtain a sample of plants from a larger group of plants. We are just making a judgment about these specific plots of plants.

Steps for Finding Population Standard Deviation

1) Find the mean of the data set
2) Find the difference between each data value and the mean
3) Square the differences from step 2
4) Sum the squared differences from step 3
5) Divide the sum from step 4 by the number of data points
6) Find the square root of the result from step 5

The formula for this process is a bit complex. It is included below for your reference, but we will use the process as described above without directly using the formula.

Let $\bar{x}$ = the mean of the data set

Let $x_i$ = a data value in the data set

Let $n$ = the number of data values

Let $\sigma$ = the population standard deviation

$\sum_{i=1}^{n} \text{an operator that means to add up the n values computed}$

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}}$$
Problem 8: Worked Example – The Population Standard Deviation of a Data Set

Three friends grew water spinach plants and measured the plant heights after 30 days rounded to the nearest centimeter. The table below shows the results for one friend. Find the standard deviation of the plant heights in Plot A. Round to two decimal places as needed.

<table>
<thead>
<tr>
<th>Height in cm</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>47</td>
<td>1</td>
</tr>
<tr>
<td>48</td>
<td>2</td>
</tr>
<tr>
<td>49</td>
<td>4</td>
</tr>
<tr>
<td>50</td>
<td>6</td>
</tr>
<tr>
<td>51</td>
<td>4</td>
</tr>
<tr>
<td>52</td>
<td>2</td>
</tr>
<tr>
<td>53</td>
<td>1</td>
</tr>
</tbody>
</table>

Solution: Observe that the data in the table is the same data from the first graph from the beginning of 8.3. The graph was perfectly symmetric and the mean is 50 cm.

Step 1) Mean of data set = 50
Step 2) Column 2: Find the difference between each data value and the mean
Step 3a) Column 3: Square the differences
Step 3b) Column 5: Since the frequencies are greater than 1 for some of the data values, multiply the frequency of each data value by the squared difference to find the total.
Step 4) Bottom Cell Column 5: Find the sum of the total squared differences
Step 5) Divide the sum from Step 4 by the total number of data points. 42 ÷ 20 = 2.1
Step 6) Find the square root of 2.1. \(\sqrt{2.1} \approx 1.45\)

<table>
<thead>
<tr>
<th>Height in cm</th>
<th>Step 2: Difference from mean</th>
<th>Step 3a: Squared difference from mean</th>
<th>Frequency</th>
<th>Step 3b: Total Squared Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>47</td>
<td>47 – 50 = -3</td>
<td>(-3 )² = 9</td>
<td>1</td>
<td>1 × 9 = 9</td>
</tr>
<tr>
<td>48</td>
<td>48 – 50 = -2</td>
<td>(-2 )² = 4</td>
<td>2</td>
<td>2 × 4 = 8</td>
</tr>
<tr>
<td>49</td>
<td>49 – 50 = -1</td>
<td>(-1 )² = 1</td>
<td>4</td>
<td>4 × 1 = 4</td>
</tr>
<tr>
<td>50</td>
<td>50 – 50 = 0</td>
<td>( 0 )² = 0</td>
<td>6</td>
<td>16 × 0 = 0</td>
</tr>
<tr>
<td>51</td>
<td>51 – 50 = 1</td>
<td>( 1 )² = 1</td>
<td>4</td>
<td>4 × 1 = 4</td>
</tr>
<tr>
<td>52</td>
<td>52 – 50 =2</td>
<td>( 2 )² = 4</td>
<td>2</td>
<td>2 × 4 = 8</td>
</tr>
<tr>
<td>53</td>
<td>53 – 50 = 3</td>
<td>( 3 )² = 9</td>
<td>1</td>
<td>1 × 9 = 9</td>
</tr>
<tr>
<td></td>
<td>Total: 20</td>
<td>Total: 9+8+4+0+4+8+9 = 42</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Answer: The standard deviation of plant heights in Plot A is 1.45 cm.
Problem 9: Media Example – The Population Standard Deviation of a Data Set

Three friends grew water spinach plants and measured the plant heights after 30 days rounded to the nearest centimeter. The table below show the results for all three plots of land.

<table>
<thead>
<tr>
<th>Height in cm</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>47</td>
<td>1</td>
</tr>
<tr>
<td>48</td>
<td>2</td>
</tr>
<tr>
<td>49</td>
<td>4</td>
</tr>
<tr>
<td>50</td>
<td>6</td>
</tr>
<tr>
<td>51</td>
<td>4</td>
</tr>
<tr>
<td>52</td>
<td>2</td>
</tr>
<tr>
<td>53</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Height in cm</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>46</td>
<td>2</td>
</tr>
<tr>
<td>47</td>
<td>4</td>
</tr>
<tr>
<td>48</td>
<td>6</td>
</tr>
<tr>
<td>49</td>
<td>8</td>
</tr>
<tr>
<td>50</td>
<td>12</td>
</tr>
<tr>
<td>51</td>
<td>8</td>
</tr>
<tr>
<td>52</td>
<td>6</td>
</tr>
<tr>
<td>53</td>
<td>4</td>
</tr>
<tr>
<td>54</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Height in cm</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>44</td>
<td>1</td>
</tr>
<tr>
<td>45</td>
<td>1</td>
</tr>
<tr>
<td>46</td>
<td>2</td>
</tr>
<tr>
<td>47</td>
<td>3</td>
</tr>
<tr>
<td>48</td>
<td>4</td>
</tr>
<tr>
<td>49</td>
<td>6</td>
</tr>
<tr>
<td>50</td>
<td>10</td>
</tr>
<tr>
<td>51</td>
<td>6</td>
</tr>
<tr>
<td>52</td>
<td>4</td>
</tr>
<tr>
<td>53</td>
<td>3</td>
</tr>
<tr>
<td>54</td>
<td>2</td>
</tr>
<tr>
<td>55</td>
<td>1</td>
</tr>
<tr>
<td>56</td>
<td>1</td>
</tr>
</tbody>
</table>

a) Find the standard deviation of the plant heights in Plot B by hand using the table below. Round to two decimal places as needed.

<table>
<thead>
<tr>
<th>Height in cm</th>
<th>Step 2: Difference from mean</th>
<th>Step 3a: Squared difference from mean</th>
<th>Frequency</th>
<th>Step 3b: Total Squared Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>46</td>
<td></td>
<td></td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>47</td>
<td></td>
<td></td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>48</td>
<td></td>
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<td>50</td>
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<td>12</td>
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<td></td>
<td>6</td>
<td></td>
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<tr>
<td>53</td>
<td></td>
<td></td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>54</td>
<td></td>
<td></td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Total: Total:

b) Find the standard deviation of the plant heights in Plot C using the STAT function of your calculator. Round to two decimal places as needed.

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**Problem 10: You Try – The Population Standard Deviation of a Data Set**

A group of adults were asked how many children they have in their families. The frequency chart below shows the number of adults who indicated they had each number of children.

<table>
<thead>
<tr>
<th>Number of Children</th>
<th>Frequency of Adults</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

a) Find the mean of the data set.

b) Find the standard deviation of the set by hand. Check your results on your calculator.

**III. Population Standard Deviation and Distribution from the Mean**

One useful result of the population standard deviation is that almost all (or frequently all) of the data points will lie within 3 standard deviations of the mean given below.

**3 – Standard Deviation Rule:** For any data set, almost all of the data values (or frequently all of the data values) will lie within 3 standard deviation of the mean.
Problem 11: Worked Example – The 3 –Standard Deviation Rule

Consider the data set from Plot A in the Worked Example Problem 8. The mean of the data set is 50 and the standard deviation is 1.45. The graph of this data set is shown below.

Use the 3 Standard Deviation rule to find the interval that almost all or all of the data points should lie between. State which data points do or do not lie within this range.

Solution: Based on the 3 standard deviation rule, all or almost of all of the data should fall within $3 \times 1.45 = 4.35$ cm of the mean of 50 cm. In particular,

Approximate Lower bound:   $50 – 3 \times 1.45 = 45.65$ cm
Approximate Upper bound:   $50 + 3 \times 1.45 = 54.35$ cm

The graph shows that the minimum data value is 47 and the maximum data value is 53. So all of the data points are within 3 standard deviations of the mean.

Answer: The interval that is within 3 standard deviations of the mean is 45.65 to 54.35. All the data points or 100% of the data fall within this range.

Problem 12: You Try – The 3 –Standard Deviation Rule

Consider the data set from Plot C in the Worked Example Problem 8. The mean of the data set is 50 and the standard deviation is 2.54. The graph of this data set is shown below.

Use the 3 Standard Deviation rule to find the interval that almost all or all of the data points should lie between. State which data points do or do not lie within this range.
SECTION 8.4: BOXPLOTS AND THE 5−NUMBER SUMMARY

Now we will look at 5 numbers to gain an understanding of a set of data. Five numbers can tell us more about a distribution than one number can, yet we are still reducing the number of values under consideration to understand large amounts of information. This is a common theme in statistics. We reduce the amount of information we consider to gain a better understanding of it, but lose information along the way. So we try to have as many tools as possible to make sense of data.

I. The Five Number Summary

In this section, we will look at what we call the 5−number summary and its corresponding graph called a boxplot to gain insight into center and shape. Let’s start with some definitions.

Definitions:

1) The **minimum** is the smallest value represented in a data set.

2) The **maximum** is the largest value represented in a data set.

3) The **median**, or Q2, is a point where 50% of the data lie both above and below the point.

4) The **first quartile**, or Q1, is the median of the lower 50% of the data or the point where 25% of the data lie to the left of the point.

5) The **third quartile**, or Q3, is the median of the upper 50% of the data or the point where 75% of the data lie to the left of the point.

6) The **five−number summary** is the list of the 5 values above in the following order; minimum, first quartile, median, third quartile, maximum.

Shorthand: Min, Q1, Med, Q3, Max
Problem 13: Worked Example – Finding the 5-Number Summary

The data below represent the average annual number of births during the year 2017 per 1,000 persons in the population at midyear for 10 countries in North America. Find the 5 – number summary for the birth rate.

<table>
<thead>
<tr>
<th>Country</th>
<th>Birth Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Belize</td>
<td>24</td>
</tr>
<tr>
<td>Canada</td>
<td>10.3</td>
</tr>
<tr>
<td>Costa Rica</td>
<td>15.5</td>
</tr>
<tr>
<td>El Salvador</td>
<td>16.2</td>
</tr>
<tr>
<td>Guatemala</td>
<td>24.1</td>
</tr>
<tr>
<td>Honduras</td>
<td>22.4</td>
</tr>
<tr>
<td>Mexico</td>
<td>18.3</td>
</tr>
<tr>
<td>Nicaragua</td>
<td>17.7</td>
</tr>
<tr>
<td>Panama</td>
<td>17.9</td>
</tr>
<tr>
<td>United States</td>
<td>12.5</td>
</tr>
</tbody>
</table>

Solution:

1) First we will order the data from least to greatest. Since we are interested in the birth rate, we do not need to list the corresponding country.

10.3, 12.5, 15.5, 16.2, 17.7, 17.9, 18.3, 22.4, 24, 24.1

Now that the data are in order, we know the minimum is the first value, 10.3, and the maximum is the last value, 24.1.

2) Next we will find the median. We will also list the ordered number of the data although this is not required.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10.3</td>
<td>12.5</td>
<td>15.5</td>
<td>16.2</td>
<td>17.7</td>
<td>17.9</td>
<td>18.3</td>
<td>22.4</td>
<td>24</td>
<td>24.1</td>
</tr>
</tbody>
</table>

We divide 10 by 2 to get 5. Since there are an even number of data values, the median is the mean of the 5th and 6th entries.

\[
\text{Median} = \frac{17.7 + 17.9}{2} = 17.8
\]
3) Now we will look at the list of the numbers less than the median. The median of the lower half of the data set is the first quartile, Q1.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>10.3</td>
<td>12.5</td>
<td>15.5</td>
<td>16.2</td>
<td>17.7</td>
</tr>
</tbody>
</table>

We can observe that the third value is the median data point of the 5 values below the median. The first quartile is the data value that corresponds to the 3rd entry.

First Quartile = 15.5

4) Now we will look at the list of the numbers greater than the median. The median of the upper half of the data set is the third quartile, Q3.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>17.9</td>
<td>18.3</td>
<td>22.4</td>
<td>24</td>
<td>24.1</td>
</tr>
</tbody>
</table>

We can observe that the eighth value is the median data point of the 5 values above the median. The third quartile is the data value that corresponds to the 8th entry.

Third Quartile = 22.4

5) Now list the 5 – number summary from least to greatest.

5 – Number Summary: 10.3, 15.5, 17.8, 22.4, 24.1
**Problem 14: Media Example – Finding the 5-Number Summary**

Note: This is the same data set as Problem 11, but we will use a calculator to find the 5 number summary.

The data below represent the average annual number of births during the year 2017 per 1,000 persons in the population at midyear for 10 countries in North America. Find the 5 – number summary for the birth rate using the STAT function of your calculator.

<table>
<thead>
<tr>
<th>Country</th>
<th>Birth Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Belize</td>
<td>24</td>
</tr>
<tr>
<td>Canada</td>
<td>10.3</td>
</tr>
<tr>
<td>Costa Rica</td>
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</tr>
<tr>
<td>El Salvador</td>
<td>16.2</td>
</tr>
<tr>
<td>Guatemala</td>
<td>24.1</td>
</tr>
<tr>
<td>Honduras</td>
<td>22.4</td>
</tr>
<tr>
<td>Mexico</td>
<td>18.3</td>
</tr>
<tr>
<td>Nicaragua</td>
<td>17.7</td>
</tr>
<tr>
<td>Panama</td>
<td>17.9</td>
</tr>
<tr>
<td>United States</td>
<td>12.5</td>
</tr>
</tbody>
</table>

**Problem 15: You Try – Finding the 5-Number Summary**

The data below represent the average annual number of births during the year 2017 per 1,000 persons in the population at midyear for 10 countries in Asia. Find the 5 – number summary for the birth rate by hand and check your results using your calculator.

<table>
<thead>
<tr>
<th>Country</th>
<th>Birth Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>China</td>
<td>12.3</td>
</tr>
<tr>
<td>India</td>
<td>19</td>
</tr>
<tr>
<td>Malaysia</td>
<td>19.1</td>
</tr>
<tr>
<td>Nepal</td>
<td>19.5</td>
</tr>
<tr>
<td>Pakistan</td>
<td>21.9</td>
</tr>
<tr>
<td>Russia</td>
<td>11</td>
</tr>
<tr>
<td>Singapore</td>
<td>8.6</td>
</tr>
<tr>
<td>Thailand</td>
<td>11</td>
</tr>
<tr>
<td>United Arab Emirates</td>
<td>15.1</td>
</tr>
<tr>
<td>Yemen</td>
<td>28.4</td>
</tr>
</tbody>
</table>
II. Boxplots

The graph below is called a boxplot. It corresponds to the 5-number summary from the previous example. It gives us an idea of the distribution using only 5 numbers. In the next example, we will learn to interpret a boxplot.

Problem 16: Media Example – Interpreting a Boxplot

Robin and Marian are arborists and measured the heights of the 312 fledgling saplings in the forest to examine their health and growth. Their data are displayed in the boxplot below.

a) What is the 5-number summary of the boxplot and what does each represent?
b) What is the range of heights of the tallest 25% of saplings?

c) For what value are 50% of the sapling’s heights both above and below the value?

d) What percentage and how many of the 312 saplings are 7.3 cm or taller?

 e) Are the number of saplings between 7.3 cm and 13.2 cm in height greater than, less than, or equal to the number of saplings between 13.2 cm and 25.5 cm in height? Explain your answer.

f) If we had all the data and made a histogram, would the data be approximately symmetric, right-skewed or left skewed? How do you know?
Problem 17: You Try – Interpreting a Boxplot

A nutritionist gathered data on the heights of the 83 women in her nutrition study. The data are displayed in the boxplot below.

![Boxplot of Heights of 83 Adult Women](image)

a) What is the 5-number summary of the boxplot and what does each represent?

b) What is the range of heights of the tallest 25% of women?

c) For what value are 50% of the women’s heights both above and below the value?

d) What percentage of the women are at least 64 inches tall?

e) Approximately what number of the 83 women are less than 69 inches tall?

f) If we had all the data and made a histogram, would the data be approximately symmetric, right-skewed or left skewed? Explain.
UNIT 9 – SETS AND VENN DIAGRAMS

In this Unit we will study sets and set operations. Sets are a fundamental concept of mathematics and will give us the language and tools to formalize probability in the upcoming units.

Section 9.1: Define sets, elements, subsets and proper subsets

Section 9.1: Determine whether an element belongs to a set

Section 9.2: Define subsets, proper subsets, and the empty set

Section 9.2: Determine whether a set is proper subset or subset of another set

Section 9.3: Define and perform set operations

Section 9.4: Given a list of set elements, represent the set elements in a 2 set Venn Diagram

Section 9.4: Given a list of set elements, represent the number of elements in a 2 set Venn Diagram

Section 9.5: Define the disjoint partitions of a 2 set Venn Diagram

Section 9.5: Count the number of elements in subsets using a completed Venn Diagram for 2 sets

Section 9.5: Determine the number of elements in the disjoint partitions of a Venn Diagram for 2 sets and use the information to find the number of elements in subsets

Section 9.5: Solve applied problems involving sets using Venn Diagrams for 2 sets including fractions or percentages of the universal set or subsets

Section 9.6: Define the disjoint partitions of a 3 set Venn Diagram

Section 9.6: Count the number of elements in subsets using a completed Venn Diagram for 3 sets and use the information to find the number of elements in subsets

Section 9.6: Determine the number of elements in the disjoint partitions of a Venn Diagram for 3 sets

Section 9.6: Solve applied problems involving sets using Venn Diagrams for 3 sets including fractions or percentages of the universal set or subsets
SECTION 9.1: SETS AND ELEMENTS

In everyday language, you may use the word set to describe a grouping of some type of things. For example, you may have a set of flatware or a set of encyclopedias. The idea is you have a group of things that have some common attribute that you combine into a whole in which you categorize the entire group.

The mathematical idea of a set is similar. In particular, we say a set is a collection of distinct objects. We call the distinct objects in a set elements of the set. By distinct, we mean that an element can only belong to a set once. For example, if a set is defined as people in your household with a car, and you belong to your household and have a car, you would be an element of the set, but only listed once even though you meet both criteria. In short, there are no repetitions of the same element in sets.

Set Definitions: We can define a set using a description or by simply listing the elements of the set. For example,

1) **Set as a Description:** The set A is the first 5 letters in the English alphabet.
2) **Set as a List:** A = \{ a, b, c, d, e \} or equivalently A = \{ c, a, e, d, b \}
3) **Elements of the set:** a, b, c, d, e

Note that in the list examples for a set, we used a capital letter to name the set and curly brackets to enclose the elements listed. This is standard notation for writing a set as a list. In addition, notice that the elements of the set can be listed in different ways and it does not change the set itself. We are only concerned with what elements are in the set or not in the set. We may prefer to write the elements in alphabetical order to keep track of the elements more easily, but it is not necessary.

Notice that in the verbal description of a set, the type of alphabet was identified as the English alphabet. You may have assumed this if it had not been written explicitly. However, without this identification, the description could have referred to Greek or Cyrillic alphabets. The point being, we are as specific as possible when describing sets with words so that our set is well-defined.

A set is well-defined if it clearly defines its elements with certainty. This means a reasonable person could objectively determine whether an element belongs to the set or not based on the definition. The example below shows examples of sets that are well-defined and not well-defined.
**Not well-defined:** The set C is the best three ice cream flavors.

**Well-defined:** The set C contains the three highest selling ice cream flavors by volume in the United States in 2016.

Note the difference between the two set descriptions. The “best” ice cream is a subjective term. Different people will have different responses. However, the highest selling ice cream flavors in a year can be determined objectively given the correct data and process of analyzing the results.

**Element Language and Notation:** Finally, we may use different language and notation to convey the relationships that an element is in a set or not in a set. For example,

Given \( A = \{ a, b, c, d, e \} \), we may say that

1) \( b \) is an element of \( A \), \( b \) is a member of \( A \), \( b \) belongs to \( A \), or \( b \in A \)

2) \( z \) is not an element of \( A \), \( z \) is not a member of \( A \), \( z \) does not belong to \( A \), or \( z \notin A \)

You should be familiar with these alternate phrases and representations as they will be used interchangeably.

**Problem 1:** Worked Example – Sets and Elements

a) Use the description below to create a list of the set.

Description: The set \( B \) is the set of positive even whole numbers less than 14.

**Solution:** First consider what the specific requirements mean. Positive numbers are greater than 0. Even numbers are divisible by 2. Whole numbers do not include fractions or decimals. Numbers less than 14 does not include the number 14 (a number is not less than itself).

**Answer:** \( B = \{ 2, 4, 6, 8, 10, 12 \} \)

b) Let \( A = \{ \text{Amy, Beth, Cal, Don, Ed} \} \). Determine whether the following are elements of the set \( A \). State your answer in words and use the notation, \( \in \) or \( \notin \) to represent your answer.

Question 1: Is Don a member of \( A \)?

**Answer:** Don is an element of the set \( A \). \( Don \in A \)

Question 2: Is Alice a member of \( A \)?

**Answer:** Alice is not an element of \( A \). \( Alice \notin A \)
Problem 2: You Try – Sets and Elements

a) Use the description below to create a list of the set.
   Description: The set D is the whole numbers greater than 4 and less than or equal to 10.

b) Let A = { dog, cat, fish, turtle }. Determine whether the following are elements of the set A
   State your answer in words and use the notation, $\in$ or $\notin$ to represent your answer.

Question 1: Is frog an element of A?

Question 2: Is dog an element of A?

SECTION 9.2: SUBSETS, PROPER SUBSETS AND THE EMPTY SET

We have discussed the relationship between an element and a set. In particular, an element is either a member of a set or not a member of set. Similarly, we can define a relationship between two sets.

Definition: A set A is a subset of a set B if every element of A is also an element of B. We denote this relationship symbolically as $A \subseteq B$. A set A is not a subset of B if there exists an element in A that is not an element in B. If A is not a subset of B, we write $A \not\subseteq B$.

Example: Consider the sets below.

$S =$ all the residents of Scottsdale, Arizona

$M =$ all the residents of Maricopa County, Arizona

$A =$ all the residents of the state of Arizona.

Question: What subset relationships exist between these sets?

Solution:

1) $S \subseteq A$ All the residents of Scottsdale, Arizona are residents of the state of Arizona

2) $S \subseteq M$ All the residents of Scottsdale, Arizona are residents of Maricopa County

3) $M \subseteq A$ All the residents of Maricopa County, Arizona are residents of Arizona
Notice if we interchange the set positions, the same relationships do not hold. We can show that one set is not a subset of another set if we can identify at least one element in the first set that is not an element in the second set. The relationships below state a justification that identifies such an element in the first set that is not in the second set.

1) \( A \nsubseteq S \) All the residents of the state of Arizona are not residents of Scottsdale
   Reasoning: Someone who lives in Globe lives in Arizona, but not Scottsdale

2) \( M \subseteq S \) All the residents of Maricopa County are not residents of Scottsdale
   Reasoning: Someone who lives in Phoenix lives in Maricopa County, but not Scottsdale.

3) \( A \subseteq M \) All the residents of the state of Arizona are not residents of Maricopa County
   Reasoning: Someone who lives in Pima County lives in Arizona, but not Maricopa County.

You may have noticed that the subset symbol, \( \subseteq \), has a line underneath it, whereas the symbol for not a subset, \( \nsubseteq \), does not contain a line underneath it. This is because the definition of a subset permits the possibility that the two sets are equal. This is similar to the mathematical relationship \( 3 \leq 3 \) which reads 3 is less than or equal to 3. Three is not less than three, but 3 does equal 3. So the inequality is true because it includes both options: equality and less than.

**Definition:** If we want to state that a set \( A \) is a subset of set \( B \), but they are definitely not equal we call \( A \) a proper subset of \( B \) and use the notation \( A \subset B \).

Observe that in our previous example, the proper subset relationship holds because the sets are not equal and we can also write the relationship as shown below. Either statement is correct because a proper subset is also a subset by definition. The proper subset notation is shown below.

1) \( S \subset A \) All the residents of Scottsdale, Arizona are residents of the state of Arizona
2) \( S \subset M \) All the residents of Scottsdale, Arizona are residents of Maricopa County
3) \( M \subset A \) All the residents of Maricopa County, Arizona are residents of Arizona

Finally, we will define one last important set that we will be helpful in our study of subsets.

**Definition:** The empty set is the set that does not contain any elements. We use the symbol, \( \emptyset \), to denote the empty set.

**FACT:** For any non-empty set \( A \), \( \emptyset \subset A \). This means the empty set is a proper subset of every non-empty set. The empty set is a subset of itself, but not a proper subset of itself, \( \emptyset \subseteq \emptyset \)
Problem 3: Media Example – Subsets, Proper Subsets, and the Empty Set

Use the sets $U$, $A$, $B$, $C$, $D$, and $E$ as defined below to answer the questions.

$U = \{ a, b, c, d, e, f, g, h, i, j \}$  
$A = \{ a, c, d, i, j \}$  
$B = \{ a, d, e, h, i, j \}$  
$C = \{ a, c \}$  
$D = \{ a, d, e, h, i, j \}$  
$E = \{ a, h, j \}$

a) Which sets are proper subsets of $U$?

b) Is $A$ a proper subset of $C$? Is $C$ a proper subset of $A$?

c) Is $B$ a subset of $D$? Is $B$ a proper subset of $D$?

d) $E$ is a proper subset set of which sets?

e) The empty set is a proper subset of which sets?

Problem 4: You Try – Subsets, Proper Subsets, and the Empty Set

Use the sets $U$, $A$, $B$, $C$, $D$, and $E$ as defined below to answer the questions.

$U = \{ a, b, c, d, e, f, g, h, i, j \}$  
$A = \{ a, c, d, f, g \}$  
$B = \{ a, d, e, h, i, j \}$  
$C = \{ a, c, d, f, g \}$  
$D = \{ a, d, f \}$  
$E = \{ a, d, f, g, j \}$

a) Is $U$ a proper subset of $E$?  
b) Is $D$ a proper subset of $E$?

c) Is $C$ a proper subset of $A$?  
d) Is $\emptyset$ a proper subset of $B$?
SECTION 9.3: OPERATIONS ON SETS

Just as we can perform operations on numbers such as addition, subtraction, multiplication and division, we can perform operations on sets. First we will need the concept of what we call a **Universal Set**.

**Definition:** The **Universal set** for a given problem or scenario are all the possible elements under consideration in the specific situation. The Universal set is either explicitly stated in the problem or can be implied from the context of the problem. Typically, we label the Universal set with the letter U.

**Examples:**

a) In the previous You Try problem, you were given the sets below. The first set is labeled U and represents all the possible elements under consideration in the problem or the Universal Set. Notice that every other set is a subset of U. The element b is in U, but not any other sets. This is a possibility.

U = { a, b, c, d, e, f, g, h, i, j }   A = { a, c, d, f, g}   B = { a, d, e, h, i, j }  
C = { a, c, d, f, g }   D = { a, d, f }   E = {a, d, f, g, j }

b) Let U = the set of all animals, Let M = the set of all mammals, Let R = the set of all reptiles

Notice that mammals and reptiles are subsets of the set of all animals. They aren't all the animal possibilities, but they are contained in the classification of the animal kingdom.

Now that we have defined the Universal Set, we can define four different set operations as shown below.

**Set Operations:** Let U be the universal set and let A and B be subsets of U.

1) The **intersection** of two sets, \( A \cap B \), means **A and B**. This set contains any elements in U that are both in A and in B.

2) The **union** of two sets, \( A \cup B \), means **A or B**. This set contains any elements in U that are in A, in B, or in both A and B.

3) The **complement** of a set A, \( A' \), means **not A**. This set contains any elements in U that are not in the set A.

4) The **difference or relative complement** of the set A in relation to B, \( B - A \), means **B and not A**. This set contains any elements in B that are not in the set A.

In this course, we will mostly use the language or words associated to the operation rather than the set notation. However, you should be familiar with both.
Unit 9 – Sets and Venn Diagrams

Problem 5: Media Example – Operations on Sets
List the elements in the sets described below. Then find the number of elements in the set.

\[ U = \{ a, b, c, d, e, f, g, h, i, j \} \quad A = \{ b, d, e, h, j \} \quad B = \{ b, c, d, f, h, i \} \quad C = \{ c, d, f, i \} \]

a) The elements of \( U \) that are in the set \( A \) and \( B \).

b) The elements of \( U \) that are in the set \( A \) or \( C \).

c) The elements of \( U \) that are not in the set \( B \).

d) The elements of \( U \) that are in the set \( B \) and not in the set \( A \).

Problem 6: You Try – Operations on Sets
List the elements in the sets described below. Then find the number of elements in the set.

\[ U = \{ a, b, c, d, e, f, g, h, i, j \} \quad A = \{ a, d, e, f, g \} \quad B = \{ a, e, f, g, h, i \} \quad C = \{ c, f, g, h \} \]

a) The elements of \( U \) that are in the set \( B \) and \( C \).

b) The elements of \( U \) that are in the set \( A \) or \( B \).

c) The elements of \( U \) that are in the set \( C \) and not in the set \( B \).
**Problem 7: Media Example – Operations on Sets**

List the elements in the sets described below. Then find the number of elements in the set.

\[ U = \{ a, b, c, d, e, f, g, h, i, j \} \quad A = \{ b, d, e, h, j \} \quad B = \{ b, c, d, f, h, i \} \quad C = \{ c, d, f, i \} \]

a) The elements of \( U \) that are in the set \( A \) and the set \( B \) and the set \( C \).

b) The elements of \( U \) that are not in the set \( A \) or \( B \).

c) List the elements of \( U \) that are not in the set \( A \) or in the set \( B \).

d) List the elements of \( U \) that are in the set \( B \) and not in \( A \), or in the set \( C \).

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**Problem 8: You Try – Operations on Sets**

List the elements in the sets described below. Then find the number of elements in the set.

\[ U = \{ a, b, c, d, e, f, g, h, i, j \} \quad A = \{ a, d, e, f, g \} \quad B = \{ a, e, f, g, h, i \} \quad C = \{ c, f, g, h \} \]

a) The elements of \( U \) that are in the set \( A \) or \( B \) and not in the set \( C \).

b) The elements of \( U \) that are in the set \( B \) and \( C \) or in the set \( A \).
SECTION 9.4: INTRODUCTION TO 2 SET VENN DIAGRAMS

It can be useful to represent set relationships visually. Typically, we represent the Universal set with a rectangle and any subsets of the universal set with circles. These type of diagrams are called **Venn Diagrams** and represent the relationships between the universal set and its subsets. For example, suppose the universal set $U = \{a, b, c, d, e, f, g, h, i, j\}$ and the set $A = \{a, d, e, h\}$. We can create diagrams representing these relationships as shown below.

**Diagram 1**: This diagram shows the basic relationship that $A$ is a subset of $U$. The universal set is the rectangle and is labeled in the upper right corner with the letter $U$. The circle is inside the rectangle and represents $A$ which is a subset of $U$. The circle is labeled with the letter $A$ above the circle. Sometimes we put the label below the circle if there are multiple circles.

**Diagram 2**: This diagram has all of the characteristics of Diagram 1, but it also lists the elements in $A$ inside the circle labeled $A$, and the elements in $U$, but not in $A$, listed in the rectangle $U$ but outside the circle labeled $A$. The elements outside the $A$ circle are determined as follows.

$U = \{a, b, c, d, e, f, g, h, i, j\}$
$A = \{a, d, e, h\}$
$A' = \{b, c, f, g, i, j\}$  (Recall $A'$ means $U$ and not $A$)

We don't often use the representation in Diagram 2. It may be useful if we have small sets like $A$ and $U$, but if there were a lot of elements in $U$, say 30, the diagram would be difficult to create and read. Also note that the elements in $A'$ are in the lower right hand corner. These elements could be placed anywhere in the rectangle as long as they are not in the $A$ circle. We typically put them in the lower right hand corner as a means of organizing the diagram.

**Diagram 3**: This diagram has all of the characteristics of Diagram 1, but it doesn't have the elements of the set listed inside the diagram like Diagram 2. Instead, it has the number 4 inside the circle, and the number 6 in the rectangle, but outside the circle. These numbers are indicating that the number of elements in the subsets $A$ and $A'$. You should verify this by counting the number of elements in Diagram 2 and seeing how they correspond to 4 and 6.
Now let’s suppose we have another set $B$ that is a subset of $U$ where $B = \{b, d, h, i, j\}$ with $U$ and $A$ as previously defined. The Venn Diagrams for $B$ in relation to $U$ are shown below.

Now suppose we would like to represent the relationships between the sets $A$, $B$, and $U$ all in one diagram. The Venn Diagrams below are in the form of Diagram 2 and show the elements in each set. The third diagram shows how we represent the relationships between all $U$ and the two subsets.

Observe that the circles for $A$ and $B$ overlap. The part that overlaps contains the elements that are in both $A$ and $B$ or the intersection of the sets $A$ and $B$. The diagrams below show the elements in each set, the number of elements in each part, and what the part describes in words. The third diagram shows how a Venn diagram can partition the set relationships into disjoint sets. Two sets are disjoint if their intersection is the empty set. The four disjoint pieces highlighted considered separately are non-overlapping and combined equal the entire set $U$. 
Problem 9: Worked Example – 2 Set Venn Diagrams

Let \( U = \{a, b, c, d, e, f, g, h, i, j\} \)
\( A = \{c, d, e, f, h, j\} \)
\( B = \{a, e, f, h, i\} \)

a) Write the elements in the corresponding parts of the Venn diagram.

Solution: First find the elements that are in both \( A \) and \( B \). A systematic way of doing this is to go through the list of elements in \( A \) from left to right, and highlight or circle them if they are also in \( B \) as shown below.

\[
A = \{c, d, e, f, h, j\} \\
B = \{a, e, f, h, i\}
\]

Next write any elements that are in \( A \) but not \( B \) in the other portion of \( A \) that doesn’t overlap with \( B \). And follow a similar process for elements in \( B \) and not \( A \).

\[
A = \{c, d, e, f, h, j\} \\
B = \{a, e, f, h, i\}
\]

Finally, list any elements in \( U \) that haven’t already been written. These are the elements in \( U \) that are neither \( A \) nor \( B \). A systematic way of achieving this is to start with the set \( U \), and cross out any elements that have already been written in \( A \), \( B \), or \( A \) intersect \( B \). These are highlighted below so whatever isn’t highlighted is in \( U \), but outside of the \( A \) and \( B \) circles.

\[
U = \{a, b, c, d, e, f, g, h, i, j\}
\]

The elements \( b \) and \( g \) are not highlighted so they are outside the \( A \) and \( B \) circles.
b) Write the number of elements in the corresponding parts of the Venn diagram.

Solution: Count the number of elements in each of the four disjoint pieces and write the corresponding number of elements in the associated part.
3 elements in A, but not B: c, d, j
3 elements in A and B: e, f, h
2 elements in B, but not A: a, i
2 elements neither in A nor B: b, g

Problem 10: You Try – 2 Set Venn Diagrams

Let \( U = \{a, b, c, d, e, f, g, h, i, j\}\)

A = \{a, d, g, h, j\}
B = \{c, d, e, f, h, j\}

a) In Diagram 1, write the elements in the corresponding parts of the Venn diagram.

b) In Diagram 2, write the number of elements in the corresponding parts of the Venn diagram.
SECTION 9.5: COUNTING AND 2 SET VENN DIAGRAMS

In this section, we will look at using Venn Diagrams in the context of set operations. First we will generalize the components of the 2 Set Venn diagram by analyzing its combined and disjoint parts. Recall that two sets are disjoint if their intersection is the empty set. This analysis will help us determine the number of elements in different sets and operations on sets.

Diagram A shows our verbal description of the disjoint partitioning of a 2 set Venn Diagram. Diagram B labels these parts with lowercase Roman Numerals so we can identify different set operations.

Diagram A

Diagram B

Below are descriptions of the four disjoint region descriptions and also the multiple regions that a set or a set operation contains. This is not a list to memorize, but a list to use to make sense of the Venn diagram as a representation of sets. Recall that the definition of a set states that every element is either in a set or not in a set. As you read through the lists focus on this idea to make sense of which elements belong to the different areas.

Disjoint Region Descriptions:
1) Region i: A and not B
2) Region ii: A and B
3) Region iii: B and not A
4) Region iv: not In A and not in B

Set or Set Operation Descriptions
1) An element in the set A is either in Region i or Region ii
2) An element in the set A’ (not A) is either in Region iii or Region iv
3) An element in the set B is either in Region ii or Region iii
4) An element in the set B’ (not B) is either in Region i or Region iv
5) An element in the set A or B is either in Region i, Region ii, or Region iii

We will now use these ideas to aid us in counting elements in different sets or set operations.
Problem 11: Media Example – Counting Elements in Subsets of 2 Set Venn Diagrams

The Venn diagram below displays the number of elements in each corresponding part of the diagram. Use this information to find the number of elements in the subsets of U given below.

a) How many elements are in A and B?

b) How many elements are in A or B?

c) How many elements are not in B?

d) How many elements are in B but not in A?

Problem 12: Media Example – Counting Elements in Subsets of 2 Set Venn Diagrams Application

A bag of marbles contains blue and white marbles. Each marble is labeled with a number between 1 and 5. The Venn diagram below gives the counts of the marbles based on whether they are blue or labeled with an even number.

a) How many marbles are blue?

b) How many marbles are white?

c) How many marbles are there all together?

d) How many marbles are blue and labeled with an even number?

e) How many marbles are white and labeled with an odd number?

f) How many marbles are labeled with an odd number?
Problem 13: You Try – Counting Elements in Subsets of 2 Set Venn Diagrams

I. The Venn diagram below displays the number of elements in each corresponding part of the diagram. Use this information to find the number of elements in the subsets of U given below.

a) How many elements are in A and B?

b) How many elements are in A or B?

c) How many elements are not in B?

d) How many elements are in B but not in A?

II. Talia sells homemade ice cream at a local art fair. A customer can purchase chocolate or vanilla ice cream with or without a topping. The Venn diagram below shows how many orders she sold based on whether the ice cream was chocolate or had a topping.

a) How many orders were chocolate?

b) How many orders were vanilla?

c) How many orders were there all together?

d) How many orders were vanilla with a topping?

e) How many orders were chocolate without a topping?

f) How many orders did not have a topping?
In our next examples, we will perform the opposite process as the last examples. Specifically, we will be given the number of elements in certain subsets and we will use the information to find the number of elements in the disjoint regions of the Venn diagram.

**Problem 14: Media Example – Determining the Number of Elements in 2 Set Venn Diagrams**

A deck of 52 cards is categorized by whether they are Red cards and whether they are Face cards. Fill in the Venn diagram to show how many cards are in each category. (Note: Aces are not considered Face cards)

**Problem 15: Media Example – Determining the Number of Elements in 2 Set Venn Diagrams**

Katie recorded the hair and eye color of 68 people in her class. She then categorized them based on whether they had brown hair (or not) and blue eyes (or not). Use the information below to complete the Venn diagram and answer the additional questions.

28 people had brown hair
25 people had blue eyes
7 people had brown hair and blue eyes

a) What percent of people surveyed have brown hair?

b) What percent of people with brown hair also have blue eyes?

c) What percent of people with blue eyes also have brown hair?
Scott surveyed 47 friends on Facebook and asked them where they were born and where they went to college. He then categorized the data by whether they were born in Arizona (or not) and whether they went to college in Arizona (or not). Use the information he collected below to complete the Venn diagram and answer the additional questions.

32 people were born in Arizona
23 people went to college in Arizona
15 people were born in Arizona and went (or go) to college in Arizona

a) What percent of people surveyed went (or go) to college in Arizona?

b) What percent of people surveyed were neither born in Arizona nor went to college in Arizona?

c) What percent of people born in Arizona went to college in Arizona?

d) What percent of people who went to college in Arizona were born in Arizona?
SECTION 9.6: COUNTING AND 3 SET VENN DIAGRAMS

In this section, we will extend Venn diagrams with 2 sets to include a third set. We will begin by looking at the eight disjoint pieces of a 3 set Venn diagram and what they represent. An image of a 3 set Venn diagram is below. Notice that we can start with a 2 set Venn diagram for A and B and add a 3rd circle for the set C below that overlaps with A and B as shown.

3 Set Venn Diagram

Recall that is useful to think of the disjoint pieces as either belonging or not belonging to each set. This can be determined by observing the disjoint piece and noting what circles it is in and not in. The table below shows a summary of these findings.

<table>
<thead>
<tr>
<th>Region</th>
<th>i</th>
<th>ii</th>
<th>iii</th>
<th>iv</th>
<th>v</th>
<th>vi</th>
<th>vii</th>
<th>viii</th>
</tr>
</thead>
<tbody>
<tr>
<td>In set A</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>In set B</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>In set C</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
</tbody>
</table>

Problem 17: Media Example – The Disjoint Pieces of a 3 Set Venn diagram

Determine the regions that make up the sets.

a) C
b) not C
c) C only
d) A and B
e) A or C
f) A or C and not B
g) B and A or not C
Problem 18: You Try – The Disjoint Pieces of a 3 Set Venn Diagram

Determine the regions that make up the sets.

a) B

b) not A

c) A only

d) B and C

e) A or B

f) A or B and not C

g) B and C or not A

Problem 19: Media Example – Counting Elements in Subsets of 3 Set Venn Diagrams

The Department of Transportation surveyed 125 people and asked whether they ever used alternative forms of transportation to go to work. They provided three options, Bus, Bike, and Walk. The participants checked any alternatives they had ever used. The results are in the Venn diagram below. Answer the questions below based on the Venn diagram. Round to two decimal places as needed.

a) How many people of all the surveyed people responded Walk?

b) How many of all the surveyed people did not respond Bus?

c) What percent of all the surveyed people responded Bus and Walk?

d) What percent of all the surveyed people responded Bike and Walk but not Bus?
Problem 20: You Try – Counting Elements in Subsets of 3 Set Venn Diagrams

An exercise science student surveyed 123 people that use the treadmill and asked whether they Jog, Run, or Walk on the treadmill. The participants checked any forms they had ever used. The results are in the Venn diagram below.

![Venn Diagram](image)

a) How many of all the surveyed people responded Jog?

b) How many of all the surveyed people did not respond Walk?

c) What percent of all the surveyed people responded Jog or Run?

d) What percent of all the surveyed people responded Jog or Walk but not Run?
Problem 21: Media Example – Determining the Number of Elements in 3 Set Venn Diagrams

162 students were surveyed on their current enrollment in Math, Biology, and Chemistry. The results are listed below. Use this information to complete the Venn diagram. Then answer the questions below based on the Venn diagram. Round to two decimal places as needed.

54 students are enrolled in Math.
67 students are enrolled in Biology.
68 students are enrolled in Chemistry.
16 students are enrolled in Math and Bio.
22 students are enrolled in Bio and Chem.
18 students are enrolled in Math and Chem.
6 students are enrolled in all three courses.

a) How many of all the surveyed people took Math and Bio?

b) How many of all the surveyed people did not take Chemistry?

c) What percent of all the surveyed people took Chemistry, but not Bio?

d) What percent of all the people who took Chemistry also took Bio?

e) What percent of all the people who took Bio also took Chemistry?
Problem 22: You Try – Determining the Number of Elements in 3 Set Venn Diagrams

129 people were surveyed on the types of volunteering in which they participate. They were asked about donating Time, donating Money, and donating Goods and Services. The results are given below. Use this information to complete the Venn diagram. Then answer the questions below based on the Venn diagram. Round to two decimal places as needed.

73 respondents have donated their Time.
58 respondents have donated their Money.
69 respondents have donated Goods and Services.
33 respondents have donated their Time and Money.
27 respondents have donated their Money and Goods and Services.
37 respondents have donated their Time and Goods and Services.
16 respondents have donated in all three capacities.

a) How many of all the surveyed people donated Goods and Services?

b) How many of all the surveyed people did not donate Time?

c) What percent of people who donated Time also donated Money?

d) What percent of people who donated Money also donated Time?
UNIT 10 – INTRODUCTION TO PROBABILITY

In this Unit we will learn elementary concepts of probability. Probability is a branch of mathematics that helps us quantify the chances of what may happen in the future scientifically.

Section 10.1: Use likelihood language to describe a numeric probability

Section 10.2: Determine whether a probability is subjective, empirical, or theoretical

Section 10.3: Determine the short term and long term relative frequency of an experiment given data and their relationship

Section 10.4: Compute theoretical probabilities for events

Section 10.5: Compute theoretical probabilities for events involving union or intersection

Section 10.6: Compute theoretical probabilities for the complement of an event

Section 10.7: Compute the odds in favor or against an event

Section 10.8: Compute an expected value
SECTION 10.1: UNCERTAINTY, LIKELIHOOD, AND PROBABILITY

“We sail within a vast sphere, ever drifting in uncertainty, driven from end to end. When we think to attach ourselves to any point and to fasten to it, it wavers and leaves us; and if we follow it, it eludes our grasp, slips past us, and vanishes forever. Nothing stays for us.”

– Blaise Pascal (1623–1662) Mathematician, Philosopher, Scientist, Inventor, Theologian

Life is uncertain. This is both a fantastic and terrible part of the human condition. You may wish you knew everything that would happen in the future, but this would make life pretty boring. Think about it.

Humanity has tried to predict the future for thousands of years. Some people were considered prophets with a direct line to knowledge of future events. Rituals were used to try to divine the future. We want to know what will happen next, sometimes out of pure curiosity, or planning, decision making, or financial gain.

The application of mathematics influenced the field of probability during the Enlightenment around the 16th and 17th centuries. Through scientific experiment, we realized some things were more likely to happen than others. People transitioned from trying to figure out what they thought was certain to happen (the future is written but unknown) to trying to quantify the likelihood or chance of the different outcomes that could occur (the future is not written but we can discern the possibilities and their chances of happening). This was a significant change in perspective and helped drive the development of scientific inquiry.

As you work through the units on probability, try to keep the idea of uncertainty in mind. Frequently, you will be computing fractions, ratios and percents, and you may wonder what is new or different about the mathematics you are learning. In mathematics, we typically have one answer that is always correct (given an agreed understanding of the question). In probability, you will find “correct answers”, but they won't necessarily correspond exactly to what happens in the future. So as you work the problems, reflect on the idea that your answers are measures of likelihood and possibilities, not certainty.

The Language of Likelihood

We use the language of uncertainty almost daily. You may tell a friend, “I'll probably be a little late”, or “It's possible I left my cell phone at the restaurant.” Words like “possible” and “probably” are referring to uncertainty. We suspect something happened or will happen, but we are uncertain. Probability is a means of quantifying uncertainty. We assign an outcome or event, a numerical value between 0 and 1 or 0% and 100% to express how likely it is the event will occur.
Before we begin to compute probabilities mathematically, we will develop some more descriptive language that corresponds to numerical probabilities and their relative likelihood. The number line below provides language we can use to describe the likelihood of an event based on numerical values between 0 and 1 as fractions or decimals, and percentages between 0% and 100%. This is not universally accepted language for describing the probabilities, but an approximate form to give us an understanding of numerical probabilities.

Note that there three exact probabilities to which we assign their own name. If a probability is 0 or 0%, we say it is **impossible**. If a probability is 1 or 100% we say it is **certain**. If a probability is ½, 0.5, or 50%, we say it is **as likely as not likely** or alternatively, the probabilities that the event will happen or not happen are **equally likely**.

**Problem 1: Worked Example – The Language of Likelihood**

Use the likelihood chart to categorize the given probabilities using language that involves the likelihood of the event.

a) The probability of rain today is 65%.

**Solution:** 65% is between 50% and 70% which is in the somewhat likely category.

**Answer:** It is somewhat likely that it will rain today.

b) The probability that a person will acquire a genetic disease is \( \frac{1}{4} \).

**Solution:** \( \frac{1}{4} \) equals 0.25 as a decimal or 25% and is between 10% and 30% which is in the unlikely category.

**Answer:** It is unlikely that a person will acquire a genetic disease.
c) The probability that a diagnostic medical test is accurate is 0.9785

**Solution:** 0.9785 is a decimal between 0.9 and 1 which is in the very likely category.

**Answer:** It is very likely that the diagnostic medical test is accurate.

d) A six sided die is labeled with the numbers 1, 2, 3, 4, 5, and 6. The probability that you roll the die and it lands on 8 is 0.

**Solution:** A probability of 0 is an impossible event.

**Answer:** It is impossible that the rolled die will land on 8.

**Problem 2: You Try – The Language of Likelihood**

Use the likelihood chart to categorize the given probabilities using language that involves the likelihood of the event.

a) The probability that a person who bought a lottery ticket will win the jackpot is 0.34%

**Solution:**

**Answer:**

b) The probability that a person rolls a six sided die and gets a number greater than 4 is $\frac{1}{3}$.

**Solution:**

**Answer:**

c) The probability that a person flips a coin and it lands on heads or tails is 100%.

**Solution:**

**Answer:**
SECTION 10.2: TYPES OF PROBABILITY

There are three types of probability people use. The first is a personal probability or a subjective probability. A personal probability is when someone assigns a quantity based on their experience, but without any data collection to back up their results. These probabilities are based on experience, but are not scientifically determined.

For example, you may have a friend who you know very well and you can frequently predict their behavior. Perhaps they are habitually late, have always remembered your birthday, or go to sleep before 10 p.m. You may assign numbers to these events such as:

- There is a 75% chance my friend will be late picking me up.
- There is a 3% chance my friend will forget my birthday.
- Since it is 11 p.m., there is a 99% chance that my friend is already asleep.

These values give us a feel for the likelihood of an event and are useful in everyday life, but they are not determined by a scientific process and there isn’t a way to measure their accuracy.

A second type of probability is a theoretical probability. These probabilities use mathematics and not observations or experiments to determine the likelihood of an event. For example, if you flip a fair coin, the probability it lands on heads is ½. This probability is found by determining that there are two outcomes, heads or tails, and since we are given that the coin is “fair” each outcome is equally likely to occur. Therefore, the probability of heads is 1 out of 2 or ½. We will compute many of these probabilities in the next few units.

The third type of probability is an empirical or experimental probability. This type of probability is used frequently in science to test hypotheses. For example, if a pharmaceutical company wants to determine if a newly developed drug cures a disease, they will design an experiment to determine the percentage of people in the experiment (subjects) that are cured by their treatment. The actual experiment is much more complex than this to ensure scientific validity, and many variables need to be accounted for that are not mentioned here. However, observe that this is different than a personal probability because it is intentionally performing an experiment and recording the results to quantify a probability as opposed to recollecting past experiences to make a judgement. In the next section, we will look at some very basic experiments to see the power of empirical probabilities and how their results can provide us insight into likelihood.
SECTION 10.3: EXPERIMENTS AND RELATIVE FREQUENCY

In this section, we will look at an example of an empirical probability using *relative frequency*. Recall that frequency is the number of times an event occurs, and relative frequency is the frequency divided by a total amount such as the number of times an experiment is run or the number of people in an experiment.

We will also use a *random process*. Some examples of random processes are flipping a coin, rolling a die, or picking a name out of a hat (returning the original name before each new pick). The main components of a random process are that each time you perform the process or experiment the possible outcomes and chances are identical to the other experiments, and the previous outcome does not affect the next outcome (this is why we would put the name back in the hat before picking a new name).

Notice that the word random does not have the same meaning in probability as it does in everyday language. Some synonyms given by Microsoft Word for the word random are, “accidental, haphazard, arbitrary, unsystematic, and unplanned.” In probability, these words are very misleading. We are very systematic and planned when we *perform* a random process. Our goal in planning is to ensure that we are not affecting the outcome and that each outcome is not dependent on a previous outcome. As you will see, the short term relative frequency of the outcomes of a random process are unpredictable, but the long term relative frequencies tend to be stable and close to a specific percentage. The next example will help us develop these ideas.

**Problem 3:** Media Example – Random Processes and Relative Frequency

**Part 1:** A fair coin is tossed repeatedly. The graph represents the relative frequency of heads that occur (number of heads divided by the total number of tosses) versus the number of times the coin was tossed for the first 10 tosses. The table shows these categories, and in addition, the total number of heads after each number of tosses. Use this information to answer the questions below.

**Short Term Relative Frequency**
a) What is the minimum relative frequency of the number of heads for the first 10 tosses?

b) What is the maximum relative frequency of the number of heads for the first 10 tosses?

c) What is the difference between the minimum relative frequency and the maximum relative frequency of the number of heads for the first 10 tosses? Write your answer as a percent.

Part 2: The graph below represents the relative frequency of heads that occur (number of heads divided by the total number of tosses) versus the number of times the coin was tossed for the first 1000 tosses. The table shows these categories, and in addition, the total number of heads for the 9,991st to 10,000th tosses. Use this information to answer the questions below.
Long Term Relative Frequency

<table>
<thead>
<tr>
<th>Number of Tosses</th>
<th>9991</th>
<th>9992</th>
<th>9993</th>
<th>9994</th>
<th>9995</th>
<th>9996</th>
<th>9997</th>
<th>9998</th>
<th>9999</th>
<th>10000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Heads</td>
<td>4959</td>
<td>4960</td>
<td>4960</td>
<td>4960</td>
<td>4961</td>
<td>4962</td>
<td>4962</td>
<td>4962</td>
<td>4962</td>
<td>4962</td>
</tr>
<tr>
<td>Relative Frequency</td>
<td>0.4963</td>
<td>0.4964</td>
<td>0.4963</td>
<td>0.4963</td>
<td>0.4963</td>
<td>0.4964</td>
<td>0.4963</td>
<td>0.4962</td>
<td>0.4962</td>
<td>0.4962</td>
</tr>
</tbody>
</table>

a) What is the minimum relative frequency of the number of heads for the tosses between 9,991 and 10,000?

b) What is the maximum relative frequency of the number of heads for the tosses between 9,991 and 10,000?

c) What is the difference between the minimum relative frequency and the maximum relative frequency of the number of heads for the tosses between 9,991 and 10,000? Write your answer as a percent.

d) Since the coin is fair, on average, approximately half of the tosses should be heads. So when the coin is tossed 10,000 times approximately 5000 of the tosses should be heads. What is the actual number of heads for 10,000 tosses as given by the chart?

e) What is the difference between the number of heads we should expect on average, 5000, and the actual number of heads for 10,000 tosses?

f) What is the difference between the percent we expect to be heads on average, 50%, and the relative frequency as a percent for 10,000 tosses?

In the You Try below, you will answer the same questions for the same random process, but a different experiment with different outcomes. Note the similarities and differences to the Media Example as you work through it.
**Problem 4: You Try – Random Processes and Relative Frequency**

**Part 1:** A fair coin is tossed repeatedly. The graph represents the relative frequency of heads that occur (number of heads divided by the total number of tosses) versus the number of times the coin was tossed for the first 10 tosses. The table shows these categories, and in addition, the total number of heads after each number of tosses. Use this information to answer the questions below.

**Short Term Relative Frequency**

<table>
<thead>
<tr>
<th>Number of Tosses</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Heads</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>Relative Frequency</td>
<td>1</td>
<td>1</td>
<td>0.6667</td>
<td>0.75</td>
<td>0.8</td>
<td>0.8333</td>
<td>0.7143</td>
<td>0.75</td>
<td>0.7778</td>
<td>0.7</td>
</tr>
</tbody>
</table>

a) What is the minimum relative frequency of the number of heads for the first 10 tosses?

b) What is the maximum relative frequency of the number of heads for the first 10 tosses?

c) What is the difference between the minimum relative frequency and the maximum relative frequency of the number of heads for the first 10 tosses? Write your answer as a percent.
Part 2: The graph below represents the relative frequency of heads that occur (number of heads divided by the total number of tosses) versus the number of times the coin was tossed for the first 1000 tosses. The table shows these categories, and in addition, the total number of heads for the 9,991st to 10,000th tosses. Use this information to answer the questions below.

Long Term Relative Frequency

<table>
<thead>
<tr>
<th>Number of Tosses</th>
<th>9991</th>
<th>9992</th>
<th>9993</th>
<th>9994</th>
<th>9995</th>
<th>9996</th>
<th>9997</th>
<th>9998</th>
<th>9999</th>
<th>10000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Heads</td>
<td>5049</td>
<td>5049</td>
<td>5050</td>
<td>5051</td>
<td>5051</td>
<td>5051</td>
<td>5052</td>
<td>5052</td>
<td>5052</td>
<td>5052</td>
</tr>
<tr>
<td>Relative Frequency</td>
<td>0.5054</td>
<td>0.5053</td>
<td>0.5054</td>
<td>0.5054</td>
<td>0.5054</td>
<td>0.5053</td>
<td>0.5054</td>
<td>0.5053</td>
<td>0.5053</td>
<td>0.5052</td>
</tr>
</tbody>
</table>

a) What is the minimum relative frequency of the number of heads for the tosses between 9,991 and 10,000?

b) What is the maximum relative frequency of the number of heads for the tosses between 9,991 and 10,000?

c) What is the difference between the minimum relative frequency and the maximum relative frequency of the number of heads for the tosses between 9,991 and 10,000? Write your answer as a percent.

d) Since the coin is fair, on average, approximately half of the tosses should be heads. So when the coin is tossed 10,000 times approximately 5000 of the tosses should be heads. What is the actual number of heads for 10,000 tosses as given by the chart?
e) What is the difference between the number of heads we should expect on average, 5000, and the actual number of heads for 10,000 tosses?

f) What is the difference between the percent we expect to be heads on average, 50%, and the relative frequency as a percent for 10,000 tosses?

Summary and Analysis of Random Processes and Relative Frequency

**Short Term Relative Frequency:** The image below shows the short term relative frequency for 10 coin tosses for the Media Example and You Try Problem. These were both randomly generated. Observe the differences between the two graphs. The Media Example began with 3 tosses of tails and a relative frequency of 0. The relative frequency then increases to 0.5 after 6 tosses and varied as the coin flips increased. In contrast, the You Try problem began with 2 tosses of heads and a relative frequency of 1. The relative frequency then decreases to 0.67 after 3 tosses and varied as the coin flips increased. This illustrates an important fact: The short term frequency of random processes are unpredictable.
Long Term Relative Frequency: Now let's consider the long term relative frequency. The image below shows the long term relative frequency for 1000 coin tosses for the Media Example and You Try Problem. These were both randomly generated. Observe the differences between the two graphs. The graphs appear to have different behavior or patterns for the first 100 or 200 tosses. However, after many tosses, both graphs appear to stabilize and center around or near the value 0.5. This illustrates an important fact: The long term frequency of random processes are stable.

A consequence of these facts is that in the short run, you are not likely to get predictable results or determine the underlying behavior of the random process, but in the long run, you can observe stabilized results. In addition, the long term relative frequency approximates the theoretical probability of the random process. This fact is called the Law of Large Numbers.

One real world application of this fact is how casinos earn money. A few people playing can win large jackpots that cost the casino a lot of money, but in the long run enough people will lose to counteract this effect. Similarly, an insurance company may have to pay some large death premiums, but a large amount of people will live and continue to make payments. We will revisit these ideas in a later section when we discuss expected value.

In the next sections, we will discuss theoretical probabilities and odds which are computed directly, and not based on experiments. We will see that the long term relative frequencies we discussed in this section are good estimators of these theoretical probabilities.
SECTION 10.4: INTRODUCTION TO THEORETICAL PROBABILITY

Recall from Section 10.2 that a **theoretical probability** uses mathematics and not observations or experiments to determine the likelihood of an event. In Section 10.3, we found in Problem 3 that in one experiment, if we flipped a coin 10,000 times the long term relative frequency of the number of times the coin landed on heads was 0.4962. However, *theoretically*, if you flip a fair coin, the probability it lands on heads is \( \frac{1}{2} \). This probability is found by determining that there are two outcomes, heads or tails, and since we are given that the coin is “fair” each outcome is equally likely to occur. Therefore, the probability of heads is 1 out of 2 or \( \frac{1}{2} \).

This will be our strategy for finding theoretical probabilities. We will determine all the possible outcomes of an experiment. We will create a list or set of these outcomes so that each outcome is equally likely. The number of elements in this set, \( S \), will be our total and the denominator of the ratio. Then we will look specifically at the event in which we are interested. This event such as landing on heads or picking a red card will be a subset of all the possible outcomes. The number of elements in this set, \( E \), will be our part and the numerator of the ratio. The information below summarizes these ideas.

1) Let \( S \) be a set called the **sample space** that contains all of the possible outcomes of an experiment written so that all of the outcomes in \( S \) are equally likely to occur.

2) Let \( E \) be a set called the **event space**. \( E \) is a subset of \( S \) that contains all of the equally likely outcomes of \( S \) in which we want to find a probability.

3) The **theoretical probability** of the event \( E \) with sample space \( S \) is

\[
\text{Probability of the event } E = \frac{\text{number of elements in } E}{\text{number of elements in } S} = \frac{\text{number of equally likely ways } E \text{ can occur}}{\text{number of equally likely ways } S \text{ can occur}}
\]

**Problem 5: Worked Example – Event Space, Sample Space, and Theoretical Probability**

1) A fair single six sided die is rolled. The sides are labeled with 1, 2, 3, 4, 5, or 6 dots.

a) What is the sample space for the experiment of rolling a single sided die and observing the number of dots on the top side?

**Solution:** We are given that the die is “fair” meaning that each side of the die is equally likely to land up in an experiment. (This is often implied instead of stated in the problem.) So the possible equally likely outcomes are 1, 2, 3, 4, 5, or 6.

**Answer:** The sample space, \( S = \{ 1, 2, 3, 4, 5, 6 \} \)
b) What is the event space for the following event: the die lands on 3.

**Solution:** The event space is a subset of the sample space and lists the possible outcomes from S that apply to the specific event. So the event space in this case, contains only the number 3.

**Answer:** The event space $E = \{3\}$

c) What is the probability the die lands on 3?

**Solution:** The formula for the theoretical probability is the number of elements in $E$, 1, divided by the number of elements is $S$, 6.

**Answer:** The probability the die lands on 3 is $\frac{1}{6}$.

2) Sylvia has 12 marbles as shown in the diagram below. Each marble is either blue (labeled with a B), red (labeled with an R), or yellow (labeled with a Y).

![Diagram of 12 marbles]

a) What is the sample space for the experiment of choosing a marble at random and observing its color?

**Solution:** Since we are told we are choosing a marble at random it is implied that each of the 12 marbles are equally likely to be chosen. It may be tempting to list the sample space outcomes as red, blue, and yellow. However, note that there are more red marbles than blue or yellow marbles and more yellow marbles than blue marbles. So these three outcomes are not equally likely.

To remedy this disparity, we will think of each of the 12 marbles as *distinguishable*. By this we mean we will label each of the marbles with a number as well as a color so we can differentiate them from one another. There are 7 red marbles, 4 yellow marbles, and one blue marble.

**Answer:** The sample space, $S = \{R1, R2, R3, R4, R5, R6, R7, Y1, Y2, Y3, Y4, B1\}$
b) What is the event space for the following event: the marble chosen is yellow.

**Solution:** The event space is a subset of the sample space and lists the possible outcomes from $S$ that apply to the specific event. So the event space in this case, contains the four yellow marbles.

**Answer:** The event space $E = \{Y_1, Y_2, Y_3, Y_4\}$

c) What is the probability the marble chosen is yellow?

**Solution:** The formula for the theoretical probability is the number of elements in $E$, 4, divided by the number of elements is $S$, 12.

**Answer:** The probability the marble is yellow is $\frac{4}{12} = \frac{1}{3}$

(Note it is customary to simplify the fraction when possible).

**Problem 6:** You Try – Event Space, Sample Space, and Theoretical Probability

The device below is a spinner. When the arrow is spun, the arrow is equally likely to land on any part of the spinner. If the arrow lands on a black line, we spin again.

![Spinner Diagram]

a) What is the sample space for the experiment of spinning the arrow and observing the color in which it lands?

b) What is the event space for the following event: the arrow lands on white.

c) What is the probability the arrow lands on white?
Recall from our study of sets and Venn Diagrams that the intersection of two sets, $A \cap B$, refers to any elements in both A and in B. The union of two sets, $A \cup B$, refers to any elements that are only in A, only in B, or in both A and B (the intersection of A and B). Since theoretical probabilities involve determining the number of elements in a set, we will can use the ideas learned about sets to count the number of elements for event spaces.

**Problem 7: Media Example – Probabilities involving Union and Intersections**

1) Danny has 12 marbles as shown in the diagram below. Each marble is either blue or white. In addition, each marble is labeled with a number between 1 and 5.

Danny chooses one of the marbles at random. Determine the following probabilities. Enter your probabilities as fractions and simplify any fractions as needed.

a) What is the probability that Danny will choose a white and even numbered marble?

b) What is the probability that Danny will choose a white or even numbered marble?

c) What is the probability that Danny will choose a blue and odd numbered marble?

d) What is the probability that Danny will choose a blue or odd numbered marble?
2) A single card is drawn at random from a deck of 52 cards. Determine the following probabilities. Enter your final answers as reduced fractions or whole numbers. (Note: Aces are not considered Face or Number cards.)

Group A

a) What is the probability the card drawn is a 5?
b) What is the probability the card drawn is a Queen?
c) What is the probability the card drawn is a 5 and a Queen?
d) What is the probability the card drawn is a 5 or a Queen?

Group B

a) What is the probability the card drawn is a Face Card?
b) What is the probability the card drawn is a Heart?
c) What is the probability the card drawn is a Face Card and a Heart?
d) What is the probability the card drawn is a Face Card or a Heart?

Group C

Let $E_1$ and $E_2$ be two events for an experiment. Determine a formula for the probability of $E_1$ or $E_2$ based on the problems in Groups A and B.
Problem 8: You Try – Probabilities involving Union and Intersections

Two six-sided dice are rolled. The image below shows the 36 equally likely outcomes of rolling the two dice and the corresponding sum of the two dice. Use this information to determine the following probabilities. Enter your final answers as reduced fractions or whole numbers.

a) What is the probability that the first die is a 5 and the sum of the die is more than 8?

b) What is the probability that the first die is a 5 or the sum of the die is more than 8?

c) What is the probability that the two dice land on different numbers and the sum of the die is less than 6?

d) What is the probability that the two dice land on different numbers or the sum of the die is less than 6?

Results: The probability of $E_1 \cup E_2 = \text{probability of } E_1 + \text{probability of } E_2 - \text{probability of } E_1 \cap E_2$

Two events $E_1$ and $E_2$ are called **mutually exclusive**, if their intersection is empty ($E_1 \cap E_2 = \emptyset$).

If two events are mutually exclusive, the probability of $E_1 \cup E_2$ is the prob of $E_1 + \text{prob of } E_2$. 
SECTION 10.6: PROBABILITY AND COMPLEMENTS

Recall from sets and Venn diagrams, that the complement of a set $A$ is denoted by $A'$, and means not $A$. This set contains any elements in $U$ that are not in the set $A$. In terms of probability, our universal set is denoted by the sample space of equally likely outcomes $S$, and we refer to the subset as $E$, the corresponding event space for the probability.

Problem 9: Media Example – Probabilities and Complements

A single card is drawn at random from a deck of 52 cards. Determine the following probabilities. Enter your final answers as reduced fractions or whole numbers. (Note: Aces are not considered Face or Number cards.)

Group A

a) What is the probability the card drawn is a Face card?

b) What is the probability the card drawn is not a Face card?

c) Find the sum: the probability the card is a Face card plus the probability the card is not a Face card?

Group B

a) What is the probability the card drawn is a diamond?

b) What is the probability the card drawn is not a diamond?

c) Find the sum: the probability the card is a diamond plus the probability the card is not a diamond?

Group C

a) The probability of rain tomorrow is 60%. What is the probability that it will not rain tomorrow?

b) The probability of winning a lottery game is 0.0013%. What is the probability of not winning the game?

c) The probability of flipping a coin three times and getting all heads is $\frac{1}{8}$. What is the probability of flipping a coin three times and getting at least one tail?
Problem 10: You Try – Probabilities and Complements

A single card is drawn at random from a deck of 52 cards. Determine the following probabilities. Enter your final answers as reduced fractions or whole numbers. (Note: Aces are not considered Face or Number cards.)

a) What is the probability the card drawn is a Number card?

b) What is the probability the card drawn is not a Number card?

c) Find the sum: the probability the card is a Number card plus the probability the card is not a Number card?

d) The probability that a diagnostic medical test is accurate is 93.8%. What is the probability the test is not accurate?

e) Kara is going to randomly pull three candies out of a bag of assorted candies. The probability that all three candies are her favorite is 0.076. What is the probability that at least one of the candies is not her favorite?
SECTION 10.6: ODDS AND PROBABILITY

The terms probability and odds are often used interchangeably in error. Probabilities and odds do have a mathematical relationship and both measure uncertainty, but they are not identical. Recall the definition of the probability of an event E below.

\[
\text{Probability of the event } E = \frac{\text{number of elements in } E}{\text{number of elements in } S} = \frac{\text{number of equally likely ways } E \text{ can occur}}{\text{number of equally likely ways } S \text{ can occur}}
\]

We have seen that since the sample space, S, contains all possible outcomes for the experiment, the probability of any event E ranges from 0 to 1, from impossible to certain. In particular, probabilities measure a part to whole relationship. When written in fraction form, the denominator is the total or whole, and the numerator is the part of the whole that corresponds to the specific event.

With odds, we consider a part to part relationship that involves the event under consideration and its complement.

\[
\text{Odds in favor the event } E = \frac{\text{number of elements in } E}{\text{number of elements not in } E}
\]

\[
\text{Odds against the event } E = \frac{\text{number of elements not in } E}{\text{number of elements in } E}
\]

Typically, we use one of two types of ratio notation instead of fraction notation to represent odds. The first representation uses a colon “:” to separate the quantities. The second representation uses the word “to” to separate the quantities.

\[
\text{Odds in favor the event } E = \text{number of elements in } E : \text{number of elements not in } E
\]

\[
\text{Odds against the event } E = \text{number of elements not in } E : \text{number of elements in } E
\]

Note: We always write odds as a ratio of one whole number to another and simplify the results like a fraction.
Problem 11: Media Example – Odds and Probability

Part 1: A random point within the large rectangle below is chosen. If the point lies on a black line, another random point will be chosen. Determine the following odds.

![rectangle diagram]

a) What are the odds in favor of choosing a point in a blue region?

b) What are the odds against choosing a point in a blue region?

c) What are the odds in favor of choosing a point in a white region?

d) What are the odds against choosing a point in a white region?

Part II: Sharon was practicing free throw shots. She made 56 shots for every 7 shots she missed. We will randomly select the outcome (made or missed) of one of her free throw shot attempts.

a) What are the odds in favor of her making the shot? Show the unsimplified and simplified odds and interpret the results.

b) What are the odds against her making the shot? Show the unsimplified and simplified odds and interpret the results.

c) What is the probability of her making the shot? Show the unsimplified and simplified probability and interpret the results.

d) What is the probability of her missing the shot? Show the unsimplified and simplified probability and interpret the results.
**Problem 12:** You Try – Odds and Probability

Jeremy has 12 marbles as shown in the diagram below. Each marble is either blue (labeled with a B) or yellow (labeled with a Y). Jeremy chooses one of the marbles at random. Determine the following odds.

a) What are the odds in favor of Jeremy choosing a yellow marble?

b) What are the odds against Jeremy choosing a yellow marble?

c) What are the odds in favor of Jeremy choosing a blue marble?

d) What are the odds against Jeremy choosing a blue marble?

**Converting between Odds and Probability:** As we have seen, odds and probabilities are related, but not identical. Sometimes we will be given an odds relationship, and we will want to rewrite it as a probability or vice versa. Recall that an odds relationship is part to part, and probability is part to whole. In addition, since the parts in the odds are the elements in E and not E, part E + part not E = whole.

So given the odds, part E to part E', the probability of E = \( \frac{\text{part } E}{\text{part } E + \text{part } E'} \)

Given the probability of E, \( \frac{\text{part } E}{\text{whole}} \), the odds in favor of E = part E to (whole – part E)
Problem 13: Worked Example – Converting between Odds and Probability

a) The probability of randomly drawing a face card from a deck of 52 cards is \( \frac{12}{52} = \frac{3}{13} \). What are the odds in favor of and against drawing a face card?

Solution: The numerator and denominator of the probability correspond as \( \frac{\text{part } E}{\text{whole}} = \frac{3}{13} \).

Therefore, the odds in favor of \( E \) =

\[
\frac{\text{part } E}{\text{whole} - \text{part } E} = \frac{3}{13 - 3} = \frac{3}{10}
\]

The odds against \( E \) are found by reversing the quantities in the odds in favor: 10 to 3

Answer: Odds in favor of drawing a face card: 3 to 10
Odds against drawing a face card: 10 to 3

b) The probability that it is sunny in Phoenix tomorrow is 88%. What are the odds in favor of and against it being sunny tomorrow in Phoenix?

Solution: We are given the probability that it will be sunny as a percentage, specifically, 88%. We will rewrite the percent as a simplified fraction and use our odds conversion formula.

\[
88% = \frac{88}{100} = \frac{4\cdot22}{4\cdot25} = \frac{22}{25} = \frac{\text{part } E}{\text{whole}}
\]

Therefore, the odds in favor of \( E \) =

\[
\frac{\text{part } E}{\text{whole} - \text{part } E} = \frac{22}{25 - 22} = \frac{22}{3}
\]

The odds against \( E \) are found by reversing the quantities in the odds in favor: 3 to 22

Answer: The odds in favor of it being sunny in Phoenix tomorrow are 22 to 3
The odds against it being sunny in Phoenix tomorrow are 3 to 22
c) The odds in favor of a medical test being accurate are 997 to 3. What is the probability the medical test is accurate? What is the probability the medical test is not accurate? Write your answers as percents.

Solution: We are given the odds 997 to 3 which correspond to part E to part E’. The conversion formulas are shown below.

\[
\text{Probability of } E = \frac{\text{part } E}{\text{part } E + \text{part } E'} = \frac{997}{997 + 3} = \frac{997}{1000}
\]

\[
\text{Probability of } E' = \frac{\text{part } E'}{\text{part } E + \text{part } E'} = \frac{3}{997 + 3} = \frac{3}{1000}
\]

Answer: We are also asked to write our final answers as percents as shown below.

Probability the test is accurate = \(\frac{997}{1000} = 0.997 = 99.7\%\)

Probability the test is not accurate = \(\frac{3}{1000} = 0.003 = 0.3\%\)

Problem 14: You Try – Converting between Odds and Probability

a) The probability that a randomly chosen student at SCC is less than 23 years of age is 43%. What are the odds in favor of and against a randomly chosen person at SCC being less than 23 years of age?

b) The odds in favor of a kidney transplant recipient living at least 5 years after the procedure are 43 to 7. What is the probability a kidney transplant recipient will live at least 5 years after the procedure? Write your answer as a percent.
SECTION 10.7: EXPECTED VALUE

In this last section, we will use probabilities and the idea of long term frequencies from section 10.3 to learn about expected values. The expected value of an experiment is the long run weighted average of a repeated experiment. We have seen that the long term relative frequency approximates the theoretical probability of a repeated experiment. In the same way, we can use theoretical probabilities to approximate the long term average of an experiment.

Process for finding an expected value: Multiply the probability of each outcome by the outcome value, and then add the results.

Problem 15: Worked Example – Expected Value

a) A six sided die is rolled repeatedly with possible outcomes 1, 2, 3, 4, 5, 6 where each outcome is equally likely with probability $\frac{1}{6}$. What is the expected value of the number rolled?

Solution: Since each outcome has a probability of $\frac{1}{6}$, multiply each value by $\frac{1}{6}$ and add the results.

$$\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6$$

$$= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{1 + 2 + 3 + 4 + 5 + 6}{6} = \frac{21}{6} = 3.5$$

Answer: The expected number of a six sided die rolled repeatedly is 3.5. (Note the die can never land on 3.5. This is similar to means or averages that are not included in a data set.)

b) You and a friend are playing a game. You randomly draw one card from a 52 card deck. If the card is a face card, your friend gives you $3. If the card is anything else, you give your friend $1. What is your expected value for this game to the nearest cent?

Solution: There are a total of 12 face cards so the probability of drawing a face card is $\frac{12}{52} = \frac{3}{13}$, and the probability of not drawing a face card is $\frac{40}{52} = \frac{10}{13}$. Multiply these by the gains or losses.

$$\frac{3}{13} \cdot $3 + \frac{10}{13} \cdot -$1 = \frac{9}{13} - \frac{10}{13} = -$\frac{1}{13} \approx -$0.08$$

Answer: Your expected value for this game is $-0.08$, meaning, if you play these game repeatedly, on average, you will lose approximately 8 cents per game.
Problem 16: Media Example – Expected Value

a) A company offers a 5 year warranty on a product for $30. Replacing the product to the consumer who purchases the warranty costs the company $125. The quality control division estimates that 1.37% of their products will fail within the first 5 years. What is the company's expected value for each warranty sold?

b) A school is holding a raffle for a $500 prize. They sold 300 tickets at a cost of $2 per ticket. What is your expected value if you bought a single ticket? What is the school's expected value per ticket? A fair game is when the players each have an expected value of 0. Is this game fair?

Problem 17: You Try – Expected Value

a) You and a friend are playing a game. You randomly draw one card from a 52 card deck. If the card is a heart, your friend gives you $3. If the card is anything else, you give your friend $1. What is your expected value for this game to the nearest cent? A fair game is when the players each have an expected value of 0. Is this game fair?

b) A company offers a 2 year warranty on a product for $50. Replacing the product to the consumer who purchases the warranty costs the company $375. The quality control division estimates that 0.45% of their products will fail within the first 2 years. What is the company's expected value for each warranty sold?
UNIT 11 – PROBABILITY PART TWO

In this Unit, we will extend our knowledge of probability to include conditional probabilities, independence, and probabilities that involve more complex counting techniques.

Section 11.1: Compute conditional probabilities
Section 11.2: Determine whether two events are independent or dependent
Section 11.3: Find two stage probabilities for independent and dependent events
Section 11.4: Use the fundamental counting principle for two or more choices
Section 11.4: Use the fundamental counting principle to find probabilities
Section 11.5: Adapt fundamental counting principle for dependent events and permutations
Section 11.5: Calculate probabilities using permutations
Section 11.6: Adapt permutation formula to combinations
Section 11.6: Compute probabilities using combinations
SECTION 11.1: CONDITIONAL PROBABILITY

Conditional probabilities involve finding the probability of an event when you are given some additional knowledge or information about the observation or experiment. For example, suppose we have heart disease data for people over 50. We can use this data to calculate the probability of a random person over 50 having heart disease.

Now suppose the data are also partitioned by gender (male or female). We could also calculate probabilities of heart disease for a specific gender, perhaps the probability of heart disease of a person over 50 given that the person is a woman. The results could be different (and likely will in this case) if you use this new information. Similarly, the “given” information could be reversed. We could find the probability that a person over 50 is a woman given that they have heart disease. Note that although these are similar, the known information is different. In one case you know the person is a woman, and in the other, you know the person has heart disease. These probabilities are considering different subsets of people from the original population at the start and the probabilities are most likely different.

In the next example, we will compute some conditional probabilities using the method of reducing the sample space. The idea is, instead of looking at the original sample space of all equally likely outcomes, we will only consider the subset of the sample space that applies to our specific “given” information.

Problem 1: Worked Example – Conditional Probability
Compute the following conditional probabilities by reducing the sample space.
a) A single die is rolled.
i. What is the probability the die is a 6 given that the roll is an even number?

Solution: First consider the original sample space of the experiment. Then reduce the sample space using the given information that the roll is an even number.

Using the reduced samples space, \( S = \{2, 4, 6\} \), the total possible number of equally likely outcomes is 3. The event space, \( E = \{\text{die is a 6}\} \), is \( \{6\} \) which has one outcome. So the probability is 1 out of 3 or \( \frac{1}{3} \).

Answer: \( \frac{1}{3} \)
ii. What is the probability the die is an even number given that the roll is a 6?

**Solution:** First consider the original sample space of the experiment. Then reduce the sample space using the given information that the roll is a 6.

Using the reduced samples space, $S = \{6\}$, the total possible number of equally likely outcomes is 1. The event space, $E =$ the die is even, is $\{6\}$ which is also one outcome. Note that even though there are other even numbers in the original sample space, there are not any in the reduced sample space so we do not consider them in our computations. So the probability is 1 out of 1, $\frac{1}{1}$, or just 1. Note that this is a *certain event*. If we know the die is a 6, it is certainly also an even number.

**Answer:** 1

*Notice the probability of $E_1$ given $E_2$ is not necessarily equal to the probability of $E_2$ given $E_1.*

b) A card is drawn from a deck of 52 cards. What is the probability the card is a 5 given that the card is a number card? (Note: Aces are not considered number cards.)

**Solution:** First consider the original sample space of the experiment, the 52 cards, and then construct the reduced sample space with only number cards.

The number of elements in the reduced sample space is 36, and 4 of these cards are 5’s. So the probability is 4 out of 36, $\frac{4}{36} = \frac{1}{9}$.

**Answer:** $\frac{1}{9}$
c) Two cards are drawn from a deck of 52 cards.

i. What is the probability the second card is a diamond given that the first card is a diamond?

**Solution:** The sample space of all the possible outcomes for drawing two cards from a deck of cards is a bit complex. We can list all of the possible outcomes, but there are so many, it would likely make things more confusing. So let's consider the two draws separately using the given information as shown in the image below.

Note for the second card, there are only 51 possible cards that can be drawn since we are given that the first card is a diamond. Specifically, there are still 13 clubs, 13 spades, 13 hearts, but only 12 diamonds. So there are 51 cards in the sample space for the second card. Since we are considering the event, E, that the second card is also a diamond, there are 12 diamonds in the sample space that are in this event space. So the probability is 12 out of 51 or \( \frac{12}{51} = \frac{4}{17} \).

Answer: \( \frac{4}{17} \)

ii. What is the probability the second card is a club given that the first card is a diamond?

**Solution:** Notice that this problem is similar to part i except that we want the probability that the second card is a club. It may seem like this probability wouldn't change since the first card was a diamond and not a club. However, this sample space for the second card is the same as part i. The number of possible outcomes in the sample space has been reduced from 52 to 51 by the first card being chosen. There are 13 clubs in the reduced sample space. So the probability is 13 out of 51 or \( \frac{13}{51} \).

Answer: \( \frac{13}{51} \)
Problem 2: Media Example – Contingency Tables and Conditional Probability

A teacher categorized students by whether they regularly completed their homework (or not) and whether they passed the course (or not). The data are in the table below.

<table>
<thead>
<tr>
<th>Completed Homework</th>
<th>Did not Complete Homework</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passed the Course</td>
<td>52</td>
<td>11</td>
</tr>
<tr>
<td>Did not Pass the Course</td>
<td>6</td>
<td>31</td>
</tr>
<tr>
<td>Total</td>
<td>58</td>
<td>42</td>
</tr>
</tbody>
</table>

A student is randomly selected from this group of students. Use this information to find the probabilities requested. Enter all of your results as percents rounded to two decimal places.

a) What is the probability that a student completed the homework given that they passed the course?

b) What is the probability that a student passed the course given that they completed the homework?

c) What is the probability that a student completed the homework given that they did not pass the course?

d) What is the probability that a student did not pass the course given that they did not complete the homework?
Problem 3: You Try – Contingency Tables and Conditional Probability

a) Two cards are drawn from a deck of 52 cards.
   i. What is the probability the second card is a 3 given that the first card is an Ace?

   ii. What is the probability the second card is a face card given that the first card is the King of clubs?

b) A researcher collected data on adult alcohol consumption and marital status. She reported the number of adults in each category.

<table>
<thead>
<tr>
<th></th>
<th>0 drinks per month</th>
<th>1 - 60 drinks per month</th>
<th>More than 60 drinks per month</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single</td>
<td>69</td>
<td>211</td>
<td>70</td>
<td>350</td>
</tr>
<tr>
<td>Married</td>
<td>410</td>
<td>636</td>
<td>135</td>
<td>1181</td>
</tr>
<tr>
<td>Widowed</td>
<td>84</td>
<td>53</td>
<td>18</td>
<td>155</td>
</tr>
<tr>
<td>Divorced</td>
<td>30</td>
<td>69</td>
<td>23</td>
<td>122</td>
</tr>
<tr>
<td>Total</td>
<td>593</td>
<td>969</td>
<td>246</td>
<td>1808</td>
</tr>
</tbody>
</table>

An adult is randomly selected from the study. Use this information to find the probabilities requested. Enter all of your results as percents rounded to two decimal places.

i. What is the probability that a person drinks 1 – 60 drinks per month given that they are single?

ii. What is the probability that a person is single given that they drink 1 – 60 drinks per month?

iii. What is the probability that a person is divorced given that they have 0 drinks per month?

iv. What is the probability that a person has more than 60 drinks per month given that they are married?
SECTION 11.2: INDEPENDENT EVENTS

In this section, we will investigate independent events. Two events $E_1$ and $E_2$ are said to be independent if the probability of $E_1$ does not affect the probability of $E_2$ or vice versa. Two events that are not independent events are called dependent events.

For example, if you flip a coin and roll a die, the outcome of the coin and the outcome of the die have no effect on one another so the outcomes of the coin and the die are independent events. We can justify this mathematically using conditional probabilities. We can choose particular events (outcomes) of the die and coin and show that the probability of $E_1$ given $E_2$, is the same as the probability of $E_1$ verifying that $E_2$ has no effect on $E_1$.

Fact: If the probability of $E_1$ equals the probability of $E_1$ given $E_2$, then $E_1$ and $E_2$ are independent. If the probability of $E_1$ does not equal the probability of $E_1$ given $E_2$, then $E_1$ and $E_2$ are dependent.

Problem 4: Worked Example – Independent Events and Conditional Probability

A fair coin is flipped and a die is rolled. What is the probability the coin will land on heads? What is the probability the coin will land on heads given that the die lands on 3? Are the events of the coin landing on heads and the die landing on 3 independent?

Solution: First construct the sample space for the experiment of flipping a coin and rolling a die.

<table>
<thead>
<tr>
<th>Die $\rightarrow$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coin $\downarrow$</td>
<td>H, 1</td>
<td>H, 2</td>
<td>H, 3</td>
<td>H, 4</td>
<td>H, 5</td>
<td>H, 6</td>
</tr>
<tr>
<td>Heads (H)</td>
<td>T, 1</td>
<td>T, 2</td>
<td>T, 3</td>
<td>T, 4</td>
<td>T, 5</td>
<td>T, 6</td>
</tr>
<tr>
<td>Tails (T)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a) What is the probability the coin will land on heads?

Observe the sample space has 12 equally likely outcomes. The event space for the coin landing on heads is boxed in blue below and contains 6 of the equally likely outcomes. So the probability of the coin landing on heads is 6 out of 12 or $\frac{6}{12} = \frac{1}{2}$.

Answer: $\frac{1}{2}$
b) What is the probability the coin will land on heads given that the die lands on 3?

**Solution:** The outcomes where the die lands on 3 gives the reduced sample space, and are boxed in green from the original sample space in the diagram below.

So the reduced sample space is \{ H3, T3 \} and contains two outcomes. One of these outcomes, namely, H3 has an outcome with heads. So the probability of heads given the die lands on 3 is 1 out of 2 or \( \frac{1}{2} \).

Answer: \( \frac{1}{2} \)

c) Are the events of the coin landing on heads and the die landing on 3 independent?

**Solution:** Since the probability the coin lands on heads and the probability the coin lands on heads given the die is a 3 are equal (both are \( \frac{1}{2} \)), the two events are independent.

Answer: Yes the two events are independent.

**Problem 5:** Worked Example – With or Without Replacement and Independence

One card is drawn from a deck of 52 cards and the card is an Ace.

a) What is the probability that if you draw a second card the card is a 3 given that the Ace is returned to the deck?

**Solution:** Since you are given that the Ace is returned to the deck, there are 52 cards. Four out of the 52 cards are 3's so the probability of drawing a 3 is \( \frac{4}{52} = \frac{1}{13} \).

Answer: \( \frac{1}{13} \)

b) What is the probability that if you draw a second card the card is a 3 given that the Ace is *not* returned to the deck?

**Solution:** Since the Ace is not returned to the deck, there are only 51 cards. Four out of the 51 cards are 3's so the probability of drawing a 3 is \( \frac{4}{51} \).

Answer: \( \frac{4}{51} \)
c) Is drawing a second card that is a 3 independent of the first draw of the Ace when the Ace is returned to the deck?

**Solution:** The probability of drawing a club from the deck if there wasn't a first card at all is \( \frac{1}{13} \). The probability of drawing a second card that is a club when an Ace is drawn, but returned to the deck is also \( \frac{1}{13} \). Since these probabilities are the same, the second card is independent of the first when the card is returned to the deck.

**Answer:** Independent

d) Is drawing a second card that is a 3 independent of the first draw of the Ace when the Ace is not returned to the deck?

The probability of drawing a club from the deck if there wasn't a first card at all is \( \frac{1}{13} \). The probability of drawing a second card that is a club when an Ace is drawn, and not returned to the deck is \( \frac{4}{51} \). Since these probabilities are the different, the second card is dependent on the first card when the card is not returned to the deck.

**Answer:** Dependent

**Notes:** In the first example, part a, the card was returned to the deck. This is called drawing the second card with replacement. In the second example, part b, the card was not returned to the deck. This is called drawing the second card without replacement. This idea will occur frequently in our discussion of probability since we often draw multiple cards from a deck without replacement.

Also observe that when we draw cards without replacement, a single card outcome of the 52 cards cannot be repeated. For example, you could not draw the Ace of Spades twice if it is not returned to the deck. We will also use the language without repetition corresponding to without replacement or repetition allowed corresponding to with replacement.

**Summary:**

1) Without repetition or without replacement events will be dependent.

2) With repetition or with replacement events will be independent.
SECTION 11.3: PROBABILITY OF THE INTERSECTION OF EVENTS

Recall that the intersection of two sets or events, \( E_1 \cap E_2 \), means that \( E_1 \) and \( E_2 \) both occur. We have already computed some probabilities of two events occurring by examining the sample space. In this section we will develop formulas for finding these probabilities directly from the probabilities of the individual events, first for independent events and then dependent events.

Part 1: Independent Events and the Probability of their Intersection

The sample space for rolling a single die and flipping a coin is below. Let’s find the probability that the coin flip is heads, and the die roll is 5 using the sample space.

We can find the intersection of the events by determining the outcomes for each event, and then determining the intersection of the two events. The event that the coin flip is heads is highlighted in blue in the diagram below and to the right. This event space contains 6 outcomes out of 12, so the probability the coin toss is heads is \( \frac{6}{12} = \frac{1}{2} \). The event that the die is a 5 is highlighted in yellow in the diagram below the coin flip diagram. This event space contains 2 outcomes out of 12, so the probability the die is a 5 is \( \frac{2}{12} = \frac{1}{6} \). The intersection of the events, that the coin toss is a head and the die is a 5, contains one out of 12 outcomes, namely, H5. So the probability that both occur is 1 out of 12 or \( \frac{1}{12} \).

Recall from Problem 4, that the coin toss and die roll are independent events. Now that we know this concept, we can find the probability of the intersection in a different way.

The probability that the coin toss is heads and the die is a 5 equals:

\[
\text{probability the coin toss is heads } \times \text{ probability the die is a 5 } = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}
\]

Notice this corresponds to our previous result. Now we will summarize this fact in general.
Product Rule for the Intersection of Independent Events:

If two events $E_1$ and $E_2$ are independent, then

The probability of $E_1$ and $E_2 = \text{the probability of } E_1 \times \text{the probability of } E_2$

Problem 6: Media Example – Probability of the Intersection of Independent Events

Determine the following probabilities using the product rule for the intersection of independent events.

a) Two coins are flipped. Find the probability that both coins land on heads.

b) Two cards are drawn from a deck of 52 cards. The first card is replaced before drawing the second card. Find the probability that the first card is a heart and the second card is a Jack.

c) A single digit between 0 and 9 is randomly chosen, and a single letter from A to Z is randomly chosen. Find the probability that the number is 7 and the letter is a vowel.

Problem 7: You Try – Probability of the Intersection of Independent Events

Determine the following probabilities using the product rule for the intersection of independent events.

a) Two die are rolled. Find the probability that both die land on even numbers.

b) Two cards are drawn from a deck of 52 cards. The first card is replaced before drawing the second card. Find the probability that the first card is a 3 and the second card is a club.
Part 2: Dependent Events and the Probability of their Intersection

Two develop the intersection formula for two dependent events, let’s begin with a card problem. For simplicities sake, we will consider drawing 2 cards from a deck that only contains the 13 diamonds of a standard deck of cards. The 13 cards are shown below.

![Deck of 13 cards for Experiment](image)

The sample space for drawing two cards without replacement is shown in the table below. Observe that the diagonal from the top left to the bottom right is filled with X’s and shaded gray. This is because we are drawing the two cards *without replacement* so it is impossible to draw the same card both 1st and 2nd. There are 156 equally likely possible outcomes for this experiment. This table is large and tedious to create, but once we have established our rule for the intersection of independent events, we will not need to do this again.

<table>
<thead>
<tr>
<th>1st card →</th>
<th>K</th>
<th>Q</th>
<th>J</th>
<th>10</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd card ↓</td>
<td>K</td>
<td>X</td>
<td>Q,K</td>
<td>J,K</td>
<td>10,K</td>
<td>9,K</td>
<td>8,K</td>
<td>7,K</td>
<td>6,K</td>
<td>5,K</td>
<td>4,K</td>
<td>3,K</td>
<td>2,K</td>
</tr>
<tr>
<td></td>
<td>Q</td>
<td>K,Q</td>
<td>X</td>
<td>J,Q</td>
<td>10,Q</td>
<td>9,Q</td>
<td>8,Q</td>
<td>7,Q</td>
<td>6,Q</td>
<td>5,Q</td>
<td>4,Q</td>
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<td>X</td>
<td>9,2</td>
<td>8,2</td>
<td>7,2</td>
<td>6,2</td>
<td>5,2</td>
<td>4,2</td>
<td>3,2</td>
</tr>
</tbody>
</table>

Sample Space for Drawing 2 of 13 cards Without Replacement

Now let’s find a probability for this experiment.
Problem 8: Worked Example – Probability of the Intersection of Dependent Events

Two cards are drawn from a deck of 13 diamond cards without replacement. Find the probability that the first card is a face card and the second card is a number card. (Note: Aces are not considered face or number cards).

Solution 1: The sample space below shows that the first 3 columns and 9 rows in the middle have outcomes where the first card is a face card and the second card is a number card. Observe that there are 27 out of 156 possible outcomes meet these requirements.

So the probability is \( \frac{27}{156} = \frac{9}{52} \).

Solution 2: Now let’s consider finding the probabilities for the two events separately.

First we will find the probability the first card is a face card. Observe that there are 36 possible outcomes in the first three columns that have a first card that is a face card. So the probability the first card is a face card is \( \frac{36}{156} = \frac{3}{13} \).
Now we will find the probability that the second card is a number card given that the first card is a face card.

The table to the right shows the reduced sample space for the second card given that the first card is a face card. Observe that there are 36 possible outcomes, and 27 of these outcomes meet the requirement that the second card is a number card. So the probability that the second card is a number card given that the first card is a face card is \( \frac{27}{36} = \frac{3}{4} \).

Now let's observe the product of the two probabilities before we simplified the fractions.

Probability 1\(^{\text{st}}\) card is a face card: \( \frac{36}{156} \)

Probability 2\(^{\text{nd}}\) card is a number card given the 1\(^{\text{st}}\) is a face card: \( \frac{27}{36} \)

Product: \( \frac{36}{156} \times \frac{27}{36} = \frac{36 \times 27}{156 \times 36} = \frac{36 \times 27}{156 \times 36} = \frac{27}{156} = \frac{9}{13} \)

Observe that the numerator of the first probability equals the denominator of the second probability. This occurs because in the second probability, we reduce the sample space (denominator) to the outcomes from the first probability which is the first probability's numerator.

Since this gives an identical factor in the numerator and denominator, these equal factors of 36 cancel to 1 resulting in the fraction \( \frac{27}{156} \) which reduces to \( \frac{9}{13} \).

This fraction should look familiar. This was our unsimplified and simplified fractions from Solution 1 for this problem. This is not a coincidence. This result generalizes to all dependent probabilities as follows.

**Product Rule for the Intersection of Dependent Events:**

If two events \( E_1 \) and \( E_2 \) are **dependent**, then

The probability of \( E_1 \) and \( E_2 \) = the probability of \( E_1 \) \times the probability of \( E_2 \) **given** \( E_1 \)

(Note: Since for independent events, the probability of \( E_2 \) **given** \( E_1 \) equals the probability of \( E_2 \), you can generally apply this formula to the intersection of any two events.)
Problem 9: Media Example – Probability of the Intersection of Dependent Events

Determine the following probabilities using the product rule for the intersection of dependent events.

a) Two names are drawn from a hat of 15 names without replacement to determine who will be responsible for driving to a sporting event. If there are 8 women and 7 men, what is the probability that the first person chosen is a man and the second person chosen is a woman?

b) Two cards are drawn from a deck of 52 cards without replacing the first card before choosing the second card. Find the probability that the first card is a club and the second card is a red Jack.

c) Two letters are chosen from A to Z for a password. The letters must be different from one another. Find the probability that the first letter is a vowel and the second letter is a Z.

Problem 10: You Try – Probability of the Intersection of Dependent Events

Determine the following probabilities using the product rule for the intersection of dependent events.

a) Two digits are chosen from 0 to 9 for a password. The numbers must be different from one another. Find the probability that the first number is odd and the second number is 6.

b) Two cards are drawn from a deck of 52 cards without replacing the first card before choosing the second card. Find the probability that both cards are clubs.
SECTION 11.4: THE FUNDAMENTAL COUNTING PRINCIPLE AND PROBABILITY

Thus far, we have found probabilities in two main ways. First we listed all the outcomes in the sample space, determining the number of outcomes in the event space, and then wrote the ratio of these quantities. In the last section, we used the product rules for the intersection of independent or dependent events to find the probability of an experiment that had two events.

In the remaining sections of this unit, we will find probabilities of experiments that have at least two events or stages by using more complex counting techniques. This will eliminate the need to enumerate or list the entire sample space (which may become quite large). And we can use extensions of the product rules for the intersection of independent or dependent events to ease our computations further.

The first counting method we will discuss is called the fundamental counting principle. In its simplest form, it states that if you have two independent choices to make, and the first choice has $m$ options and the second choice has $n$ options, there are $m \times n$ total ways to make these two choices.

**Problem 11: Worked Example – Fundamental Counting Principle**

Suppose you are having lunch at a pizza parlor and they have a lunch special where you can purchase a slice and a soda. The options for the slice and soda are given below. How many different slice and soda pairings are possible?

**Slice Options:** Cheese, Pepperoni, White, Veggie, Sicilian, Meat, Hawaiian

**Soda Options:** Cola, Diet Cola, Lemon-lime, Seltzer

**Solution:** We can make a list of the options in a table as shown below.

<table>
<thead>
<tr>
<th></th>
<th>Cheese(Ch)</th>
<th>Pepperoni(P)</th>
<th>White(W)</th>
<th>Veggie(V)</th>
<th>Sicilian(S)</th>
<th>Meat(M)</th>
<th>Hawaiian(H)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cola</td>
<td>Ch/C</td>
<td>P/C</td>
<td>W/C</td>
<td>V/C</td>
<td>S/C</td>
<td>M/C</td>
<td>H/C</td>
</tr>
<tr>
<td>Diet</td>
<td>Ch/D</td>
<td>P/D</td>
<td>W/D</td>
<td>V/D</td>
<td>S/D</td>
<td>M/D</td>
<td>H/D</td>
</tr>
<tr>
<td>Lemon</td>
<td>Ch/L</td>
<td>P/L</td>
<td>W/L</td>
<td>V/L</td>
<td>S/L</td>
<td>M/L</td>
<td>H/L</td>
</tr>
<tr>
<td>Seltzer</td>
<td>Ch/S</td>
<td>P/S</td>
<td>W/S</td>
<td>V/S</td>
<td>S/S</td>
<td>M/S</td>
<td>H/S</td>
</tr>
</tbody>
</table>

If we count the pairings, we can see there are a total of 28. Moreover, based on the fundamental counting principle, there are 7 choice for the pizza option, and 4 choices for the soda option. Therefore, there are $7 \times 4 = 28$ total options. This corresponds to the 4 rows and 7 columns of the table creating $7 \times 4 = 28$ cells each with a different pairing.

**Answer:** There are 28 different pairings.
In the last example, we had two choices to make. The fundamental counting principle extends to any number of independent choices. To make sense of this, suppose you also had 2 salad options in the last problem. We could multiply the 28 previous outcomes by 2 by thinking of the 28 pairings as your first choice and the salad as your second choice. In general, we can do this as many times as needed.

Extension of Fundamental Counting Principle: For any whole number amount of independent choices you need to make, you can multiply the number of options for each choice to find the total number of possibilities for all the choices.

Identifying Independence

For now, we will require the choices be independent as prescribed by the counting principle. You can determine that the choices be independent in three ways:

1) Making choices from different groups. Example: Number group and letter group.

2) Making choices from the same group, but with replacement. Example: Draw a card, but put it back in the deck before drawing another card (like drawing from two different identical decks).

3) Making choices from the same group, but allow repetition. Example: Choosing two numbers but the numbers can repeat (making choices from two different sets, but of identical numbers).

Identifying Sequencing or Order

In addition, we will require that there is a sequence or order to our choices. What this means is that each choice corresponds to one specific outcome. The term “order” can be misleading because we are not necessarily concerned with how the choices occur in time although we will frequently label them with numbers or call them a first choice, second choice, etc... For example, the pizza choice corresponds to the pizza options and the soda choice to the soda options. It doesn’t matter which we chose first, but the correspondence matters.

In contrast, for a phone number, it is easier to see that the order matters. You cannot interchange the digits. If a phone number is 941 – 2738 you could not enter the numbers in any order to call someone. However, if you were choosing them randomly, you could choose the 4th digit first and it wouldn't change the total possible number of outcomes. The important idea is that the choice corresponds to its own unique spot in the phone number, the fourth spot.
The diagram below displays this idea. Note that you could make your choices in any order as long as they are labeled as indicated. We will make similar diagrams for different types of counting and this will be a useful aid for comparison.

Problem 12: Media Example – Fundamental Counting Principle and Probability

Use the fundamental counting principle to calculate the following probabilities. Write your answers as percents rounded to two decimal places.

a) A multiple choice quiz has 7 questions and 4 possible answers per question.

i. How many different ways are there to answer the 7 questions? (Assume all questions are answered)

ii. A student who didn’t study randomly guessed on each question. What is the probability the student got all of the answers correct?
iii. A student who didn’t study randomly guessed on each question. What is the probability the student got all of the answers wrong?

iv. A student randomly guessed on each question. What is the probability the student answered questions 2 and 3 correct, but the rest wrong?

b) Tom is creating a 3 character passcode. The first character must be a digit from 0 to 9, the second character must be a letter from A to Z, and the third character must be from the set of special characters \{ !, @, $, *, & \}

i. How many 3 character passcodes are possible?

ii. Tom chooses all of the characters at random. What is the probability the passcode starts with a 1?

iii. Tom chooses all of the characters at random. What is the probability the letter is a t, o, or m?

iv. Tom chooses all of the characters at random, what is the probability that the passcode is exactly “7k$”?

Use the fundamental counting principle to calculate the following probabilities. Write your answers as fractions and percents rounded to four decimal places.

Arizona license plates have 3 letters followed by 4 single digit numbers. Repetition of letters and numbers is allowed. (Although some combinations are not permitted we will ignore this fact for this problem).

a) How many possible AZ license plates are there?

b) If a license plate sequence is randomly chosen, what is the probability the letters spell JOY?

c) Zach's birthday is November 24\textsuperscript{th} or 11/24. What is the probability a randomly chosen license plate has the sequence of numbers 1124?

d) What is the probability a randomly chosen license plate is JOY – 1124?
SECTION 11.5: PERMUTATIONS AND PROBABILITY

In this section, we will discuss probabilities involving permutations. A **permutation** is an arrangement or ordering of a set of objects. Let’s begin by looking at a simple example.

**Problem 14: Worked Example – Permutations**

There are three people that are taking turns using the swing on a playground: Amy, Beth, and Carl. List all the different orders in which they can use the swing where everyone gets exactly one turn. How many different ways are there?

**Solution:** First we will list all of the ways. We will abbreviate the names by using the letters A, B, and C. It is important to use a systematic method so you don’t miss any possible arrangements. We will start by writing all the arrangements with A first, then B, and then C.

First Letter A:  ABC, ACB
First Letter B:  BAC, BCA
First Letter C:  CAB, CBA

**Answer:** List of Permutations:  ABC, ACB, BAC, BCA, CAB, CBA
Number of Permutations: Six

**Counting Permutations Directly:** Making a list of permutations can be tedious when there are more than 3 objects to permute. To count the number of possible permutations without making a list, we can use a variation on the Fundamental Counting Principle.

One difference between the fundamental counting principle and permutations, is that with permutations our choices are dependent. We begin with one group of objects, and with every choice we make, we have one less option to choose from for our next choice.

One similarity between the fundamental counting principle and permutations, is that we will require that there is a sequence or order to our choices. What this means is that each choice corresponds to one specific outcome.

The diagram below connects these ideas. The first image shows that we are making choices from one group of options (dependent). However, we can also think of making our choices from a series of distinct groups where each group has one less choice than the previous group. We do not know which option was specifically removed for the previous choice, but we know how many remain. This is similar to how we reduced the sample space in previous problems when events were dependent.
Diagram: Connecting Permutations and the Fundamental Counting Principle

Problem 15: Worked Example – Permutations and Counting

Use multiplication to find the number of permutations.

a) There are three people that are taking turns using the swing on a playground: Amy, Beth, and Carl. How many different ways are there in which they can use the swing where everyone gets exactly one turn.

Solution: This problem is identical to problem 14, but we will solve it using multiplication. There are 3 options for who goes first. There are 2 remaining options for who goes second after we choose the first person. And there is only one option for who goes third (only one person remains). So the number of ways is

\[ 3 \times 2 \times 1 = 6 \]

Answer: 6

b) Don has 12 books on his summer reading list. He wants to choose 4 books to read in order from first to last. In how many ways can he do this?

Solution: Observe that there are a total of 12 books, 4 are to be chosen, and the order in which he reads them matters. Sometimes it can be helpful to make a diagram to set up your multiplication as shown below.

\[
\begin{array}{c}
\text{12} \\
\text{1st book}
\end{array} \times \begin{array}{c}
\text{11} \\
\text{2nd book}
\end{array} \times \begin{array}{c}
\text{10} \\
\text{3rd book}
\end{array} \times \begin{array}{c}
\text{9} \\
\text{4th book}
\end{array}
\]

\[ 12 \times 11 \times 10 \times 9 = 11,880 \]

Answer: 11,880 ways
Formula for Permutations: You are given a group of \( n \) objects and want to select \( r \) of the objects where the order of the objects matters. The number of such permutations is the product of \( r \) factors, beginning at \( n \), and decreasing by 1 for each successive factor.

Notation: \( _nP_r = n \times (n-1) \times (n-2) \times (n-3) \cdots (n-r+1) \)

Note: In Example 14, \( n = r \) (both 3). This is called a factorial denoted, \( 3! \), read 3 factorial.

Problem 16: Worked Example – Permutation Formula and Calculator

a) Find the number of permutations of 10 objects taken 4 at a time.

Solution: Note that \( n = 10 \) and \( r = 4 \). Starting at 10, count down 4 numbers decreasing by 1 each time and multiply the 4 factors.

\[ _{10}P_4 = 10 \times 9 \times 8 \times 7 = 5040 \]

Answer: 5040

Calculator:

TI 30XS Multiview Screen: \( 10 \text{ nPr } 4 \quad 5040 \)

Key Strokes

\[ \begin{array}{cccccc}
1 & 0 & \text{prb} & \text{enter} & 4 & \text{enter} \\
\end{array} \]

b) Compute \( _8P_5 \)

Solution: Note that \( n = 8 \) and \( r = 5 \). Starting at 8, count down 5 numbers decreasing by 1 each time and multiply the 5 factors.

\[ 8 \times 7 \times 6 \times 5 \times 4 = 6720 \]

Answer: 6720

Calculator:

TI 30XS Multiview Screen: \( 8 \text{ nPr } 5 \quad 6720 \)

Key Strokes

\[ \begin{array}{cccc}
8 & \text{prb} & \text{enter} & 5 & \text{enter} \\
\end{array} \]
**Problem 17: Media Example – Permutations and Probability**

Use permutations to calculate the following probabilities. Write your answers as fractions and percents rounded to four decimal places.

**a) The student council has 14 students. They need to choose a President, Vice President, Secretary and Treasurer for student council.**

i. In how many ways can the 4 different positions be appointed from the 14 students?

ii. There are 4 seniors in the high school. If the students are randomly assigned to the positions, what is the probability the President is a senior?

**b) James is creating a 4 digit passcode using the digits 0 through 9.**

i. How many 4 digit passcodes are possible if the numbers can be repeated?

ii. James chooses all of the characters at random. What is the probability that no digits are repeated?

**c) Three cards are chosen without replacement from a deck of 52 cards and placed in order from first to third.**

i. How many outcomes are possible?

ii. What is the probability that the first card is a club?

iii. What is the probability that the draw is Ace, King, Queen?
Problem 18: You Try – Permutations and Probability

Arizona license plates have 3 letters followed by 4 single digit numbers. Repetition of letters and numbers is allowed. (Although some combinations are not permitted we will ignore this fact for this problem). Write your answers as fractions and decimals rounded to four decimal places.

i. How many possible AZ license plates are there?

ii. If a license plate sequence is randomly chosen, what is the probability that no letter is repeated?

iii. If a license plate sequence is randomly chosen, what is the probability that no number is repeated?

iv. If a license plate sequence is randomly chosen, what is the probability that no number is repeated and no letter is repeated?
SECTION 11.6: COMBINATIONS AND PROBABILITY

In this section, we will discuss probabilities involving combinations. A combination is an unordered selection of a set of objects. Like permutations, combinations are chosen from a single group of objects. In addition, combinations do not allow for repetition since the items are chosen without replacement, and therefore, each choice is dependent. However, recall that permutations distinguished between the sequence or order of its objects, but combinations do not. This lack of ordering in combinations is the essential difference between permutations and combinations.

For instance, with permutations, 123 and 321 are considered different possibilities or outcomes just as the numbers one hundred twenty-three and three hundred twenty-one are different. With combinations, our result is a subset of the original set of objects and our only concern is what objects were chosen or not chosen. So with combinations, 123 and 321 are not different because they both contain the same three numbers. This may remind you of our study of sets. The sets \{1, 2, 3\} and \{3, 2, 1\} are equal since they contain the same elements. Their ordering within the set brackets is irrelevant.

The diagram below provides a way to visualize combinations in comparison to permutations. The top of each diagram begins with how we developed the idea of permutations. The objects from a permutation are chosen from one group and some amount of choices are made from the options with one less option for every subsequent choice. For combinations, we can think of starting this way, but at the end, we will take all of the choices and think of them as one unordered group of objects. This is represented by the bin on the bottom labeled “all choices”. We no longer consider their ordering relevant.
Problem 19: Worked Example – Developing Combinations from Permutations

There are 8 kids on the playground that want a turn using the swing: Amy, Beth, Carl, Dan, Ed, Fin, Greg, and Hope. However, there is only enough time for 3 kids to have a turn before the playground closes.

a) How many ways are there to choose 3 of the 8 children to get the 1\textsuperscript{st}, 2\textsuperscript{nd}, and 3\textsuperscript{rd} turn on the swing?

Solution: Observe that this problem asks how to choose an ordering of three of the eight children. So this is a permutation with the group being the 8 children, and 3 children chosen to be first, second and third.

\begin{align*}
1\textsuperscript{st} \text{ child on swing: } & \text{8 options, } 2\textsuperscript{nd} \text{ child on swing: } 7 \text{ options, } 3\textsuperscript{rd} \text{ child on swing: } 6 \text{ options} \\
8P_3 &= 8 \times 7 \times 6 = 336
\end{align*}

Answer: There are 336 ways to choose 3 of the 8 children to go 1\textsuperscript{st}, 2\textsuperscript{nd}, and 3\textsuperscript{rd}.

b) Now suppose we do not care what order the children get to use the swing. We just want to know all the ways that we can choose 3 of the 8 children to get a turn on the swing. How many ways can we choose 3 of the 8 children?

Solution: This is the number of \textit{combinations} of 3 children being chosen from a group of 8 since the ordering of the children on the swing does not matter. Let’s use this opportunity to adapt the permutation formula to find the number of combinations.

Recall from Problem 14, we counted the number of ways three children could each take a turn using the swing in order. We abbreviated the names Amy, Beth, and Carl, by using the letters A, B, and C, and found the following six permutations.

List of Permutations: ABC, ACB, BAC, BCA, CAB, CBA

We noted that this could be found directly by computing $3P_3 = 3 \times 2 \times 1 = 6$ and that an alternative notation when the number of objects equals the number of choices in a permutation is called a factorial, and denoted 3!.

For combinations, all six of these permutations would be considered one possible combination, namely the set \{A, B, C\}. Observe that any possible combination of three elements has $3!$ corresponding permutations. Therefore, to find the number of combinations, we can divide the number of permutations by $3!$.

\begin{align*}
\text{Answer: Number of Combinations} &= \frac{8P_3}{3!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = \frac{336}{6} = 56
\end{align*}
**Formula for Combinations:** You are given a group of \( n \) objects and want to select \( r \) of the objects where the order of the objects does not matter. The number of such combinations is the product of \( r \) factors, beginning at \( n \), and decreasing by 1 for each successive factor, divided by \( r \) factorial.

\[
\frac{n!}{(n-r)!r!} = \binom{n}{r}
\]

**Problem 20:** Worked Example – Combination Formula and Calculator

a) Find the number of combinations of 10 objects taken 4 at a time.

**Solution:** Note that \( n = 10 \) and \( r = 4 \). Starting at 10, count down 4 numbers decreasing by 1 each time and multiply the 4 factors, then divide by the product of \( 4! = 4 \times 3 \times 2 \times 1 \)

\[
\binom{10}{4} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = \frac{5040}{24} = 210
\]

**Answer:** 210

**Calculator:**

TI 30XS Multiview Screen: \( 10 \ \text{nCr} \ 4 \ = \ 210 \)

b) Compute \( \binom{12}{6} \)

**Solution:** Note that \( n = 12 \) and \( r = 6 \). Starting at 12, count down 6 numbers decreasing by 1 each time and multiply the 6 factors, then divide by the product of \( 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 \)

\[
\binom{12}{6} = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{665,280}{720} = 924
\]

**Answer:** 924

**Calculator:**

TI 30XS Multiview Screen: \( 12 \ \text{nCr} \ 6 \ = \ 924 \)
Problem 21: Media Example – Combinations and Probability

Use combinations to calculate the following probabilities. Write your answers as percents rounded to three decimal places.

a) A committee has 13 members. Four members must be chosen for a subcommittee.

i. In how many ways can the 4 members be chosen for a subcommittee from the 13 members?

ii. There are 3 members in the human resources department. If the subcommittee must have exactly one member from human resources, what is the probability that a randomly chosen subcommittee will meet this requirement?

iii. There are 4 members in the IT department. If the subcommittee must have at least 1 person from IT, what is the probability that a randomly chosen subcommittee will meet this requirement?
b) A poker hand has 5 cards and the order of the cards does not matter.

i. How many 5 card poker hands are there?

ii. A flush is a 5 card poker hand where all the cards have the same suit. What is the probability of drawing a flush that is all clubs?

iii. What is the probability of drawing a flush of any suit?

iv. A four of a kind is a 5 card poker hand where 4 of the cards have the same rank (A, 2, 3 ...) and one other card called the “kicker”. What is the probability of drawing a four of a kind with four aces and any other card as a kicker?

v. A full house is a 5 card poker hand where 3 of the cards are of one rank (A, 2, 3 ...) and the 2 other cards are of another rank. What is the probability of drawing a full house that is 3 tens and 2 jacks?
**Problem 22:** You Try – Combinations and Probability

Use combinations to calculate the following probabilities. Write your answers as percents rounded to three decimal places.

a) An organization has 15 employees. Three employees must be chosen for a committee.
   i. In how many ways can the 3 employees be chosen from the 15 employees?

   ii. The organization has 9 women and 6 men. What is the probability that a 3 person committee is all men?

   iii. The organization has 9 women and 6 men. What is the probability that a 3 person committee is all women?

b) A poker hand has 5 cards and the order of the cards does not matter.
   i. What is the probability that a randomly selected hand of 5 cards has exactly 2 Aces and 3 cards that aren't Aces?

   ii. What is the probability that a randomly selected hand of 5 cards has exactly 2 Aces, 2 Kings, and 1 card that is neither an Ace nor a King?
UNIT 12 – THE NORMAL DISTRIBUTION
In this Unit, we will learn about a special probability distribution called the normal distribution. We will learn about the normal distribution’s properties and use them to solve application problems.

Section 12.1: Compare histograms with the normal curve approximations
Section 12.2: Approximate the area under a normal curve and compare to a histogram
Section 12.3: Compare normal curves with different means and standard deviations
Section 12.4: Sketch a normal curve using its mean and standard deviation
Section 12.5: Use the empirical rule to approximate areas under the normal curve
Section 12.5: Use the symmetry and the empirical rule to approximate areas under the normal curve
Section 12.6: Convert between data values and z-scores
Section 12.7: Use z-scores and technology to find areas under the normal curve
Section 12.8: Calculate percentiles
SECTION 12.1: APPROXIMATING A HISTOGRAM WITH A NORMAL CURVE

In this section, we will see how the bell curve or normal distribution can approximate a histogram like the ones we have studied in previous units. We will look at some weather statistics for Scottsdale, AZ, and see how a normal distribution may approximate the data.

The US Climate Data website states that the average high temperature in Scottsdale, AZ, in July is 104°F. If you have ever spent a summer in Scottsdale, you know the high temperature can be a lot larger or smaller. Meaning, there is variability in the high temperature in the month of July, and 104°F represents the average. A Maricopa weather website states the record high in Phoenix was 121°F in 1995 and the record low was 63°F in 1912. These are extreme values that are very rare, but can occur.

Artie has collected his own data on the high temperatures in July for a few years. He wants to compare his data to the US weather data, and also wants to determine the standard deviation so he can measure the variability in his data.

He uses his data to create the relative frequency histogram shown below. The horizontal axis represents the daily high temperature rounded to the nearest degree Fahrenheit. The vertical axis represents the relative frequency written as a percentage. Each bar is labeled above the bar to the nearest hundredth of a percent.
The histogram is shaped approximately like a *bell curve* or has an *approximately normal distribution*. A normal distribution is a special probability distribution in statistics that occurs commonly and has great importance in the field of inferential statistics. The graph below shows a normal curve in red over the histogram data.

Observe that the red normal curve approximates the data well by sharing a similar bell shape. All normal distributions have this general bell shape, and can be uniquely determined by their mean and standard deviation. The normal curve that approximates this data are given below and has a mean of 103.76°F and a standard deviation of 1.73°F. The horizontal axis is labeled with the mean and 3 standard deviation in each direction of the mean. The right dashed lines correspond to the height of the curve at these values.
Let’s make a few general observations about the normal curve.

1) The curve is symmetric with one highest frequency or mode.

2) As we learned in Unit 8, symmetric distributions with one mode have the property that their three measures of center are equal. So the mean, median, and mode are all equal to the data value with the highest frequency in the center.

3) Since the median is in the center, half of the data are above the mean, and half the data are below the mean.

4) Most of the data are near the mean. The further you move away from the mean in either direction, the less data there are.

5) As we learned in Unit 8, the 3 standard deviation rule applies to the normal curve. All or almost all of the data lie within 3 standard deviations of the mean. We will fine tune this approximation later in this Unit.

6) The normal curve never actually touches the horizontal axis (zero height), although it gets closer to the horizontal axis as you move further away from the mean. This is because the probability of extreme values such as the records of 121°F in 1995 and 63°F in 1912 are not impossible. They are just very unlikely. So the curve is always above the horizontal axis.

7) Most of the data we will work with are approximately normally distributed and not exactly normally distributed. There is often variability in data sets, and the normal curve is a very good approximation, but may not be exactly like the normal curve.

**Problem 1: Worked Example – Properties of the Normal Curve**

Use the normal curve and our observations of normal curves to answer the questions.

a) Determine the mean, median, and mode of the normal curve.

**Answer:** All three measure of center are at the center of the curve, 103.76°F.

b) According to the 3 Standard Deviation Rule, all or almost all of the data lie between what two values?

**Answer:** Almost all or all of the data lie between 98.57°F and 108.95°F.
SECTION 12.2: APPROXIMATING AREA UNDER A NORMAL CURVE

Thus far, when given a histogram, we looked at the relative frequency, or height of the bar, to determine the percentage of data that corresponded to the data value labeled at the base of the bar. We also used the relative frequency to determine the probability of a certain data value when randomly making a selection from the data set. Most often, we use a bar with a base value of 1 as we did with the Daily High Temperature data.

Another way to think of this relative frequency or probability is that the area of the bar is the width of the bar times the height. For example, the bar centered at 102°F has a relative frequency of 14% so the area of the bar is the product $1 \times 14\% = 14\%$ which is the relative frequency of the data. Since the temperatures were rounded to the nearest degree, the bar for 102°F would correspond to all the values between 101.5°F – 102.5°F which were rounded up or down to 102°F.

When we determine relative frequencies or probabilities for the normal curve, we can use a similar idea by finding the area under the normal curve between two data values on the horizontal axis. The images below show the areas under the curve shaded for the data values 101.5°F – 102.5°F for both the histogram and the normal curve. The left graph shows both on the same graph where the purple shading is where the bar and normal curve overlap, the pink area is area under the normal curve, but not the histogram, and the blue area is area under the histogram, but not the normal curve. The diagram on the right show the two areas side by side.

We know the area of the blue histogram rectangle is 14%, but what is the area of the pink shaded portion under the normal curve for the same data values? If you look at the graph on the left hand side, it appears the blue area in the histogram that is not under the normal curve is pretty close to the pink area under the normal curve that is not in the blue histogram. So they appear to be close. The pink area in the diagram on the right is not a rectangle so we cannot
find its area using length times width. However, we can approximate the area shaded in pink by noticing it's **approximately** a trapezoid. It's not exactly a trapezoid since the top of the shape is a curve and not a straight line.

Recall the area of a trapezoid is the average (mean) of its two parallel side lengths times the distance between the parallel sides. The parallel sides have lengths of 9.82% and 17.69% and the distance between these sides is 1 (from 102.5 – 101.5).

Area Trapezoid = \( \frac{9.82\% + 17.69\%}{2} \times 1 = 13.755\% \approx 13.76\% \)

To find the **exact area** under the normal curve by hand, we would need to use methods from calculus since the top of the curve is “curvy” and not a straight line. Or in this course, we will use standard values and normal curve calculators on the computer in future sections. The actual area under the normal curve between 101.5 and 102.5, rounded to two decimal places is 13.75%. So our trapezoid approximation was very close. Furthermore, the area under the normal curve approximated the histogram percentage of 14% closely as well.

Let's make a few general observations about the area under the normal curve.

1) The area under the normal curve between two data values approximates the relative frequency or probability of the set of outcomes between the two data values for data that is approximately normally distributed.

2) The normal curve has continuous data points on the horizontal axis (the data are not only whole numbers, but every number in between). Each value on the horizontal axis has a corresponding height that creates a smooth curve instead of rectangular bars.

3) The vertical height of the normal curve **is not** the relative frequency or probability for its corresponding data value. An individual data value has no width on the horizontal axis, and therefore the distance to the vertical height is a line. Lines do not have area or we say the area is 0. Therefore, we can only find the relative frequency for an interval of values.

4) We can approximate areas using trapezoids and this is a good way to visualize the areas. In practice, we will use some standard areas introduced in the next section, or calculators and computer programs for more complex areas.
SECTION 12.3: COMPARING NORMAL CURVES

We have noted that normal distributions are bell shaped and can be determined uniquely by their mean and standard deviations. Let’s look at a few normal distributions next to each other for comparison.

Comparing Normal Curves: Group A
The four graphs below are all normal curves with means of 90. However, they have different standard deviations. Recall that the standard deviation describes how spread out the data are.

Since Graph A is the least spread out, it has the smallest standard deviation. Graph B has the 2nd smallest standard deviation. Graph C has the 2nd largest standard deviation in the group. And Graph D has the largest standard deviation.

Comparing Normal Curves: Group B
The two graphs below are both normal curves with the same standard deviation. The graph on the left has a mean of 6, and the graph on the right has a mean of 20. Their shapes look identical. Their only difference is their location on the number line. We could shift the graph on the right 14 units to the left, and it would line up exactly with the graph on the left.
Comparing Normal Curves: Group C

The two graphs below are actually mathematically identical to the two graphs above. They are both normal curves with the same standard deviation. The graph on the left has a mean of 6, and the graph on the right has a mean of 20 just like those in Group B.

The graphs below look narrower and taller compared to the graphs above. You might think at first that the graphs below have a smaller standard deviation than the graphs above. However, if you look at the horizontal axis, the numbers are identical. However, the scaling on the horizontal axis is different. The numbers are listed closer together compared to the graphs in Set B. This makes the curves appear different in comparison when they aren’t.

Note: Normal curves have the property that you can stretch them and/or shift them and they will still be normal curves. This will be an important result for our studies, and will help us make generalizations about normal curves. A lot of the curves you will see will look identical. To note their differences, identify their standard deviation and mean.
SECTION 12.4: SKETCHING NORMAL CURVES

In Section 12.1, we were given a sketch of the normal curve for the Daily High Temperature data. We will use this sketch as a means of determining the important attributes to list and mark when you are sketching a normal curve in general.

Problem 2: Worked Example – Sketching a Normal Curve

Sketch the normal curve with mean 103.76 and standard deviation 1.76

Solution:
To sketch a normal curve given a mean and standard deviation:
1) Draw the general shape of a bell curve with a single highest point, and the curve approaching the horizontal axis to the left and right.
2) Locate the highest point of the curve, and label the mean on the horizontal axis.
3) Draw three equally spaced tick marks to the left and right of the mean. The distances between the consecutive tick marks equals the standard deviation. The third tick marks to the right and left should be almost where the graph appears identical to the horizontal axis.
4) Determine the labels for the tick marks as follows: mean + # of sd's × sd
   • +1 sd is one standard deviation to the right of the mean or 103.76 + 1.73 = 105.49
   • +2 sd is two standard deviations to the right of the mean. You can find 103.76 + 2 × 1.73, or add one more standard deviation to the +1 sd answer: 105.49 + 1.73 = 107.22
   • +3 sd is three standard deviations to the right of the mean. You can find 103.76 + 3 × 1.73, or add one more standard deviation to the +2 sd answer: 107.22 + 1.73 = 108.95
   • −1 sd is one standard deviation to the left of the mean or 103.76 − 1.73 = 102.03
   • −2 sd is two standard deviations to the left of the mean. You can find 103.76 − 2 × 1.73, or subtract one more standard deviation to the −1 sd answer: 102.03 − 1.73 = 100.3
   • −3 sd is three standard deviations to the left of the mean. You can find 103.76 − 3 × 1.73, or subtract one more standard deviation to the −2 sd answer: 100.3 − 1.73 = 98.57

Answer:
Problem 3: You Try – Sketching a Normal Curve

a) Use the image below to label the normal curve with mean 57 and standard deviation 3. Show your work to the left.

![Normal Curve Image]

b) Sketch the normal curve with mean 22 and standard deviation 3.4. Make your own curve that is similar in shape to the one above.

c) Sketch the normal curve with mean 0 and standard deviation 2.45. Make your own curve that is similar in shape to the one above.
SECTION 12.5: THE EMPIRICAL RULE

In Unit 8, we learned about the 3-standard deviation rule. It stated that for any distribution, all or almost all of the data lie within 3 standard deviations of the mean. For a distribution that is exactly normal, we can determine a more precise measurement and also include measurements for the percent of data within one or two standard deviations of the mean.

**Empirical Rule:** For a normal distribution,

- Approximately 68% of the data lie within 1 standard deviation of the mean.
- Approximately 95% of the data lie within 2 standard deviations of the mean.
- Approximately 99.7% of the data lie within 3 standard deviations of the mean.

**Note:** We state that these are approximations because we could carry out the percentages to more decimal places. However, since most of the data we will see are approximately normally distributed instead of exactly normally distributed, the approximately term serves this purpose as well.

The graph on the left below shows a normal curve where the area under the curve is shaded blue within 1 standard deviation of the mean, pink between 1 and 2 standard deviations of the mean, and green within 2 and 3 standard deviations of the mean. The corresponding percentages according to the empirical rule are labeled for the intervals below the graph. Note the percentages are cumulative. Meaning, within 2 standard deviations of the mean includes both the blue and pink shaded regions. The rectangle on the right shows how these areas accumulate if we were to transfer the percent area to a more familiar shape.
Problem 4: Media Example – The Empirical Rule

The scores for a math exam are approximately normally distributed with a mean score of 78.8% and a standard deviation of 5.2%. Label the normal curve below and then apply the empirical rule to answer the questions.

According to the empirical rule,

a) Approximately 68% of the exam scores will lie between _____ and _____

b) Approximately 95% of the exam scores will lie between _____ and _____

c) Approximately 99.7% of the exam scores will lie between _____ and _____

Problem 5: You Try – The Empirical Rule

The heights of a group of 9-year olds boys are approximately normally distributed with a mean height of 52.2 inches and a standard deviation of 2.51 inches. Label the normal curve below and then apply the empirical rule to answer the questions.

According to the empirical rule,

a) Approximately 68% of the heights will lie between _____ and _____

b) Approximately 95% of the heights will lie between _____ and _____

c) Approximately 99.7% of the heights will lie between _____ and _____
**Problem 6: Media Example – Using Symmetry and the Empirical Rule**

The scores for a standardized test are approximately normally distributed with a mean score of 72 and a standard deviation of 8 as shown in the graph below. Use the symmetry of the normal curve and apply the empirical rule to answer the questions.

![Normal Distribution Graph]

a) What percent of students scored greater than 72?

b) What percent of students scored less than 72?

c) What percent of students scored between 64 and 72?

d) What percent of students scored between 72 and 80?

e) What percent of students scored between 56 and 72?

f) What percent of students scored between 72 and 96?

g) What percent of students scored are less than 80?

h) What percent of students scored greater than 96?
Problem 7: You Try – Using Symmetry and the Empirical Rule

The heights of a group of 3-year old girls are approximately normally distributed with a mean height of 38 inches and a standard deviation of 2 inches as shown in the graph below. Use the symmetry of the normal curve and apply the empirical rule to answer the questions.

![Normal Distribution Graph]

- a) What percent of the heights are greater than 38?
- b) What percent of the heights are less than 38?
- c) What percent of the heights are between 38 and 42?
- d) What percent of the heights are between 36 and 38?
- e) What percent of the heights are between 34 and 40?
- f) What percent of the heights are greater than 36?
- g) What percent of the heights are less than 32?
SECTION 12.6: THE STANDARD NORMAL DISTRIBUTION

In Section 12.2, we saw that normal curves can be shifted to the right or left and their general shape will not change. We also saw that narrowing or stretching the scale on the horizontal axis can make the distribution appear wider or narrower, but maintains the relative relationship of the data values and corresponding area. In this section we will use these ideas to define a special normal distribution called the **Standard Normal Distribution**. The standard normal distribution is like any other bell curve, but specifically, it has a mean of 0 and a standard deviation of 1 as shown below.

![Standard Normal Distribution](image)

Observe that the horizontal value of the standard normal distribution corresponds to the number and direction of standard deviations away from the mean. We call this value a **z-score**. In this section, we will find z-scores for any data value with a normal distribution with any mean or standard deviation. This will give us a common unit of measurement for determining the placement of a data value within a distribution and a method for determining areas using z-scores that are not integers.

**Problem 8: Worked Example – z-scores and the Standard Normal Distribution**

Complete the statements below.

a) A z-score of 2.3 is _______ standard deviations _______ the mean.

   **Solution:** Since a z-score represents the number of standard deviations from the mean, 2.3 is 2.3 standard deviations to the right of the mean of 0.

   **Answer:** A z-score of 2.3 is 2.3 standard deviations above the mean.

b) A z-score of −1.7 is _______ standard deviations _______ the mean.

   **Solution:** Since a z-score represents the number of standard deviations from the mean, −1.7 is 1.7 standard deviations to the left of the mean of 0.

   **Answer:** A z-score of −1.7 is 1.7 standard deviations below the mean.
Next we will develop a general formula for finding the z-score for any data value from any normal distribution.

Consider the normal distribution below with a mean of 9 and a standard deviation of 3.

To transform this data to match the standard normal distribution, we can first subtract the mean 9 from all of the data values. This is equivalent to shifting the entire graph 9 units to the left.

1st Step: Data values – 9

Observe that the new distribution below has a mean of 0 and the general shape hasn't changed. The standard deviation remains at 3.
To imagine transforming the data so that the standard deviation is 1 instead of 3, think of the individual standard deviations listed on the graph. Since 1 standard deviation above the mean is 3 above the mean, and we want a standard deviation of 1 for a standard normal curve, we can think of shifting 3 to where 1 lies on the graph. Similarly, since 6 is 2 standard deviations above the mean, and this is 2 on the standard normal distribution, we can shift 6 to where 2 lies on the graph. Following this process with all the standard deviation values in the first graph below, shrinks the graph horizontally so it is \( \frac{1}{3} \) of its original width, or equivalently, dividing the horizontal values by the standard deviation of 3.

**2nd Step:** Divide previous shifted data values by 3:
\[
\text{data values} \div 3
\]

The 2
\text{nd} graph below shows this transformation. The two curves are stacked so you can see how the 1
\text{st} graph is compressed horizontally and its new horizontal width is \( \frac{1}{3} \) of its previous width.

**Note:** The 2
\text{nd} graph is less wide relative to the previous graph. We do not need to draw the standard normal distribution this skinny.
We performed this process on the entire normal distribution with mean 9 and standard deviation 3. In general, we will perform this process on a few data values of interest in a given problem to find z-scores.

**Summary:** General process for finding a z-score of a data value.

Given a normal distribution, its mean, and its standard deviation, and a data value, the z-score can be found using the formula below.

\[ z = \frac{\text{data value} - \text{mean}}{\text{standard deviation}} \]

Conversely, given a normal distribution, its mean, and its standard deviation and a z-score, the data value can be found using the formula below.

\[ \text{data value} = \text{mean} + \text{z-score} \times \text{standard deviation} \]

**Problem 9: Media Example – Converting between z-score’s and Data Values**

Find the z-scores or data values for the given normal distributions. Round to two decimal places as needed.

a) A set of data are normally distributed with mean = 80 and standard deviation = 12. Find the z-scores for the following data values and interpret your results.

i. 83  
ii. 74  
iii. 96  
iv. 50

b) A set of data are normally distributed with mean = 20 and standard deviation = 4.5. Find the data values for the following z-scores.

i. \( z = 1.26 \)  
ii. \( z = -0.8 \)  
iii. \( z = 2.33 \)  
iv. \( z = -2.4 \)
**Problem 10:** You Try – Converting between z-score's and Data Values

Find the z-scores or data values for the given normal distributions. Round to two decimal places as needed.

a) A set of data are normally distributed with mean = 24 and standard deviation = 6. Find the z-scores for the following data values and interpret your results.

i. 28

ii. 32

iii. 20

iv. 8

b) A set of data are normally distributed with mean = 70 and standard deviation = 10. Find the data values for the following z-scores.

i. $z = 2.17$

ii. $z = -1.32$

iii. $z = 0.84$

iv. $z = -2.84$
SECTION 12.7:  Z-SCORES AND AREAS UNDER THE CURVE

We have used the Empirical rule and symmetry properties to find various areas under the curve. However, our ability to find these areas only applied to areas involving integer standard deviations. Now we will use z-scores and computer applications to find areas that have bounds with z-scores that include decimals.

Problem 11:  Media Example – z-score’s and Areas under the Curve

The National Sleep Foundation recommends that adults between 18 – 64 years of age sleep between 7 and 9 hours per night. A researcher collected data on the amount of sleep that students in college slept per night. The data were approximately normally distributed with a mean of 7.8 hours and a standard deviation of 1.3 hours as shown in the diagram below.

Find the z-scores for the boundary values of the problems below and make a rough sketch of the area under the Standard Normal Curve you are finding. Then use the Online Normal Distribution Calculator to find the requested percentages or probabilities.

a) What percent of students in the study slept between 8 hours and 9 hours per night?

b) What percent of students in the study slept less than 7 hours per night?
c) If a student is chosen at random, what is the probability that the student slept more than 10 hours per night?

d) If a student is chosen at random, what is the probability that the student slept less than 4 hours or more than 11 hours per night?

**Summary for using z-scores and the online calculator to find areas:**

1) Determine the boundary value(s) for your problem.
2) Find the z-score(s) for the boundary value(s) rounded to two decimal places.
3) Make a rough sketch labeled at your boundary values on the standard normal distribution, and shade the area you want to find.
4) Determine if you are finding an area
   a) Above a value (greater than)
   b) Below a value (less than)
   c) Between two values (greater than one boundary value and less than another)
   d) Outside two values (greater than one boundary value or less than another)

5) Using the [Online Normal Distribution Calculator](#), click the button for above, below, between, or outside.

6) Enter your z-scores in the box(es).
7) Make sure the Normal Curve image from the calculator and matches your sketch.
8) Click the recalculate button
9) The area is given next to the recalculate button.
Problem 12: You Try – z-score’s and Areas under the Curve

The National Sleep Foundation recommends that children ages 3 – 5 sleep between 11 and 13 hours per night. A researcher collected data on the amount of sleep that children in this age range slept per night. The data were approximately normally distributed with a mean of 12.3 hours and a standard deviation of 1.2 hours.

Find the z-scores for the boundary values of the problems below and make a rough sketch of the area under the Standard Normal Curve you are finding. Then use the Online Normal Distribution Calculator to find the requested percentages or probabilities.

a) What percent of children in the study slept between the recommended 11 hours and 13 hours per night?

b) If a child is chosen at random, what is the probability that the child slept less than 10 hours per night?
SECTION 12.8: PERCENTILES

A **percentile** is the data value where a given percent falls below the data value. Percentiles are similar to quartiles which we studied when we made boxplots. When we found quartiles, we partitioned the data into four groups or **quartiles**.

- The 1\(^{\text{st}}\) quartile or Q1 was the data value where 25% of all the data fell below the value.
- The 2\(^{\text{nd}}\) quartile or Q2 was the data value where 50% of the data fell below the value.
- The 3\(^{\text{rd}}\) quartile or Q3 was the data value where 75% of the data fell below the value.

We also labeled the minimum and maximum on a boxplot. The minimum was the data value where 0% of the data fell below the value. The maximum was where 100% of the data was equal to or less than the value.

For percentiles, the data are partitioned into 100 equal groups instead of 4. We find the data values that have **whole number percents** less than a given data value such as 20%, 99%, or 83%. Usually, some rounding is involved since many of the z-scores that correspond to the percentiles are rounded to 2 decimal places. Since there is no maximum data value for a normal curve, we do not find a maximum or a 100\(^{\text{th}}\) percentile. Similarly, there is no minimum value so we do not find a 0\(^{\text{th}}\) percentile.

The chart below shows z-scores (rounded to two decimal places) for percentiles that are multiples of 5, and 99% since this is a common percentile of interest. Since areas under the curve are preserved with respect to z-scores these percents apply to all normal curves.

<table>
<thead>
<tr>
<th>Percent below</th>
<th>z-score</th>
<th>Percent below</th>
<th>z-score</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>-1.64</td>
<td>55%</td>
<td>0.13</td>
</tr>
<tr>
<td>10%</td>
<td>-1.28</td>
<td>60%</td>
<td>0.25</td>
</tr>
<tr>
<td>15%</td>
<td>-1.04</td>
<td>65%</td>
<td>0.39</td>
</tr>
<tr>
<td>20%</td>
<td>-0.84</td>
<td>70%</td>
<td>0.52</td>
</tr>
<tr>
<td>25%</td>
<td>-0.67</td>
<td>75%</td>
<td>0.67</td>
</tr>
<tr>
<td>30%</td>
<td>-0.52</td>
<td>80%</td>
<td>0.84</td>
</tr>
<tr>
<td>35%</td>
<td>-0.39</td>
<td>85%</td>
<td>1.04</td>
</tr>
<tr>
<td>40%</td>
<td>-0.25</td>
<td>90%</td>
<td>1.28</td>
</tr>
<tr>
<td>45%</td>
<td>-0.13</td>
<td>95%</td>
<td>1.64</td>
</tr>
<tr>
<td>50%</td>
<td>0.00</td>
<td>99%</td>
<td>2.33</td>
</tr>
</tbody>
</table>
First let’s look at some areas to the left of a data value in context to get a feel for percentiles for the normal curve. An adult intelligence scale is approximately normally distributed with a mean of 100 and standard deviation of 10. The graph below shows the normal curve with this mean and standard deviation. In addition it shows **the percent of data below** each standard deviation value rounded to 1 decimal place. Since areas under the curve are preserved with respect to standard deviations these percents apply to all normal curves with respect to their standard deviation values.

Interpreting the area to the left of a data value: According to the graph,
- 99.9% of scores are below 130, or below 3 standard deviations above the mean.
- 97.7% of scores are below 120, or below 2 standard deviations above the mean.
- 84.1% of scores are below 110, or below 1 standard deviation above the mean.
- 50% of scores are below 100, or below the mean.
- 15.9% of scores are below 90, or below 1 standard deviations below the mean.
- 2.3% of scores are below 80, or below 2 standard deviations below the mean.
- 0.1% of scores are below 70, or below 3 standard deviations below the mean.

Only 50% is an actual percentile since the other percents aren’t whole numbers. This still gives us a good sense of where the percentiles lie on a normal curve. For instance, the 70th percentile is between a score of 100 and 110 and between the z-scores of 0 and 1 because 70% is between 50% (mean) and 84.1% (mean + 1 sd). In the next example, we will use the z-score chart for percentiles to attain more precise results for the data.
Problem 13: Media Example – Finding Percentiles with a Table

An adult intelligence scale is approximately normally distributed with a mean of 100 and standard deviation of 10. The graph below shows the normal curve with this mean and standard deviation. In addition it shows the percent of data below each standard deviation value rounded to 1 decimal place. The chart below show z-scores for some common percentiles.

For the questions below, use the graph to determine what number of standard deviations and data values the given percentiles lie between. Then use z-scores and the chart to compute the actual percentile. Also interpret the percentile in the context of the problem.

a) 75\textsuperscript{th} percentile

b) 20\textsuperscript{th} percentile

c) 80\textsuperscript{th} percentile

d) 99\textsuperscript{th} percentile
Unit 12 – The Normal Distribution

Problem 14: You Try – Finding Percentiles with a Table

According to Nielsen Media, adults spend an average of 11 hours per day interacting with Media content. Suppose the amount of time interacting with media content is approximately normally distributed with a mean of 11 hours and a standard deviation of 1.5 hours. The graph below shows the normal curve with this mean and standard deviation. In addition it shows the percent of data below each standard deviation value rounded to 1 decimal place. The chart below shows z-scores for some common percentiles.

<table>
<thead>
<tr>
<th>Percent below</th>
<th>z-score</th>
<th>Percent below</th>
<th>z-score</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>-1.64</td>
<td>55%</td>
<td>0.13</td>
</tr>
<tr>
<td>10%</td>
<td>-1.28</td>
<td>60%</td>
<td>0.25</td>
</tr>
<tr>
<td>15%</td>
<td>-1.04</td>
<td>65%</td>
<td>0.39</td>
</tr>
<tr>
<td>20%</td>
<td>-0.84</td>
<td>70%</td>
<td>0.52</td>
</tr>
<tr>
<td>25%</td>
<td>-0.67</td>
<td>75%</td>
<td>0.67</td>
</tr>
<tr>
<td>30%</td>
<td>-0.52</td>
<td>80%</td>
<td>0.84</td>
</tr>
<tr>
<td>35%</td>
<td>-0.39</td>
<td>85%</td>
<td>1.04</td>
</tr>
<tr>
<td>40%</td>
<td>-0.25</td>
<td>90%</td>
<td>1.28</td>
</tr>
<tr>
<td>45%</td>
<td>-0.13</td>
<td>95%</td>
<td>1.64</td>
</tr>
<tr>
<td>50%</td>
<td>0.00</td>
<td>99%</td>
<td>2.33</td>
</tr>
</tbody>
</table>

For the questions below, use the graph to determine what number of standard deviations and data values the given percentiles lie between. Then use z-scores and the chart to compute the actual percentile. Also interpret the percentile in the context of the problem.

a) 25th percentile

b) 70th percentile

c) 30th percentile

d) 95th percentile
APPENDIX A – CREATING GRAPHS IN EXCEL

In this Unit, you will learn to create graphs in charts in Microsoft Excel and label them properly.

Section A.1: Create a Pie Chart
Section A.2: Create a Line Graph
Section A.3: Create a Bar Chart
Section A.4: Create a Histogram
APPENDIX A – MEDIA LESSON

SECTION A.1: CREATE A CIRCLE GRAPH IN EXCEL

Problem 1: Worked and Media Example – Pie Charts

The graph below is the end result of creating a Pie Chart in Excel. The steps and screenshots for creating this graph are below. There is also a video in the MOER.


1) Data in Excel Spreadsheet

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Category</td>
</tr>
<tr>
<td>2</td>
<td>Food</td>
</tr>
<tr>
<td>3</td>
<td>Housing</td>
</tr>
<tr>
<td>4</td>
<td>Telephone</td>
</tr>
<tr>
<td>5</td>
<td>Apparel</td>
</tr>
<tr>
<td>6</td>
<td>Transport</td>
</tr>
<tr>
<td>7</td>
<td>Healthcare</td>
</tr>
<tr>
<td>8</td>
<td>Other</td>
</tr>
</tbody>
</table>
2) Highlight Data by clicking and dragging

3) Click on Insert. You can click directly on the type of chart. I usually click on *Recommended Charts* to get a quick picture of the options.

4) Click on the Pie chart image on the left (2nd down) then Click OK.
5) The image of the resulting graph is below. Notice that the data are highlighted in different colors. The title is the label of the second column, and the categories are color coded and labeled on the bottom.

6) You’ll frequently want to adapt the title
7) I wanted to format the title a bit. Use the buttons boxed in maroon.

8) I want the pie slices to be labeled with the percent values. Follow the steps in the image below.
9) Here’s the result. Clean up the data labels by clicking and dragging.

10) Moved data label results below.
11) They are rounding to the nearest percent. The data were listed as a tenth of a percent. Right click on a data callout and click Format data.

12) Adjust by unchecking Percentage and checking Value.
13) Here's the result.

14) Feel free to play with the color choices for the pie slices, different font colors, or pie chart designs (see image below).

Be careful not to go too crazy with the options! It's fun, but if the reader can't make sense of the graph, it's not worth it. Note that unless this chart is printed in color, it's hard to see the different categories with the legend. That's one reason I prefer Callouts. Colors are good for PowerPoint presentations.
15) I have been using paint to paste the images. To paste the final image, right click on the graph and click Copy. You can paste the result directly into Word for your assignment or PowerPoint for a presentation. You can even edit the chart in Word or PowerPoint if you want to adjust anything. I made all of the callouts bold and black in Word. It was set to a few shades lighter than black. I like this better, but it's your choice!

![Pie Chart](image)

**Problem 2: You Try – Create a Pie Chart**

You are going to perform the same process with the same categories, but for Millennials. The data are already in an Excel Spreadsheet for you, but they are also pasted below.

<table>
<thead>
<tr>
<th>Category</th>
<th>Millennials: 1981 - Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>Food</td>
<td>13.0%</td>
</tr>
<tr>
<td>Housing</td>
<td>34.9%</td>
</tr>
<tr>
<td>Telephone services</td>
<td>2.5%</td>
</tr>
<tr>
<td>Apparel and services</td>
<td>3.6%</td>
</tr>
<tr>
<td>Transportation</td>
<td>17.3%</td>
</tr>
<tr>
<td>Healthcare</td>
<td>5.1%</td>
</tr>
<tr>
<td>Other</td>
<td>23.6%</td>
</tr>
</tbody>
</table>

Make sure to include an informative title including year and generation, percents labeled for each category to one decimal place, and make sure everything on the graph is easy to read with identifiable categories.
SECTION A.2: CREATE A LINE GRAPH IN EXCEL

Problem 3: Worked and Media Example - Line Graphs

The graph below is the end result of creating a Line Graph in Excel. The steps and screenshots for creating this graph are below. There is also a video in the MOER.

Data Source CDC: https://www.cdc.gov/growthcharts/who/girls_length_weight.htm

1) First few data values in Excel Spreadsheet

<table>
<thead>
<tr>
<th>Age (in months)</th>
<th>50th Percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7.49</td>
</tr>
<tr>
<td>0.5</td>
<td>8.37</td>
</tr>
<tr>
<td>1.5</td>
<td>10.02</td>
</tr>
<tr>
<td>2.5</td>
<td>11.53</td>
</tr>
<tr>
<td>3.5</td>
<td>12.92</td>
</tr>
<tr>
<td>4.5</td>
<td>14.19</td>
</tr>
<tr>
<td>5.5</td>
<td>15.36</td>
</tr>
</tbody>
</table>

2) Highlight Data by clicking and dragging.
3) Click on Insert. You can click directly on the type of chart. I usually click on *Recommended Charts* to get a quick picture of the options.

4) Click on Scatter with Straight Lines and markers then click OK.

A scatter with straight lines and markers chart is used to compare at least two sets of values or pairs of data. Use it when there are few data points and the data represents separate measurements.
5) You’ll see the graph below.

![Graph of 50th Percentile Weight (in lbs)](image)

6) Click in the Title box and make the title more specific

![Updated Graph with Specific Title](image)

7) We want to add horizontal and vertical axis labels that include units. Click on DESIGN, Add Chart Element, Axis Titles, and Primary Horizontal (or Vertical) (as shown below).
8) The axis spaces will appear. Click in the spaces and type your labels.

![Graph of CDC: Girls Birth to 3 Years 50th Percentile Weight (in lbs)](image)

9) Here is the complete graph. I also clicked on all of the labels and titles and made the font black and bold.

![Graph of CDC: Girls Birth to 3 Years 50th Percentile Weight (in lbs)](image)

**Problem 4: You Try – Line Graph**

You are going to perform the same process with the same variables, but for Boys. The data are already in an Excel Spreadsheet for you. The data source is at this [CDC Link](#) and specifically uses the 50th percentile weights for boys converted to pounds from kilograms. Make sure to include an informative title like above but for boys, label both the horizontal and vertical axes, and make sure everything on the graph is easy to read.
SECTION A.3: CREATE A BAR GRAPH IN EXCEL

Problem 5: Worked and Media Example – Bar Graphs

The graph below is the end result of creating a Bar Graph in Excel. The steps and screenshots for creating this graph are below. There is also a video in the MOER.

Note: Data are for persons age 25 and over. Earnings are for full-time wage and salary workers.

1) Data in Excel Spreadsheet

<table>
<thead>
<tr>
<th>Education attained</th>
<th>earnings ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Doctoral degree</td>
<td>1,623</td>
</tr>
<tr>
<td>Professional degree</td>
<td>1,730</td>
</tr>
<tr>
<td>Master's degree</td>
<td>1,341</td>
</tr>
<tr>
<td>Bachelor's degree</td>
<td>1,137</td>
</tr>
<tr>
<td>Associate's degree</td>
<td>798</td>
</tr>
<tr>
<td>Some college, no degree</td>
<td>738</td>
</tr>
<tr>
<td>High school diploma</td>
<td>678</td>
</tr>
<tr>
<td>Less than a high school diploma</td>
<td>493</td>
</tr>
<tr>
<td>All workers</td>
<td>850</td>
</tr>
</tbody>
</table>
2) Highlight Data by clicking and dragging

3) Click on Insert. You can click directly on the type of chart. I usually click on **Recommended Charts** to get a quick picture of the options. I chose the second option Clustered column then Click OK
5) The image of the resulting graph is below.

![Graph showing median usual weekly earnings by educational attainment in 2015.]

6) You’ll frequently want to adapt the title. I changed it to Earnings by Educational Attainment 2015.
7) To add horizontal and vertical axis titles, Click on DESIGN, Add Chart Element, Axis Titles, and Primary Horizontal (or Vertical).

8) Add labels to the bars so we can see their exact value. Click on any bar and then right click. Select Add Data Labels.
9) Lastly, I would like all the fonts bold and darker. I clicked on each type of text and clicked the blackest color and the bold button. I also made the axis titles size 12 instead of 10. I dragged the corner of the graph to make it larger so the horizontal titles aren't slanted.

**Problem 6: You Try – Bar Chart**

You are going to perform the same process with the same horizontal educational values, but with the vertical axis representing unemployment rate. The data are already in an excel spreadsheet for you, but you can also view the data at this [Bureau of Labor Statistics Link](https://www.bls.gov).

Make sure to include an informative title like above but for unemployment rate, label both the horizontal and vertical axes, and make sure everything on the graph is easy to read.
**SECTION A.4: CREATE A HISTOGRAM IN EXCEL**

**Problem 7: Worked and Media Example – Histograms**

Recall that a Bar Chart is for categorical data and a Histogram is for quantitative data. With a bar chart, the bars do not touch, and with a histogram the bars do touch. So the easiest way to create a histogram in Excel, is to create a Bar Chart and then adjust the width between the bars to 0. You can find other ways online using the Data Analysis Tool Pak if you are interested.

The graph below is the end result of creating a Histogram in Excel. The steps and screenshots for creating this graph are below. There is also a video in the MOER.

Note that the horizontal axis is measured in quintiles. We studies quartiles which arranged the data from least to greatest and partitioned it into 4 equal groups of size 25% each. Quintiles are similar, but partition the data into 5 equal groups of 20% each.

1) First follow the steps from Problem 5 to create a Bar Chart of the Data as shown below.

2) Adjust the title and add axis labels.
3) To remove the space between the bars, right click on a bar and select “Format Data Series” as shown below.

4) On the left hand side, change the Gap Width from 219% to 0%.

Type 0 and click anywhere to adjust
5) The resulting graph will not have gaps between the bars, but it’s difficult to tell one bar from the next. We can either put a border on each bar in a different color or make the bars different colors. We will do the latter in the next step.

6) In Format Data Series, click on the icon indicated below and check the box “Vary colors by point”.

![Format Data Series](image)
The resulting graph is below.

7) Now add data labels as explained in Problem 5 and make sure everything is well placed and readable. I had to click and drag the data label for 0% - 20% downwards to see it. It was overlapping with the horizontal labeling. (Note this value is negative)
Appendix A – Creating Graphs in Excel

Problem 8: You Try – Histogram

You are going to perform the same process with the same variables, but for the year 2011. The data are located in your excel assignment, and the data source is at this Census Bureau Link.

Make sure to include; an informative title like above but for unemployment rate, label both the horizontal and vertical axes, and make sure everything on the graph is easy to read.
APPENDIX B – THE MARGIN OF ERROR AND POLLING

Lesson Objectives

Section B.1: Determine the population of interest for a poll
Section B.1: Define a simple random sample from a population
Section B.2: Determine the possible sample percentages for a small sample size
Section B.2: Interpret values from the distribution of sample percentages
Section B.2: Interpret a 95% Confidence Interval from the distribution of sample percentages
Section B.2: Compute the Margin of Error for a 95% Confidence Interval based on sample size.
Section B.2: Compute a 95% Confidence Interval based on the Margin of Error
Section B.3: Distinguish between accuracy and precision
Section B.3: Determine the relationship between Margin of Error and Sample Size
Section B.3: Compute the sample size need for a given Margin of Error
Section B.4: Interpret poll results of a sample proportion
Section B.4: Compare poll results of two sample proportions
Section B.5: Interpret poll results from the media with missing information
APPENDIX B – MEDIA LESSON
SECTION B.1: A SIMPLE POLL: POPULATIONS AND SAMPLES

You have likely heard the results of a poll or survey before. A news source or polling company may say, “Our poll indicates that Candidate A is in the lead with 53.2% of the votes” or “A recent poll states that 82% of Americans are in favor of reduced drug prices”. You may have even heard a statement about the margin of error such as, “54% of those surveyed chose Candidate B plus or minus 4 points”. (The “plus or minus” is what we refer to as the Margin of Error.) We are bombarded with these statistics frequently and it is important to be able to interpret these results correctly. They are frequently stated as fact, or by stating “poll”, you are supposed to infer a great deal of information that is not explicitly stated. In this section, we will learn how polls and surveys are frequently conducted and how to interpret their results.

Let’s consider a simple voting example. Suppose Arizona is voting on a proposition. A proposition is legislation that is approved or rejected by eligible voters instead of by the legislature. Eligible Arizona voters can vote in one of two ways; “Yes” in support of the proposition or “No” to reject the proposition.

Frequently, people want to get an idea of how people will vote before Election Day so a polling company such as Pew Research Center, Quinnipiac University Polling Institute, or Gallup Poll may conduct a survey. (Here is a list of how the Statistician Nate Silver of FiveThirty-Eight rates these polling companies based on their methodology). Methodology is the scientific process the polling company uses to collect their data. This can be quite complex, so we will look at a simple example in a somewhat ideal situation.

I. What is the population of interest?

Now the pollster has a question they would like to investigate, “What percent of AZ voters will vote in favor of the proposition?” Next the pollster needs to determine who they need to survey by asking, “What is my population of interest?” The population of a survey is the total set of observations or people that correspond to the survey question. This may seem simple, but we need to be careful. Consider the populations below:

1) All citizens of Arizona.

2) All voting aged citizens of Arizona.

3) All eligible voters in Arizona.

4) All registered voters in Arizona.

5) All likely voters in Arizona. (Define “likely” by previous voting history.)
We would actually like to poll from the population of people who will actually vote in the election on this proposition, but this is difficult to determine. People frequently say they will vote and then don’t. Pollsters often take Exit Polls where they ask people leaving the polls how they voted. These polls must be done on Election Day when we'll know the results soon anyway, but humans really like to try to predict the future!

So we will assume for the sake of simplicity, that we will use the population, all registered voters in Arizona, and that we have a list of these people.

II. How will we sample the population?

The next step is to find a sample of the population to survey. A sample is a subset or part of the population. Based on the Arizona Secretary of State's Website, there are 3,632,377 registered voters in Arizona eligible for the Primary Election in 2018 (with an estimated AZ population of 7.123 million in 2018). Although there are many complex methods for choosing a sample from this population, we will assume we use a simple random sample. A simple random sample is a sample obtained where each member of the population has an equal chance or probability of being chosen. You can think of putting everyone’s name in a hat, shaking up the names really well, and then choosing your sample.

Now we know our population and how we will sample from it. We will ask some number of respondents whether they will vote for or against the proposition. If the respondent will vote in favor of the proposition, we will count this as 1 in favor. If they will vote against the proposition, we will count this as 0 in favor. At the end, we will sum the results and divide by the number of people we sampled. Meaning, if 500 people were sampled, and 260 vote in favor, we could state that, $\frac{260}{500} \cdot 100\% = 52\%$, or 52% of the people surveyed responded that they are in favor of the proposition.

The number of people sampled was made up to illustrate one way we may interpret the survey. In the next section, we will discuss sample size, how to choose it, and how it affects the estimate.
Problem 1: Media Example - Polls, Surveys, and Populations

The Scottsdale Math Department wanted to know what percent of students who enrolled in MAT 112 in the Fall of 2015 passed the course.

The Research Department searched the college's database and reported that 84% of students met this criterion.

Note: Passing is defined as earning a grade of A, B or C.

1) What is the population of the study?

   a) All students at SCC who enrolled in math in the Fall of 2015
   b) All students at SCC who enrolled in MAT 112
   c) All students at SCC
   d) All students at SCC who enrolled in MAT 112 in the Fall of 2015

2) What trait is the study measuring?

   a) Enrolling in MAT 112
   b) Passing MAT 112
   c) Earning a C in MAT 112

3) Is the percentage 84% a sample percentage or population percentage?

   a) Sample percentage
   b) Population percentage
SECTION B.2: THE SAMPLING DISTRIBUTION OF A PERCENTAGE

To determine how a sample percentage compares to the true population percentage, Statisticians looked at what would happen if we knew the true value of the percent we were looking for and how much error we could expect based on the sample size. In what follows, we will do the same.

Arizona’s currently has 3,632,377 registered voters. This is a lot of people! Collecting data is expensive and requires a lot of time and effort so we don’t want to survey all of them. The good news is, when the population is sufficiently large we do not need to consider the size of the population when choosing a size of a sample. Meaning, we don’t need 10% of the population (363,238) or some amount that depends on the size of the population. We will use statistical theorems and results that are independent of the population size.

I know the phrase sufficiently large is somewhat cryptic and vague. A good rule of thumb is that the sample should have at least 5 results for each outcome. In our example, this would mean that in our sample, at least 5 people should respond in favor of the proposition and at least 5 people should respond against the proposition.

I. The Sampling Distribution of a Proportion (or Percentage)

For argument’s sake, we will assume that we know the true percentage of people that will vote in favor of the proposition. Let’s say that this percent is 52%. In statistics, this percent is frequently called a proportion where we think of 52% as a fraction of the number of votes in favor of the proposition over the total number of votes \( \frac{52}{100} \). (Note that in math we call this a ratio or fraction, and a proportion is a statement that two ratios are equal. So the terminology differs in this context.) We typically use the letter \( p \) to denote the population proportion. When we take a sample from a population to estimate the population percentage \( p \), we label the resulting sample percentage \( \hat{p} \) pronounced “p hat”.

Appendix B – The Margin of Error and Polling

So if we assume 52% of the voters vote in favor of the proposition, or $p = 52\%$, what can we expect a sample of this population to look like? First let’s start with a really small sample size of 3. We denote the sample size of a survey with the letter $n$. So here, $n = 3$.

If we sample three people, we can get 0, 1, 2, or 3 yes votes in our sample. The table below shows how this may happen with “0” representing a vote of No and “1” representing a vote of Yes.

<table>
<thead>
<tr>
<th>Number of Yes Votes</th>
<th>Arrangement of Yes Votes in Sample</th>
<th>Proportion or Percentage of Yes Votes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero Yes votes</td>
<td>0 0 0</td>
<td>$\hat{p} = \frac{0}{3} = 0%$</td>
</tr>
<tr>
<td>One Yes vote</td>
<td>1 0 0 or 0 1 0 or 0 0 1</td>
<td>$\hat{p} = \frac{1}{3} \approx 33.3%$</td>
</tr>
<tr>
<td>Two Yes votes</td>
<td>1 1 0 or 0 1 1 or 1 0 1</td>
<td>$\hat{p} = \frac{2}{3} \approx 66.7%$</td>
</tr>
<tr>
<td>Three Yes votes</td>
<td>1 1 1</td>
<td>$\hat{p} = \frac{3}{3} = 100%$</td>
</tr>
</tbody>
</table>

Observations: There are a few important things to notice from this table.

1) Each of the estimates for the percentage of voters who will vote yes, $\hat{p}$, is far from the true value. Recall the true value is $p = 52\%$. A sample size of 3 is not useful!

2) We are more likely to get a sample with 1 or 2 yes votes than 0 or 3 votes (3 ways to get 1 or 2 yes votes as opposed to 1 way to get 0 or 3 yes votes)

3) Even though there are an equal number of ways to get 1 or 2 yes votes, we are more likely to get 2 yes votes than 1 yes vote since the chance of a yes vote is 52% and the chance of a no vote is 48%.
The true chance or probability of getting each of these samples is displayed in the histogram below. The horizontal axis represents the proportion or percentage of yes votes in the sample. The vertical axis represents the percent of all possible samples that would give us the sample on the horizontal axis. (The mathematics of how these chances are computed are beyond the scope of the course, but you may look them up by researching Binomial Distributions or probabilities if you are interested.)

**Interpretation:** The true percent of population: p = 52%  Sample Size: n = 3

If we collect a sample size of 3 from a population with a true population percent of 52%:

11.06% of the time, the sample proportion, \( \hat{p} \), will equal 0%.

35.94% of the time, the sample proportion, \( \hat{p} \), will equal 33.33%.

38.94% of the time, the sample proportion, \( \hat{p} \), will equal 66.67%.

14.06% of the time, the sample proportion, \( \hat{p} \), will equal 100%.

So we know that a sample size of 3 isn’t useful. What sample size would give us a good or better estimate? We will look at sample sizes of increasing sizes and look for patterns in the distributions.
Appendix B – The Margin of Error and Polling

Sample Size = 10

The histogram below shows the distribution of sample percentages for a sample size of 10. The mode percentage is 50% which is closest to the true value of 52%, but with this sample size, we still could not even obtain the true value from the sample.

Sample Size = 100

The histogram below shows the distribution of sample percentages for a sample size of 100. Notice that the distribution is narrowing and centered at the true population percentage of 52%. Listing all of the percentages for the different percent of voters from the sample who support the proposition is no longer useful. However, notice that 96.49% of the possible samples will give a sample percent between 42% and 62% which is the true population percentage plus or minus 10%.
Sample Size = 1000

The next histogram shows the distribution of sample percentages for a sample size of 1000. Notice that as the sample size increases, more of the sample percentages are close to the true value of the population percentage. The histogram itself may not appear narrower because the scaling of the horizontal axis changed to focus in on the values where most of the sample percentages lie. Also notice there doesn't appear to be any histogram bars. These rectangles have such a small width that the distribution looks smooth. The larger the sample size, the smoother the distribution becomes. In fact, it can be closely approximated by a Normal Distribution or Bell Curve that you may have heard or learned about in prior courses.

Problem 2: Media Example - Distribution of Sample Percentages
Appendix B – The Margin of Error and Polling

a) The histogram below shows the distribution of sample percentages for a sample size of n=8 and a population percentage of p=43%. The blue dashed line shows the location of the population percentage. Complete the statement below based on this information.

If we find a sample of size 8 and the true population percentage is 43%, the percent of samples that lie between or are equal to 38% and 63% is ________

b) The histogram below shows the distribution of sample percentages for a sample size of n=100 and a population percentage of p = 51% for sample percentages ranging from 46% to 56.

a) What percent of samples of size 100 equal the true population percentage of 51%?

b) What percent of samples of size 100 are within 3% of the true population percentage of 51%?
II. 95% Confidence Intervals and the Margin of Error

We noticed that as the sample size increased from 0 to 10 to 100 and then to 1000, more of the sample percentages lie “close” to the true value of the population proportion. Now we would like to quantify this closeness. First we will start with a definition and then we will state a formula that quantifies the precision.

**Definition:** A *95% Confidence Interval* for the population percentage is an interval of values centered at the true population percentage that includes 95% of all possible sample percentages. It depends on the true population percentage and the sample size.

**Interpretation:** For the last example with \( n = 1000 \) and \( p = 52\% \), the 95% Confidence Interval is \((48.8\%, 55.2\%)\). We can state that for all possible samples of size 1000, 95 out of 100 will have a sample percentage between 48.8% and 55.2%.

**Note:** We could use other percentages for the Confidence Interval. Some popular ones are 99%, and 90%. We will use 95% for convenience and because it is frequently used in practice.

**Definition:** The *Margin of Error* for a population proportion with a 95% Confidence Interval is the difference between the true population proportion and either end point of the 95% Confidence Interval. Equivalently, the Margin of Error (or MOE) is half the distance of the length of the 95% Confidence Interval.

**Example:** For the last example with \( n = 1000 \) and \( p = 52\% \), the 95% Confidence Interval is \((48.8\%, 55.2\%)\). The Margin of Error can be computed in any of the following ways:

\[
55.2\% - 52\% = 3.2\% \quad \text{or} \quad 52\% - 48.8\% = 3.2\% \quad \text{or} \quad \frac{55.2\% - 48.8\%}{2} = \frac{6.4\%}{2} = 3.2\%
\]

**Conservative Estimate for the Margin of Error for a 95% Confidence Interval**

There is a formula to compute the Margin of Error based on the population proportion \( p \) and the sample size \( n \). We can simplify it greatly if we use a *conservative estimate* that only uses the sample size \( n \). Meaning the Margin of Error we compute will be slightly larger than the actual Margin of Error, but since this is erring on the side of caution, it is worth the simplification.

A conservative estimate of the MOE for a 95% Confidence Interval with sample size \( n \) is

\[
MOE = \frac{1}{\sqrt{n}} \quad \text{Since we report this as a percentage, we can use} \quad MOE = \frac{1}{\sqrt{n}} \cdot 100\%
\]
Appendix B – The Margin of Error and Polling

Example: For the last example, \( n = 1000 \). So the MOE = \( \frac{1}{\sqrt{1000}} \cdot 100\% \approx 3.2\% \) (rounded to one decimal place). Typically, we will use this formula to find the 95% Confidence Interval rather than creating a histogram.

The formula for the 95% Confidence Interval based on \( p \) and the MOE is \( (p - \text{MOE}, p + \text{MOE}) \).

Which for this example is \( (52\% - 3.2\%, 52\% + 3.2\%) = (48.8\%, 55.2\%) \)

Problem 3: Worked Example - MOE and 95% Confidence Interval for a Population %

The value of a population percent is \( p = 63\% \). For sample sizes of \( n = 1357 \), find the MOE and 95% Confidence Interval and interpret the CI. Round to two decimals as needed.

Solution: \( \text{MOE} = \frac{1}{\sqrt{1357}} \cdot 100\% \approx 2.71\% \)

The 95% CI is \( (63\% - 2.71\%, 63\% + 2.71\%) = (60.29\%, 65.71\%) \)

95 out of 100 samples of size \( n = 1357 \) will have sample percentages that lie between 60.29% and 65.71%.

Summary of Results:

1) As the sample size increases, more samples will have a sample percentage close to the true value.

2) We can conservatively estimate that 95% of the sample percentages with population percentage \( p \) and sample size \( n \) will lie in the interval \( \left( p - \frac{1}{\sqrt{n}}, p + \frac{1}{\sqrt{n}} \right) \). We call this the 95% Confidence Interval for the sampling distribution of the percentage.

3) The value \( \frac{1}{\sqrt{n}} \) written as a percentage is a conservative estimate for the Margin of Error of a sample percentage using a 95% confidence interval. The Margin of Error is an estimate of the sampling error that occurs when we use a sample to estimate a population.

4) Sampling error is the name for the variability of possible sample percentages. It is not an “error” that was made by the pollster, but a consequence of taking a sample from a population. It cannot be avoided when taking a sample, but can be minimized by taking a larger sample.
SECTION B.3: SAMPLE SIZE AND PRECISION

Now we know how the sample size relates to the Margin of Error. We can use this information to choose a sample size. The sample size chosen depends on how much precision we want in our estimate. The diagram below illustrates the concepts of precision and accuracy which are often confused. Suppose the diagrams below are dartboards and the true value of the percentage of voters who will vote for the proposition is in the centermost point. An estimate is precise, when multiple measurements would give you values close together. An estimate is accurate, when multiple measurements are close to the true value.

The accuracy of our estimate is not quantified, but Statisticians investigated the idea and agree that the best possible estimate we can make where we garner the most information from our sample is the sample proportion \( \hat{P} \). This doesn’t mean that we can’t get a \( \hat{P} \) that is far from the true value of the percentage. In fact, theoretically, \( p \) could equal 75% and it is possible to get an estimate of \( \hat{P} = 0\% \). It’s not likely, but it could happen. Instead it means that our best chance is to use a sample proportion to estimate the population proportion.

In terms of precision, as the sample size increases, the likelihood of the precision of our percentage estimate also increases. Theoretically, we want as much precision as possible, but increasing the sample size means surveying more people which costs time and money. Additionally, the increase in precision is not proportional to the increase in sample size. Meaning, doubling the sample size does not double the precision.
The graph below shows that as the sample size increases, the margin of error decreases. However notice that it is not a straight line. For a sample size of 100, the MOE is 10%, but to attain a MOE of 5%, you will need a sample size 4 times greater than 100 or 400.

The graph continues to flatten out as the sample size increases as shown in the graph below for sample sizes between 1000 and 11,000.

A sample size of 1000 has a MOE of 3.2%. A sample size of 6000 has a MOE of 1.3%. A sample size of 11,000 has a MOE of 1.0%.

So although increasing sample size will increase precision, the amount of increase may not warrant the cost and time of choosing an even larger sample. To determine the sample size needed if given the desired Margin of Error, we may compute $n = \frac{1}{(MOE)^2}$ where MOE is written as a decimal.
Problem 4: Media Example – Margin of Error and Sample Size

Complete the table below by computing the Margin of Error for each of the given sample sizes. Round your percent to two decimal places.
Use the conservative estimate for the Margin of Error using a 95% Confidence Interval:

\[ MOE = \frac{1}{\sqrt{n}} \]

where MOE is the Margin of Error and \( n \) is the sample size.

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>MOE</th>
</tr>
</thead>
<tbody>
<tr>
<td>n = 100</td>
<td></td>
</tr>
<tr>
<td>n = 200</td>
<td></td>
</tr>
<tr>
<td>n = 400</td>
<td></td>
</tr>
<tr>
<td>n = 800</td>
<td></td>
</tr>
</tbody>
</table>

a) As the sample size increases, the Margin of Error _________

b) Does doubling the sample size split the MOE in half?

Problem 5: Media Example – Sample Size and Margin of Error

Complete the table below by computing the Sample Size for each of the given MOE. Round to the nearest whole number.
Use the conservative estimate for the sample size using a 95% Confidence Interval:

\[ n = \frac{1}{(MOE)^2} \]

where MOE is the Margin of Error and \( n \) is the sample size.

<table>
<thead>
<tr>
<th>MOE</th>
<th>Sample Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td></td>
</tr>
<tr>
<td>2%</td>
<td></td>
</tr>
<tr>
<td>3%</td>
<td></td>
</tr>
<tr>
<td>4%</td>
<td></td>
</tr>
</tbody>
</table>

a) As the Margin of Error increases, the sample size _________

b) Does doubling the MOE split the sample size in half?
SECTION B.4: USING THE SAMPLING DISTRIBUTION TO INTERPRET POLLS

Thus far, we have discussed the sample distribution of the true percentage of the population. Recall that we assumed that we knew the true value of the percentage $p$ and then observed what the distribution of all possible samples of a given sample size would look like. However, we do not know the true percentage $p$. If we did, we wouldn't need to take a sample at all!

However, there is some great news developed by Statisticians that was mentioned in Section 9.3. The best estimate we can use for $p$ is the sample proportion $\hat{p}$. And we can use the identical Confidence Intervals and Margins of Error that we developed about $p$ for $\hat{p}$.

So we are making a slight change of perspective now, but it is an important change. If you didn't read the above paragraph carefully, you may think I am just repeating what we already did. The distinction to be made is that now, we are not assuming we know $p$. Instead we are assuming the sample proportion we obtain from a single simple random sample $\hat{p}$ is the best estimate of $p$ and that the distribution of $\hat{p}$ behaves like the sampling distribution of $p$. This result is stated below. Notice the only difference from our previous results is that $p$ is replaced with $\hat{p}$.

Result: When we use a sample percentage $\hat{p}$ to estimate the population percentage, the 95% Confidence Interval for the population percentage using $\hat{p}$ for sample size $n$ is

\[
\left( \hat{p} - \frac{1}{\sqrt{n}}, \hat{p} + \frac{1}{\sqrt{n}} \right)
\]

where $\text{MOE} = \frac{1}{\sqrt{n}}$ is a conservative estimate for the Margin of Error of the 95% CI.

Now that we have extended the results of population percentages to sample percentages, we can interpret a sample percentage. We will do this almost identically to our previous work. However, we will be given a single sample percentage.
Problem 6: Worked Example – Finding the MOE and CI from a Sample Percentage

A pollster sampled $n = 1156$ people and found that $\hat{p} = 51\%$ of likely voters sampled were in favor of a proposition on the ballot in November.

a) Find the Margin of Error for this point estimate. (Round to two decimal points as needed)

**Solution:** We will use the conservative estimate for the Margin of Error, $\text{MOE} = \frac{1}{\sqrt{n}}$

Since $n = 1156$, \[ \text{MOE} = \frac{1}{\sqrt{1156}} = \frac{1}{34} \]

Written as a percentage rounded to two decimal places: $\frac{1}{34} \cdot 100\% \approx 2.94\%$

**Answer:** The Margin of Error is 2.94%.

b) Find the 95% Confidence Interval for this point estimate and interpret it.

**Solution:** The 95% Confidence Interval for the point estimate is $\left( \hat{p} - \frac{1}{\sqrt{n}}, \hat{p} + \frac{1}{\sqrt{n}} \right)$

$\hat{p} = 51\%$

$\hat{p} - \frac{1}{\sqrt{n}} = 51\% - 2.94\% = 48.06\%$

$\hat{p} + \frac{1}{\sqrt{n}} = 51\% + 2.94\% = 53.94\%$

So the 95% Interval is (48.06%, 53.94%). The interpretation is below.

**Answer:** We are 95% confident that the true value of the percent of likely voters in favor of the Proposition lies between 48.06% and 53.94%.

**Note:** When we interpreted confidence intervals for the population proportion, we stated that 95 out of 100 samples are within the confidence interval. When we interpret the confidence interval for the single sample proportion we say we are 95% “confident” that the true value lies in the confidence interval. We use the weaker term “confident” because we were certain about the distribution of the true population distribution, but we are using an estimate when we use the single sample proportion.
Appendix B – The Margin of Error and Polling

Problem 7: Media Example – Interpreting a Sample Percentage
A market analyst sampled \( n = 729 \) people and found that \( p = 75\% \) of those surveyed preferred the "new and improved" version of a product. Find the Margin of Error and 95% Confidence Interval for this point estimate. Round any percents to two decimal places as needed.

a) Find the Margin of Error for this point estimate. The Margin of Error is:

b) Find the 95% Confidence Interval for this point estimate and interpret it below.

We are 95% confident that the true value of the percent of those who prefer the new and improved product lies between ____________ and ____________

Problem 8: Worked Example – Comparing the MOE and CI from two Sample Percentages
An exit pollster sampled \( n = 1098 \) people as they left the voting place and found that 44% of the voters sampled were in favor of Candidate A and 56% of the voters sampled were in favor of Candidate B. (Round any results below to two decimal places as needed.)

a) Find the Margin of Error for this poll.

\[
\text{MOE} = \frac{1}{\sqrt{n}} \cdot 100\% = \frac{1}{1098} \cdot 100\% \approx 3.02\%
\]

Answer: The Margin of Error is 3.02%

b) Find the 95% Confidence Interval for the percentage of Candidate A's votes and interpret it.

\[
\left( \hat{p} - \frac{1}{\sqrt{n}}, \hat{p} + \frac{1}{\sqrt{n}} \right) = (56\% - 3.02\%, \ 56\% + 3.02\%) = (52.98\%, \ 59.02\%)
\]

Answer: We are 95% confident that the true value of the percent of Candidate A's votes lies between 52.98% and 59.02%
c) Find the 95% Confidence Interval for the percentage of Candidate B’s votes and interpret it.

\[
\left( \hat{p} - \frac{1}{\sqrt{n}}, \hat{p} + \frac{1}{\sqrt{n}} \right) = \left( 44\% - 3.02\%, 44\% + 3.02\% \right) = (40.98\%, 47.02\%)
\]

**Answer:** We are 95% confident that the true value of the percent of Candidate B’s votes lies between 40.98% and 47.02%

d) Do the 95% Confidence Intervals for Candidate A and Candidate B overlap?

The largest value in Candidate B’s interval is 47.02% and the smallest value in Candidate A’s interval is 52.98%, so they do not overlap.

**Answer:** No. The confidence Intervals do not overlap.

e) Based on your previous answer, choose the appropriate interpretation:

Since the confidence intervals do not overlap, there is statistical evidence that the percentages are not equal. In addition, Candidate B’s percentage is higher, so there is statistical evidence Candidate B is in the lead.

i. There is not enough statistical evidence to determine which candidate is in the lead.
ii. There is statistical evidence that Candidate A is in the lead.
iii. There is statistical evidence that Candidate B is in the lead.

**Answer:** iii. There is statistical evidence that Candidate B is in the lead.
Problem 9: Media Example – Analyzing Poll Results I

An exit pollster sampled n=664 people as they left the voting place and found that 49.2% of the voters sampled were in favor of Candidate A and 50.8% of the voters sampled were in favor of Candidate B.

Find the Margin of Error for this sample size using the conservative estimate and the 95% Confidence Interval for each of these percentages. Round any percents to one decimal place as needed.

a) Find the Margin of Error for this poll. The Margin of Error is:

b) Find the 95% Confidence Interval for the percentage of Candidate A’s votes and interpret it below.

We are 95% confident that the true value of the percent of Candidate A’s votes lies between _____ and ______

c) Find the 95% Confidence Interval for the percentage of Candidate B’s votes and interpret it below.

We are 95% confident that the true value of the percent of Candidate B’s votes lies between _____ and ______

d) Do the 95% Confidence Intervals for Candidate A and Candidate B overlap?

e) Based on your previous answer, choose the appropriate interpretation:

i. There is statistical evidence that Candidate A is in the lead.
ii. There is statistical evidence that Candidate B is in the lead.
iii. There is not enough statistical evidence to determine which candidate is in the lead.
Problem 10: Media Example – Analyzing Poll Results II

A pollster sampled n=1182 people and found that 662 of the voters sampled were in favor of Candidate A and 520 of the voters sampled were in favor of Candidate B. Round any percents to one decimal place as needed.

a) Find the sample proportion \( \hat{p} \) for Candidate A:

b) Find the sample proportion \( \hat{p} \) for Candidate B:

c) Find the Margin of Error for a 95% Confidence Interval for this poll.

The Margin of Error is:

d) Find the 95% Confidence Interval for the percentage of Candidate A's votes and interpret it below.

We are 95% confident that the true value of the percent of Candidate A's votes lies between _____ and _______

e) Find the 95% Confidence Interval for the percentage of Candidate B's votes and interpret it below.

We are 95% confident that the true value of the percent of Candidate B's votes lies between _____ and _______

f) Do the 95% Confidence Intervals for Candidate A and Candidate B overlap?


g) Based on your previous answer, choose the appropriate interpretation:

i. There is statistical evidence that Candidate A is in the lead.

ii. There is statistical evidence that Candidate B is in the lead.

iii. There is not enough statistical evidence to determine which candidate is in the lead.
Most polls on the internet, television, and other media sources do not report the level of the confidence interval. Some polls may report the Margin of Error. Usually, you are just given a percentage or sample proportion. Reporting the results of a poll this way can mislead readers into believing the percentage given is the true value and not an estimate.

The chart below was obtain from the Pew Research Center. They include a long article analyzing their results and reference their data source IPUMS, Integrated Public Use Microdata Series. IPUMS reports how they obtained their samples in excruciating detail. Their methodology is not the simple random sample we discussed. They use more sophisticated methods of sampling because it is often more difficult and costly to obtain a simple random sample.

Notice that the chart displays that 36% of millennial women in 2017 completed at least a bachelor’s degree. This percentage was obtained from a sample and is an estimate. A common standard is using a 95% Confidence Interval and a Margin of Error of 3%. We don’t know this for certain without digging into the pages of documentation, but this is a better way to interpret these percentages than assuming the given value is the true value.

So when interpreting data, be a skeptic, but not a cynic. Check the sources to see if they are reputable. If you are going to make important decisions based on the information, follow all of the links the report sources and check that the references are valid. As the famous statistician George Box stated, “All models are wrong, but some are useful.” So we may not be getting the exact answer, but our methods promote obtaining results that are as accurate and precise as possible.
Appendix B – The Margin of Error and Polling

Problem 11: Media Example – Analyzing Poll Results from the Internet

The Pew Research Center reported the poll results below on 4/30/2018 about how adults access the internet. The 2nd table was accessed from the full report and contains the Margin of Errors for each age subgroup. This chart also notes that the confidence interval for the Margin of Error is 95% which isn't mentioned in the main article. Use the information in the tables to answer the questions below.

![Poll Results Table]

The following table shows the unweighted sample sizes and the error attributable to sampling that would be expected at the 95% level of confidence for different groups in the survey:

<table>
<thead>
<tr>
<th>Group</th>
<th>Unweighted sample size</th>
<th>Plus or minus ...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total sample</td>
<td>2,002</td>
<td>2.4 percentage points</td>
</tr>
<tr>
<td>Ages 18-29</td>
<td>352</td>
<td>5.8</td>
</tr>
<tr>
<td>30-49</td>
<td>528</td>
<td>4.7</td>
</tr>
<tr>
<td>50-64</td>
<td>544</td>
<td>4.7</td>
</tr>
<tr>
<td>65+</td>
<td>529</td>
<td>4.7</td>
</tr>
</tbody>
</table>

a) What is \( \hat{p} \) for those who are in the age category 65+ and have a smartphone, but no broadband?
Appendix B – The Margin of Error and Polling

b) What is the Margin of Error for those who are in the age category 30−49?

c) Find the 95% Confidence Interval for the percentage who are in the age category 50−64 and have broadband at home.

We are 95% confident that the true value of the percent of people in the age category 50−64 and have broadband at home lies between ______ and ________

d) Find the 95% Confidence Interval for the percentage who are in the age category 18−29 and have no broadband and no smartphone.

We are 95% confident that the true value of the percent of people in the age category 18−29 and have no broadband and no smartphone lies between lies between ______ and ________