Attributions

This workbook was designed as a supplement to the MAT 212 MOER course developed by Carla Stroud at Scottsdale Community College. The MAT 212 MOER course was developed using content from the MAT 220/221 MOER course created by Phil Clark at Scottsdale Community College.

The problems in the “Lesson Notes” sections come from videos made by James Sousa (http://www.mathispower4u.com), Phil Clark, and Khan Academy (www.khanacademy.org). Those videos are part of the “Online Lessons” in the MOER course prepared by Carla Stroud.

Content for this workbook was adapted from the following resource

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2.1: Instantaneous Rate of Change and Tangent Lines

Learning Objectives

- Calculate and interpret the average rate of change of a function between two points (from a table, graphically, and algebraically).
- Use the average rate of change of a function over small intervals to estimate the instantaneous rate of change of a function at a point.
- Use a graph to estimate the instantaneous rate of change of a function at a point.

Definitions: Average rate of change, instantaneous rate of change, secant line, tangent line

Introduction: Review Average Rate of Change

Consider the following scenario:

Suppose you drove 100 miles in two hours.
- On average, how fast were you going?
- Were you always traveling at that speed?
- Were you ever traveling at that speed?

In this section, we are going to explore the meaning of average rate of change. In particular, we will review how to calculate an average rate of change between two values from a graph, a table, and using a formula.
Calculating Average Rate of Change: Algebraic Approach

To calculate an average rate of change of a function, we can use the difference quotient of $f$ over the interval $[a, b]$.

$$\text{Average rate of change of } f = \frac{f(b) - f(a)}{b - a}$$

For a linear function, the difference quotient gives us the slope.

For a nonlinear function, the difference quotient gives us the slope of the line through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

The slope of a line can be represented by the following:

$$m = \frac{f(b) - f(a)}{b - a} = \frac{\Delta f}{\Delta x}$$

Example: Average Velocity

Suppose a large grapefruit is thrown straight up in the air at $t = 0$ seconds. The grapefruit leaves the thrower's hand at a high speed, then slows down until it reaches its maximum height, then speeds up in the downward direction and finally, SPLAT!

The height of the grapefruit, in feet, as a function of time $t$, in seconds, can be modeled by the quadratic function $H(t) = -5t^2 + 27t + 4$

1. What is the grapefruit’s average rate of change from 1 to 3 seconds? Include units in your answer.
Calculating Average Rate of Change Using a Graph

The average rate of change of $f(x)$ from $x_1$ to $x_2$ is the slope of the line through the points $(x_1, y_1)$ and $(x_2, y_2)$. The line connecting two points on a curve is called a secant line.

2. The graph below shows the temperature change throughout one day. Use the graph to answer the following questions.

a. What is the average rate of change of the temperature from 8am to 2pm?

b. What is the average rate of change of the temperature from 2pm to 6pm?
Calculating Average Rate of Change Using a Table

3. Suppose the function \( f \) represents the amount of money in Tony’s bank account as a function of time \( t \), measured in months, where \( t = 1 \) represents January.

<table>
<thead>
<tr>
<th>( t ) (months)</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) ) (dollars)</td>
<td>1200</td>
<td>1300</td>
<td>1400</td>
<td>1400</td>
<td>900</td>
<td>1150</td>
<td>1300</td>
</tr>
</tbody>
</table>

a. What was the average rate of change of Tony's bank account over the interval \([5, 8]\) (from May to August)?

b. What was the average rate of change of Tony's bank account over the interval \([6, 11]\) (from June to November)?

c. Does the answer to part b indicate that Tony's bank account balance did not change between June and November?
Instantaneous Rate of Change

Average velocity is a nice way for us to represent how the position of an object is changing over some time interval. What if we are concerned with how an object is moving at one particular time? With no interval, how do we apply our formula for the average rate of change? Well, let's do a little exploration with our grapefruit.

Alternative Notation for the Difference Quotient

Recall, we can represent the average rate of change using the difference quotient of \( f \) over the interval \([a, b]\)

\[
\frac{f(b) - f(a)}{b - a}
\]

There is an alternative notation that makes calculating an average rate of change over smaller intervals easier to see and compute.

We define the interval to be from \( a \) to \( a + h \).

Let's write the average rate of change of \( f \) over the interval \([a, a + h]\).

\[
\frac{f(b) - f(a)}{b - a} = \frac{f(a + h) - f(a)}{(a + h) - a} = \frac{f(a + h) - f(a)}{h}
\]

In this case, \( h \) represents the length of the interval.

And so, the difference quotient in this notation is given by \( \frac{f(a+h) - f(a)}{h} \)
Recall the grapefruit example:

4. The height of the grapefruit, in feet, as a function of time $t$, in seconds, can be modeled by the quadratic function $H(t) = -5t^2 + 27t + 4$

Determine the average velocity of the grapefruit over the time interval $[1, 1 + h]$ where $h$ is given as follows

a. $h = 1$

b. $h = 0.5$
2.1: Instantaneous Rate of Change and Tangent Lines

Lesson Notes

c. \( h = 0.1 \)

d. \( h = 0.01 \)

e. Using the values found in parts (a) – (e), estimate the velocity of the grapefruit at 1 second.
Tangent Lines

The instantaneous rate of change measures the rate of change at a point. The instantaneous rate of change gives the slope of the tangent line at a single point. The tangent line to a curve at a given point is the straight line that "just touches" the curve at that point.

Recall the grapefruit example:

5. The height of the grapefruit, as a function of time, $t$, can be modeled by the quadratic function: $H(t) = -5t^2 + 27t + 4$

In problem 4, we predicted the velocity (instantaneous rate of change) of the grapefruit at $t = 1$ second using smaller and smaller values of $h$. We found the instantaneous velocity to be 17 feet per second. In the graph below, sketch the tangent line at $t = 1$.
Practice Problems

1. The price of an ounce of gold, is U.S. dollars, as a function of time $t$, in days, can be modeled by the function $R(t) = 40t - 2t^2$.

Find the average rate of change of $R(t)$ over the time interval $[4, 4 + h]$ where $h$ is given as follows:
   
   a. $h = 1$
   b. $h = 0.1$
   c. $h = 0.01$
   d. $h = 0.001$
   
e. Estimate the instantaneous rate of change of the price of an ounce of gold at 4 days. Include units in your answer.

2. Find the average rate of change of $f(x) = 8x^2 - 4$ on the interval $[2, a]$.

3. Find the average rate of change of $g(x) = 5x^2 + 2x - 3$ on the interval $[4, a]$.

4. Given the function $f(x) = 6x - 1$, evaluate and simplify the following expressions
   
   a. $f(a)$
   b. $f(a + h)$
   c. $\frac{f(a+h) - f(a)}{h}$
2.2: Limits

Learning Objectives

• Read limit notation and understand what each symbol means.
• Evaluate the limit of a function numerically, graphically, and algebraically

Definitions: Limit, one-sided limit, left limit, right limit, indeterminate form

Introduction

In the last section we looked at how to use the average rate of change over an interval to estimate an instantaneous rate of change at a single point.

Average Rate of Change

• Measures the constant rate needed to acquire the change in \( f(x) \) for a given change in \( x \) (measured over an \( ___________________________ \)).
• Represented graphically as the \( ___________________________ \) between two points.
• Calculated using \( \frac{f(x_2) - f(x_1)}{x_2 - x_1} \) or \( \frac{f(x_1) - f(x_2)}{x_1 - x_2} \)

Instantaneous Rate of Change

• Measures the rate of change at a \( ___________________________ \).
• Represented graphically as the \( ___________________________ \) at a single point.
• Estimated by taking the average rate of change over a very small interval.

In order to approximate the instantaneous rate of change of a function, we need to be able to investigate the change in \( f(x) \) for a very small change in \( x \). Thus, we need to be able to make the change in \( x \) very, very, very small. In fact, we want \( \Delta x \) to get as close to 0 as possible (remember, \( \Delta x \) can’t equal 0 because then \( \frac{\Delta f(x)}{\Delta x} \) would be undefined).

In order to do this, we need a way to talk about changes in one quantity as the other quantity gets as close to a value as possible. The mathematical concept known as a limit will allow us to do this.
After we review the limit concept in this section, we will learn how to apply the concept of limit to the average rate of change in order to find the exact value of the instantaneous rate of change in the next section.

Limits: Numerical and Graphical Approaches

**Limit Definition**
Given a function \( f \), a fixed input \( x = a \), and a real number \( L \), we say that \( f \) has limit \( L \) as \( x \) approaches \( a \), and write

\[
\lim_{{x \to a}} f(x) = L
\]

- This is read "the limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \)"
- It is meant to describe the value that the function "approaches" as the input approaches some value

**Example**

Consider the function \( f(x) = x^2 \). What happens to \( f(x) \) as \( x \) gets closer to 2?

We can use a graph to view the function and a table to organize our findings.
Formal Definition of a Limit

We say that $f$ has limit $L$ as $x$ approaches $a$ from the left and write:

We say that $f$ has limit $L$ as $x$ approaches $a$ from the right and write:

A function $f$ has a limit $L$ as $x$ approaches $a$ if and only if $\lim_{x \to a} f(x) = L$.

Examples

1. Estimate the limit (a) using tables and (b) using the graph of the function.

   \[
   \lim_{x \to 3} \frac{x^2 - 9}{x - 3} =
   \]

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.99</td>
<td></td>
</tr>
<tr>
<td>2.999</td>
<td></td>
</tr>
<tr>
<td>2.9999</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.01</td>
<td></td>
</tr>
<tr>
<td>3.001</td>
<td></td>
</tr>
<tr>
<td>3.0001</td>
<td></td>
</tr>
</tbody>
</table>
2. Use the graph of the function \( f(x) \) to evaluate the following limits.

\[
\lim_{x \to -2^-} f(x) = \quad \lim_{x \to -2^+} f(x) = \quad \lim_{x \to -2} f(x) = \quad f(-2) =
\]

3. \( \lim_{x \to 2^+} \frac{1}{x - 2} = \) \quad and \quad \( \lim_{x \to 2^-} \frac{1}{x - 2} = \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.99</td>
<td></td>
</tr>
<tr>
<td>1.999</td>
<td></td>
</tr>
<tr>
<td>1.9999</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.01</td>
<td></td>
</tr>
<tr>
<td>2.001</td>
<td></td>
</tr>
<tr>
<td>2.0001</td>
<td></td>
</tr>
</tbody>
</table>
4. \[ \lim_{x \to \infty} \frac{4x^5 - 3x^2 + 3}{6x^5 - 100x^2 - 10} = \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>_</td>
</tr>
<tr>
<td>100</td>
<td>_</td>
</tr>
<tr>
<td>1000</td>
<td>_</td>
</tr>
<tr>
<td>10,000</td>
<td>_</td>
</tr>
<tr>
<td>100,000</td>
<td>_</td>
</tr>
</tbody>
</table>

If \( f(x) \) is a rational function, then we can calculate the limit of \( f(x) \) as \( x \to \pm \infty \) by

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>_</td>
</tr>
<tr>
<td>100</td>
<td>_</td>
</tr>
<tr>
<td>1000</td>
<td>_</td>
</tr>
<tr>
<td>10,000</td>
<td>_</td>
</tr>
<tr>
<td>100,000</td>
<td>_</td>
</tr>
</tbody>
</table>

5. Evaluate the limits by taking the limits of the leading terms:

a. \[ \lim_{x \to \infty} \frac{4x^5 - 3x^2 + 3}{6x^5 - 100x^2 - 10} = \]

b. \[ \lim_{x \to -\infty} \frac{3x^2 - 2x^2 + 7}{6x^4 - x^3 + 2x - 100} = \]

c. \[ \lim_{x \to \infty} \frac{4x^4 - 3x^3 + 7x^2 - 10}{250x^3 + 5x^2 - x + 1000} = \]
Limits: Algebraic Approach

If we know that the function \( f \) is continuous at the point \( a \), then we can evaluate \( \lim_{x \to a} f(x) \) by \( \boxed{ \text{_______________________________} } \). In other words:

6. Evaluate the following limits algebraically

   a. \( \lim_{x \to 2} x^2 = \)

   b. \( \lim_{x \to -2} 4 = \)

   c. \( \lim_{x \to 3} x = \)

   d. \( \lim_{x \to 3} \frac{x^2 - 4}{x - 2} = \)

   e. \( \lim_{x \to 0} \frac{0.5x^2 - 3x}{x} = \)

   f. \( \lim_{x \to 3} \frac{4x - 12}{x^3 + 2x - 15} = \)
Practice Problems

1. Evaluate the limits using tables. Verify your answer using the graph of the function.

   a. \[ \lim_{{x \to +\infty}} \frac{4x^2 + 5x}{2x^2 - 8} \]

   \[
   \begin{array}{c|c}
   x & f(x) \\
   \hline
   1.99 & \\
   1.999 & \\
   1.9999 & \\
   \end{array}
   \]

   b. \[ \lim_{{x \to 2}} \frac{4x^2 + 5x}{2x^2 - 8} \]

   \[
   \begin{array}{c|c|c|c}
   x & f(x) & x & f(x) \\
   \hline
   1.99 & & 2.01 & \\
   1.999 & & 2.001 & \\
   1.9999 & & 2.0001 & \\
   \end{array}
   \]

2. Evaluate the limits using the graph of a function or a table of values.

   a. \[ \lim_{{x \to 0}} \frac{x - 3}{x - 1} = \]

   b. \[ \lim_{{x \to 2}} \frac{1}{x - 2} = \]

   c. \[ \lim_{{x \to \infty}} \frac{6x^2 + 5x + 100}{3x^2 - 9} = \]

   d. \[ \lim_{{x \to -\infty}} \frac{x^2 - 1}{-2x^2 + 3x + 1} = \]
3. The graph of the function $f(x)$ is shown below. Use it to evaluate the following.

![Graph of the function $f(x)$]

- **a.** \(\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = f(3) = \)**
- **b.** \(\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = \lim_{x \to 5} f(x) = f(5) = \)**
- **c.** \(\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = f(0) = \)**
- **d.** \(\lim_{x \to -5^-} f(x) = \lim_{x \to -5^+} f(x) = \lim_{x \to -5} f(x) = f(-5) = \)**
- **e.** \(\lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x) = \lim_{x \to -2} f(x) = f(-2) = \)**
4. Evaluate the limits algebraically.

   a. \( \lim_{x \to 2} \frac{x^2 + 3x + 1}{x^2 - 2} = \)

   b. \( \lim_{x \to 1} \frac{x^2 - 4x + 3}{x - 1} = \)

   c. \( \lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = \)

   d. \( \lim_{x \to +\infty} \frac{2x^2 + 3x + 1}{4x^2 - 2} = \)

   e. \( \lim_{x \to -\infty} \frac{2x^2 + 3x + 1}{4x^2 - 2} = \)

   f. \( \lim_{x \to +\infty} \frac{3x + 1}{x^2 - 2} = \)

   g. \( \lim_{x \to -\infty} \frac{3x + 1}{x^2 - 2} = \)

   h. \( \lim_{x \to +\infty} \frac{x^2 + x - 1}{7x^2 + 1} = \)

   i. \( \lim_{x \to -\infty} \frac{x^3 + x - 1}{7x^2 + 1} = \)
2.3: The Derivative

Learning Objectives

- Find the derivative of a function at a point, \( f'(a) \) using the following methods
  - numerical analysis of the average rate of change over small intervals
  - the graph of the function
  - the formal definition of derivative
- Find the derivative function \( f'(x) \) using the formal definition of derivative
- Determine how the graph of the derivative function \( f'(x) \) is related to the graph of the function \( f(x) \)
- Read and understand derivative notation

Definitions: Derivative at a point, derivative function, Leibniz notation, Lagrange notation

Introduction

Our goal is to find a way to find an exact calculation of the instantaneous rate of change. As we previously discussed, we have techniques to calculate an average rate of change. We can use the average rate of change to approximate the instantaneous rate of change.

Here is a quick review of what we have learned about average rate of change and instantaneous rate of change:

Average Rate of Change

- Measures the constant rate needed to acquire the change in \( f(x) \) for a given change in \( x \) (measured over an interval)
- Represented graphically as the slope of the secant line between two points.
- Calculated using \( \frac{f(b)-f(a)}{b-a} \) or \( \frac{f(a+h)-f(a)}{h} \)

Instantaneous Rate of Change

- Measures the rate of change at a point.
- Represented graphically as the slope of the tangent line at a single point.
- Estimated by taking the average rate of change over a very small interval.
Instantaneous Rate of Change: Using a Table

In Section 2.1, we learned how to approximate the instantaneous rate of change of a grapefruit by calculating the average rate of change over smaller and smaller intervals. Let's review this procedure with the function \( f(x) = 2x^2 \)

1. Let \( f(x) = 2x^2 \) Let's approximate the instantaneous rate of change of the function at \( x = 1 \)

In order to approximate the instantaneous rate of change, we are going to use \( x = 1 \) as our starting point and calculate the average rate of change over intervals of smaller and smaller length.

<table>
<thead>
<tr>
<th>( h )</th>
<th>Average Rate of Change: ( \frac{f(1+h) - f(1)}{h} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>0.01</td>
<td></td>
</tr>
<tr>
<td>0.001</td>
<td></td>
</tr>
</tbody>
</table>
Instantaneous Rate of Change: Using Limits

We can represent this process using the limit of the difference quotient as \( h \) gets closer to 0. In order to calculate this limit, we will need to expand the numerator and simplify the rational expression.
Derivative at a Point

In the previous example, we looked at how to apply the limit concept to the average rate of change over an interval to find an instantaneous rate of change at a single point. In mathematics, we call the instantaneous rate of change the derivative.

Now that we have seen how to calculate the derivative at a point, we introduce the definition.

The derivative of $f$ at $a$ is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- We say “$f$ prime of $a$”
- $f'(a)$ is also called the instantaneous rate of change of $f$ at $a$.
- We can represent this graphically as the slope of the tangent line through the point on the graph of $f$ where $x = a$.

2. Determine the derivative of the function $f(x) = x^2 - x$ at $x = -1$
Derivative as a Function

Previously, we have been calculating the derivative of the function at a point \( x = a \). It turns out that the derivative of a function can be calculated at every point in the function's domain. In fact, the derivative is a function that takes in an input and gives an output. In this case, the outputs of the derivative are the slopes of the original function!

We are going to examine the derivative as a function, what it tells us about a function, and begin to interpret the information it gives us \( x \).

The derivative of \( f \) is defined as

\[
f'(x) = \]

3. Recall the Example \( f(x) = 2x^2 \). Find the derivative of the function \( f \) for all values of \( x \) in the domain of \( f \).
Graph of the Derivative Function

Remember that the derivative gives us the rate of change of the function, so all the information we've discussed regarding the rate of change and function behavior still hold. To re-emphasize this, we will start by taking a look at the graph of a function with the graph of its derivative.

Let's take a look at a familiar function: \( f(x) = 2x^2 \)

Below is the graph of the function \( f \)

![Graph of the function f](image)

Below is the graph of the function \( f' \)

![Graph of the function f'](image)

Note the following

\[
\begin{align*}
f(x) \text{ is increasing when } f'(x) \text{ is } & \underline{ \text{__________} } . \\
f(x) \text{ is decreasing when } f'(x) \text{ is } & \underline{ \text{__________} } . \\
f(x) \text{ has a turning point when } f'(x) \text{ is } & \underline{ \text{__________} } . 
\end{align*}
\]
Practice Problems

1. Let $f(x) = 3x^2 - 1$
   a. Use the definition of derivative to compute $f'(2)$.
   
   b. Use the definition of derivative to compute $f'(x)$. Use the formula for $f'(x)$ to verify that your answer in part (a) is correct.
2. The graph of a function $f$ is shown below together with the tangent line at the point $x = 3$.

![Graph of function with tangent line at x = 3]

a. Estimate the derivative of $f$ at $x = 3$

b. Find all values of $x$ such that $f'(x) = f'(3)$

c. Find all values of $x$ such that $f'(x) = 0$

d. (Circle one of the following) Is the slope of the secant line from $x = 3$ to $x = 3.5$
   a) greater than $f'(3)$
   b) less than $f'(3)$
   c) equal to $f'(3)$
Determine the intervals where the derivative of the function is positive and negative. Match the graph of each function (a, b, c, d) with the graph of its derivative (i, ii, iii, iv)

(a)

(b)

(c)

(d)

(i)

(ii)

(iii)

(iv)
2.5.A: Derivatives of Formulas – Building Blocks

Learning Objectives

- Power Rule, Constant Multiple Rule, Sum/Difference Rule, Exponential Functions, Natural Logarithm

Derivative of a Constant Function

Can we easily determine the derivative of a constant function?

Consider the function \( f(x) = 2 \). Sketch graph of \( f \):

What is the slope of the function?

So, given \( f(x) = 2 \), what is \( f'(x) \)?

<table>
<thead>
<tr>
<th>Derivative of a Constant Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( f(x) = c ), the ( f'(x) = 0 )</td>
</tr>
</tbody>
</table>

1. Find the derivative of the following functions

   a. \( f(x) = 7 \)  
   b. \( y = -3 \)
Derivative of a Linear Function

What makes a function linear?

Derivative of a Linear Function

\[ f(x) = mx + b, \text{ the } f'(x) = m \]

2. Find the derivative of the function \( y = -2x + 7 \).

Elementary Derivative Rules

Power Rule

\[ f(x) = x^n, \text{ the } f'(x) = nx^{n-1} \quad \text{where } n \text{ is a real number and } n \neq 0 \]

3. Find the derivative of the following functions

a. \( f(x) = x^5 \)

b. \( y = \sqrt[3]{x} \)

c. \( f(x) = \frac{1}{x^2} \)
Constant Multiple Rule

If \( f \) is a differentiable function and \( c \) is a constant, then
\[
\frac{d}{dx} [c \cdot f(x)] = c \cdot \frac{d}{dx} [f(x)] = c \cdot f'(x)
\]

Sum and Difference Rules

If \( f \) and \( g \) are differentiable functions and \( c \) is a constant, then
\[
\frac{d}{dx} [f(x) \pm g(x)] = \frac{d}{dx} [f(x)] \pm \frac{d}{dx} [g(x)] = f'(x) \pm g'(x)
\]

4. Find the derivative of the following functions

a. \( f(x) = 3x^2 \)

b. \( f(x) = \frac{1}{2} x \)

c. \( f(x) = x^4 - 5x^2 \)

d. \( f(x) = \frac{1}{8} x^4 + 2x + 5 \)

e. \( f(x) = \frac{4}{x^3} \)

f. \( f(x) = -12x^{0.25} \)
Derivative of Exponential Functions

An exponential function has the form $y = ab^x$ where $a$, the initial value, is a non-zero real number and $b$, the base, is a positive real number.

**Derivative of an Exponential Function**

$$\frac{d}{dx}(b^x) = b^x \ln(b) \quad \text{where } b \text{ is a real number and } b > 0$$

**Derivative of the Natural Exponential Function**

$$\frac{d}{dx}(e^x) = e^x$$

5. Find the derivative of the following functions

a. $f(x) = 5^x$

b. $y = 3e^x$

Derivative of Logarithmic Functions

A logarithmic function has the form $y = \log_b(x)$ where $b$, the base, is a positive real number.

**Derivative of a Logarithmic Function**

$$\frac{d}{dx}[\log_b(x)] = \frac{1}{x \ln(b)} \quad \text{where } b \text{ is a real number and } b > 0$$

**Derivative of the Natural Logarithmic Function**

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

6. Find the derivative of the following functions

a. $f(x) = 2\log_5(x)$

b. $f(x) = 2\ln(x)$
Equation of a Tangent Line

7. Find the equation of the tangent line to the graph of \( f(x) = 6x + 2 - 3e^x \) at \((0, -1)\).

8. Find the equation of the tangent to the graph of \( f(x) = 3x^3 - 3x \) at \( x = -1 \).
Summary of Formulas

Power Rule: \( \frac{d}{dx} (x^n) = nx^{n-1} \)

Special cases: \( \frac{d}{dx} (c) = 0 \) and \( \frac{d}{dx} (x) = 1 \)

Constant Multiple Rule: \( \frac{d}{dx} [c \cdot f(x)] = c \cdot f'(x) \)

Sum/Difference Rule: \( \frac{d}{dx} [f(x) \pm g(x)] = f'(x) \pm g'(x) \)

Exponential Function: \( \frac{d}{dx} [b^x] = b^x \cdot \ln b \)

Natural Exponential Function: \( \frac{d}{dx} [e^x] = e^x \)

Exponential Function: \( \frac{d}{dx} [\log_b x] = \frac{1}{x \cdot \ln b} \)

Natural Exponential Function: \( \frac{d}{dx} [\ln(x)] = \frac{1}{x} \)
Practice Problems

Find the derivative of the following functions

1. \( f(x) = 3x - 0.5 \)
2. \( y = -2x \)
3. \( f(x) = x^{-1} \)
4. \( f(x) = \frac{1}{x^5} \)
5. \( y = \frac{4}{x^2} \)
6. \( f(x) = \sqrt{x} \)
7. \( f(x) = x^3 + x \)
8. \( f(x) = 4x^2 \)
9. \( f(x) = 3x^3 + 2x^{-1} + x \)
10. \( f(x) = 12x^3 - 6x^{0.5} + 1 \)
11. \( f(x) = 2x^2 + \frac{4}{x^2} \)
12. \( f(x) = \frac{1}{\sqrt{x}} \)
13. \( f(x) = 2^x \)
14. \( f(x) = 5e^x \)
15. \( f(x) = \log_3(x) \)
16. \( f(x) = 6 \ln(x) \)
17. Find the equation of the tangent to the graph of \( f(x) = x^3 \) at \((-1, -1)\).
18. Find the equation of the tangent to the graph of \( f(x) = x^2 + 1 \) at \( x = 3 \).
2.5.B: Derivatives of Formulas – Product, Quotient, and Chain Rule

Learning Objectives

- Product Rule, Quotient Rule, Chain Rule

Introduction

In the last section we began learning methods for taking derivatives. We are going to continue to compile rules that will allow us to take derivatives of various functions. In this section we are going to look at functions that can be written as a product, a quotient, or a composition of two functions.

Often times we have a function that can be considered a product of two (or more) functions.

Consider the function \( y = 2t^2(t^3 + 4t) \)

One strategy for finding the derivative of \( y \) is to expand and simplify the function, then take the derivative using the rules from the previous lesson:

\[ y = \]

\[ y' = \]

Let's investigate another strategy. Can we simply multiply the derivatives of \( 2t^2 \) and \( t^3 + 4t \) together to find the derivative of \( y \)?

Is this true: \( y' = \)
The Product Rule

If $f$ and $g$ are differentiable functions, then

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

For the following exercises, find $f'(x)$

1. $y = (4x - 3)(2x^2 + 3x + 5)$
2. $y = (3x^2 - 2x + 7)e^x$
3. \( y = (2x^2 - 5)(3x + \sqrt{x}) \)

The Quotient Rule

If \( f \) and \( g \) are differentiable functions, then

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}
\]

For the following exercises, find \( f'(x) \)

4. \( y = \frac{5x^2 - 1}{2x^3 + 3} \)
5. \[ y = \frac{4e^x}{2x^2+1} \]

The Chain Rule

Review: Function Composition

6. If \( f(x) = x^3 \) and \( g(x) = x^2 + 4x + 1 \), find the following

   a. \( f(g(x)) \)

   b. \( g(f(x)) \)

7. Given \( h(x) = (4x^2 + 2x)^3 \), find \( f(x) \) and \( g(x) \) such that \( h(x) = f(g(x)) \)
The Chain Rule

If $f$ and $g$ are differentiable functions, then

$$\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x)$$

8. Find the derivative of $f(x) = (2x + 3)^7$

The Chain Rule in Differential Notation

If $y$ is a differentiable function of $u$ and $u$ is differentiable function of $x$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

9. Find the derivative of $f(x) = (2x + 3)^7$ using differential notation
For the following exercises, find $f''(x)$

10. $f(x) = \sqrt{x^3} - 4$

11. $f(x) = \frac{1}{(3x-2)^3}$

12. $f(x) = 3e^{\sqrt{x}}$
Practice Problems

For the following exercises, find \( f'(x) \)

1. \( y = (x^{-6} - 4)(6x^2 + 9x - 7) \)
2. \( y = (8x^3 - 5x^8)(3e^x + 2) \)
3. \( y = (2x^3 - 1)^2 \)
4. \( y = (4x^2 - 3x) \cdot \ln(x) \)
5. \( y = \frac{7x+2}{3x+5} \)
6. \( y = \frac{13x^2-15x+15}{-12x-2} \)
7. \( y = \frac{e^x}{2-5x} \)
8. \( y = (x^3 + 2x - 1)^7 \)
9. \( y = (3x + 8)^{-3} \)
10. \( y = \sqrt{9x^3 - x} \)
11. \( y = 3e^{x^2+4x} \)
12. \( y = \ln(4x^2 - 3x) \)
2.5.C: Derivatives of Formulas – Applications

Learning Objectives

- Describe the meaning of a derivative using various contexts.
- Use the derivative to find the marginal cost, marginal revenue, and marginal profit.

Introduction

Now that we have learned the formulas for finding the derivative of various functions, we can further investigate the information the derivative provides us.

In this lesson, we will see how to apply the derivative to determine the marginal cost, marginal revenue, and marginal profit as well as how to interpret our findings. Then, we will spend some more time interpreting the derivative using various contexts.

Marginal Analysis

Recall: Profit = ______________ - ______________

______________________________ is the derivative of the cost function and represents the cost to produce one more unit.

______________________________ is the derivative of the revenue function and represents the earnings made when you produce one more unit.

______________________________ is the derivative of the profit function and represents the increase in profit from producing one more unit.
1. Let $P(x) = -0.002x^2 + 4.5x - 120$ be the profit, measured in hundreds of dollars, where $x$ is the number of computers produced and sold.

   a. Determine $P(500)$ and interpret.

   b. Determine the marginal profit at $x = 500$ and interpret.

   c. Estimate the total profit after 501 computers are sold.

   d. Estimate the total profit after 502 computers are sold using the marginal profit at $x = 500$. 
2. For the given cost function $C(x) = 62500 + 500x + x^2$ find the following:

a. The cost at a production level of 1050 items

b. The average cost at a production level of 1050 items

c. The marginal cost at a production level of 1050 items
3. Let \( P(t) \) be the average monthly Social Security benefit for men, measured in dollars, where \( t \) is measured in years since 1940.

   a. Interpret the meaning of \( P(60) = 959.79 \)

   b. Interpret the meaning of \( P'(60) = 36.33 \)

4. The temperature in degrees Fahrenheit of a cup of coffee placed on a desk is given by \( f(t) \) where \( t \) is the minutes since the coffee was poured into the cup.

   a. Interpret the meaning of \( f(5) = 90 \)

   b. Interpret the meaning of \( f'(5) = -3 \)
Practice Problems

1. The cost in dollars to refurbish $x$ iPods in a month is calculated to be $C(x) = 0.25x^2 + 40x + 1000$. You charge customers $80 per iPod for the work.

   a. Find the marginal profit function.

   b. Compute the profit and marginal profit if you have refurbished 20 iPods in a month. Interpret the results.
2. For the given cost function $C(x) = 62500 + 300x + x^2$ find the following:

a. The cost at a production level of 1500 items

b. The average cost at a production level of 1500 items

c. The marginal cost at a production level of 1500 items
2.6: Second Derivative and Concavity

Learning Objectives

- Use derivative rules to find higher order derivatives.
- Use the second derivative to determine the concavity of a function and points of inflection
- Describe the meaning of a second derivative using the concept of acceleration.

Definitions: second derivative, increasing rate, decreasing rate, concave up, concave down, inflection point.

Introduction

Now that we have learned the formulas for finding the derivative of various functions, we can further investigate the information the derivative provides us.

In this lesson, you will learn how to find the second derivative of a function and how to use the second derivative to determine the shape of the original function's graph.

We have learned that the first derivative is the rate of change of a function and determines where the function is increasing and decreasing. In this lesson, we will review these concepts and build on them using the second derivative. In particular, we will see that we can use the derivative of the derivative to describe the way the rate of change changes just as we use the derivative of a function to describe the way the function change.
The Second Derivative

The _______________________________ is the derivative of the first derivative.

1. For \( f(x) = 2x^7 - 3x^3 + 8x - 2 \), determine the first and second derivative.

2. For \( f(x) = 4x^5 - \frac{1}{x^3} \), determine the first and second derivative.

3. Given \( f(x) = 2x^4 - 5e^x \), find the following
   a. \( f'(x) \)
   b. \( f'(1) \)
   c. \( f''(x) \)
   d. \( f''(1) \)
4. Find the second derivative of $f(x) = x^{2.1} - 3x^{1.2}$

5. Given $y = \sqrt[3]{x}$, find $\frac{d^2y}{dx^2}$

**Application of the Second Derivative: Acceleration**

6. The function $s(t) = -4.9t^2 + 15t$ gives the height of a projectile in meters after $t$ seconds. Determine the height, velocity, and acceleration of the projectile after 3 seconds.
Concavity and Inflection Points

How is \( f'(x) \) changing when \( f''(x) > 0 \)?

How is \( f'(x) \) changing when \( f''(x) < 0 \)?

Concave Up

- The function has an ________________ rate of change, so ________________.
- The graph opens upward

Concave Down

- The function has a ________________ rate of change, so ________________.
- The graph opens downward
Inflection Point

- Where a function’s rate of change goes from increasing to decreasing (or decreasing to increasing), so ________________.
- When the graph changes concavity.
Practice Problems

For the following exercises, find the second derivative of the function.

1. \( f(x) = 5x^2 + 3x - 2 \)
2. \( f(x) = 3e^x \)
3. \( f(x) = 8 + \frac{3}{x} \)
4. \( f(x) = \frac{2}{x^2} \)
5. \( f(x) = 2x^{3.2} - 3x^{-3.2} \)

6. Let \( s(t) = 8t^3 - 24t^2 - 360t \) be the equation of position, in meters, for a particle after \( t \) seconds. Find a function for the velocity and the acceleration of the particle. Include units with each function.

7. The function \( s(t) = 100 - 16t^2 \) gives the height of an object in feet after \( t \) seconds. Determine the velocity and acceleration of the object. Include units with each function.

8. The position of a particle moving in a straight line is given by \( s(t) = 80t - 7t^3 \).
   a. Find an expression for the particle’s acceleration.
   b. Is the particle’s velocity increasing or decreasing when \( t = 1 \)?

9. The position of a particle moving in a straight line is given by \( s(t) = t^3 - t^2 \).
   a. Find an expression for the particle’s acceleration.
   b. Is the particle’s velocity increasing or decreasing when \( t = 1 \)?
Given the graph of a function, determine (a) the exact location of the inflection point and (b) the intervals where the function is concave up and concave down.

10.  

11.  

12. Use the graph below to determine the sign of $f'(x)$ and $f''(x)$ at each $x$ value.

   a. At $x = -3$:

   b. At $x = -1$:

   c. At $x = 1$:

   d. At $x = 3$: 
2.7: Optimization

Learning Objectives

- Use derivatives (first and second order) to find maximum and minimum values of a function

Definitions: local maximum/minimum, local extreme, optimization, critical number, critical point, global maximum/minimum, global extreme

Local Maxima and Minima

Consider \( f(x) = (x - 1)^2 \)

Describe the behavior of the function \( f(x) \)

Local extrema denote the points were the output of a function goes from increasing to decreasing or from decreasing to increasing. There are two kinds of a local extrema: local maximum and local minimum.

The function \( f(x) \) has a __________________________ at \( a \) if \( f(a) \geq f(x) \) for all \( x \) near \( a \).

The function \( f(x) \) has a __________________________ at \( a \) if \( f(a) \leq f(x) \) for all \( x \) near \( a \).
Finding Maxima and Minima of a Function

The extrema occur at places where the derivative changes sign. Now, that can happen where the derivative is zero or where the derivative does not exist. Both of these cases are considered critical points.

If $a$ is in the domain of $f$ and either $f'(a) = 0$ or $f'(a)$ does not exist, then $a$ is called a __________________________ of $f$.

The point $(a, f(a))$ is called a __________________________.

Examples: Finding Critical Points

1. Find the critical point(s) of $f(x) = 18x - \frac{2}{3}x^3$

Verify your answer using the graph of $f(x)$
2. Find the critical point(s) of the function \( f(x) = x^3 - 6x^2 + 9x + 2 \)

3. Find the critical point(s) of \( f(x) = 2x^3 - 30x^2 + 126x - 1 \)
The First Derivative Test

When we find a critical point, is there a way we can determine if the critical point occurs at a maximum, minimum, or neither WITHOUT looking at the graph of the function?

The First Derivative Test for Local Extrema

Suppose that \( c \) is a critical point of a continuous function \( f(x) \). Then

1. If \( f'(x) \) changes from positive to negative at \( x = c \), then \( f \) has a **local maximum** at \( c \).
2. If \( f'(x) \) changes from negative to positive at \( x = c \), then \( f \) has a **local minimum** at \( c \).
3. If \( f'(x) \) does not change sign at \( x = c \), then \( f \) does not have a local maximum or minimum at \( c \).

4. Use the First Derivative Test to find and classify the local extrema of the function

\[
f(x) = x^3 - 6x^2 + 9x + 2
\]
5. Use the First Derivative Test to find and classify the local extrema of the function

\[ f(x) = 3x^4 - 4x^3 - 12x^2 + 5 \]

6. Use the First Derivative Test to find and classify the local extrema of the function

\[ f(x) = x^5 + 5x^4 - 20x^3 + 4 \]
7. Use the First Derivative Test to find and classify the local extrema of the function
\[ f(x) = \frac{x^3}{4} - 3x \]

8. Use the First Derivative Test to find and classify the local extrema of the function
\[ f(x) = 12x^5 + 15x^4 - 40x^3 \]
Concavity and Inflection Points

Recall the following information about concavity from section 2.6:

If \( \frac{d^2y}{dx^2} > 0 \) for every \( x \) in an interval, then \( f \) is concave up on that interval.

If \( \frac{d^2y}{dx^2} < 0 \) for every \( x \) in an interval, then \( f \) is concave down on that interval.

We say \( c \) is an inflection point of the function \( f \) if the concavity of the graph changes at \( c \). So \( f''(c) = 0 \) or \( f''(c) \) does not exist.

9. Determine where the graph of the function \( f(x) = 2x^3 - 36x^2 + 120x + 10 \) is concave up, concave down, and state any points of inflection.

10. Determine where the graph of the function \( f(x) = x^4 - 4x^3 + 3 \) is concave up, concave down, and state any points of inflection.
The Second Derivative Test

The Second Derivative Test for Local Extrema

Suppose $c$ is a critical point of $f$ such that $f'(c) = 0$ and $f''(c)$ exists.

1. If $f''(c) < 0$, then $f$ is concave down and has a **local maximum** at $c$.
2. If $f''(c) > 0$, then $f$ is concave up and has a **local minimum** at $c$.
3. If $f''(c) = 0$, then $f$ may have a local maximum, minimum, or neither at $c$.

11. Use the Second Derivative Test to find and classify the local extrema of the function
   \[ f(x) = -x^3 + 3x^2 - 4 \]

12. Use the Second Derivative Test to find and classify the local extrema of the function
   \[ f(x) = 18x - \frac{2}{3}x^3 \]
13. Use the Second Derivative Test to find and classify the local extrema of the function
\[ f(x) = 4x^3 - 15x^2 - 18x + 12 \]

14. Use the Second Derivative Test to find and classify the local extrema of the function
\[ f(x) = x^4 - \frac{4}{3}x^3 \]
**Global Maxima and Minima**

Global extrema are the greatest and smallest values a function can achieve over the entire domain.

<table>
<thead>
<tr>
<th>Global Extrema</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$ has an <strong>global maximum</strong> at $a$ if $f(a) \geq f(x)$ for all $x$ in the domain of $f$.</td>
</tr>
<tr>
<td>$f$ has an <strong>global minimum</strong> at $a$ if $f(a) \leq f(x)$ for all $x$ in the domain of $f$.</td>
</tr>
</tbody>
</table>

15. Find and classify the global extrema of the function $f(x) = x^3 - 3x^2$ on $[0, 4]$.

16. Find and classify the absolute extrema of the function $f(x) = \frac{1}{4}x^4 - x^3$ on $[-1, 4]$. 
Learning Objectives

- Use Find maximum and minimum values of application problems.

Applied Optimization

The process of finding maxima or minima is called optimization.

**Steps for Solving Optimization Problems**

1. **Identify the unknown(s), possibly with the aid of a diagram.**
   Assign variables for the unknown quantities as well as the quantity that is to be optimized.
   Think about: What is given? What are we trying to optimize? Are there any constraints?

2. **Identify the objective function.**
   This is the quantity you are asked to maximize or minimize.

3. **Identify the constraint(s).**
   These can be equations relating variables or inequalities expressing limitations.

4. **If applicable, eliminate extra variables in the objective function.**
   Solve the constraint equation for one variable and substitute into the objective function.

5. **Find the absolute maximum or minimum of the objective function.**
   Find the critical point(s). Test the critical points and endpoints.

6. **Interpret your answer using the context of the problem.**
Fencing an Area

1. A fence is to be built adjacent to building using the building as one side of the fence. The total area to be enclosed by the fence and building is 10,800 square feet. The fencing parallel to the building costs $3 per foot while the fencing for the sides perpendicular to the building is $2 per foot. The company wants the dimensions of the fence that will minimize its costs.
2. A rectangular field is going to be enclosed by a fence. Fencing costs $4 per foot for two opposite sides and $5 per foot for the other two sides. What are the dimensions of the field that will minimize the cost to enclose an area of 800 square feet?
3. A rectangular field with an area of 800 square meters is going to be enclosed by a fence. The field has a building on one side, so we only need to fence three sides. What are the dimensions of the field that will minimize the amount of fencing needed?
4. A rectangular field is going to be enclosed by a fence. One side of the field is against a mountain and will not need any fencing. What are the dimensions of the field that will give us the largest area to enclose if we want to use all 600 feet of fencing?
5. If you have 1,200 yards of fencing available, what is the largest rectangular enclosed field you can make if you only need to fence three sides due to a cliff?
Maximizing Profit

6. Suppose the Sunglasses Hut Company has a profit function given by

\[ P(q) = -0.03q^2 + 3q - 36 \]

where \( q \) is the number of thousands of pairs of sunglasses sold and produced, and \( P(q) \) is the total profit, in thousands of dollars, from selling and producing \( q \) pairs of sunglasses.

a. How many pairs of sunglasses (in thousands) should be sold to maximize profits? (If necessary, round your answer to three decimal places).

b. What is the maximum profit (in thousands) that can be expected?
7. The revenue (in dollars) for selling $x$ items is given by $R(x) = 60x - 0.5x^2$ and the costs (in dollars) of producing $x$ items is given by $C(x) = 3x + 8$. Find the quantity that gives the maximum profit.
Minimizing Average Cost

8. Suppose that \( C(x) = 4x^3 - 56x^2 + 1400x \) is the cost of manufacturing \( x \) items. Find a production level that will minimize the average cost of making \( x \) items.
9. The total cost of $x$ items is given by $C(x) = 78400 + 800x + x^2$. Find the production level that minimizes the average cost and find the minimum average cost.
2.11: Implicit Differentiation and Related Rates

Learning Objectives

- Find the derivative of an implicitly defined function.
- Solve related rates problems (application of implicit differentiation)

Definitions: Explicit function, implicit equation/function

Implicit Differentiation

So far in this course, the functions we have been given to differentiate were either given explicitly, such as \( y = x^2 + e^x \), or it was possible to get an explicit formula for them, such as solving \( y^3 - 3x^2 = 5 \) to get \( y = \sqrt[3]{5 + 3x^2} \).

However, there are equations relating \( x \) and \( y \) which are either difficult or impossible to solve explicitly for \( y \), such as \( y + e^y = x^2 \). Equations that are written in this form are implicitly defined equations.

We can still find the derivative of \( y \) with respect to \( x \) by using **implicit differentiation**.

1. Assume \( y \) is a function of \( x \). Calculate the following

   a. \( \frac{d}{dx} (5) \)

   b. \( \frac{d}{dx} (x^3) \)

   c. \( \frac{d}{dx} (3y) \)

   d. \( \frac{d}{dx} (y^7) \)

   e. \( \frac{d}{dx} (x^3y^2) \)
Examples: Implicit Differentiation

For the following exercises, find $\frac{dy}{dx}$ using implicit differentiation.

2. $2x^2 - 3y^3 = 5$

3. $y^3 + 5y - 2x^4 - x = 12$
4. \( y^3 + x^2 y^5 - x^4 = 27 \)

For the following exercises (a) find \( \frac{dy}{dx} \) using implicit differentiation, (b) find the slope of the tangent line at the given point, and (c) find the equation of the tangent line at the given point.

5. \( x^2 + 4y^2 = 20 \) \hspace{1cm} (2, 2)
6. \[ y + y^3 - x = 7 \] \((-5, 1)\)

7. \[ -3x^2 - 2xy - 2y^3 = -31 \] \((-3, 2)\)
Related Rates

In previous sections, we have seen how we can use mathematics to model phenomenon in the world around us. If a situation is modeled with a continuous function, we can use calculus to analyze how things are changing and then possibly use that information to make predictions.

So far in this course, we have considered how the output is changing with respect to a change in the input. Now we will consider the interaction between the rates of change of the input and output with respect to a third quantity. What we will see is that the relationship between the input and output is also reflected in the relationship between their rates of change (with respect to a third quantity). Thus we are going to look at the relationship between the rate of the input and the rate of the output.

For the following exercises, rewrite the statements using mathematical notation.

8. In 2015, the average price of children’s shoes is $15 and increasing by $2 per year.

9. The monthly costs for the television manufacturer are currently $32,840 and decreasing by $124 each unit produced.
A company's revenue from selling $x$ units of an item is given as $R(x) = 1600x - 2x^2$. If sales are increasing at the rate of 30 units per day, how rapidly is revenue increasing when 200 units have been sold?
11. The revenue for selling $x$ items is given by $R(x) = 80x - 0.25x^2$ and the cost is given by $C(x) = 5x + 8$. If sales are increasing at a rate of 40 items per day, how fast is the profit changing when 50 items have been sold?

12. A large sheet of ice in the shape of a circle has a radius of 350 miles in 2008 but is shrinking at a rate of 1.5 miles per year. How fast is the area changing in 2008?
13. If the radius of a beach ball is increasing at a rate of 1.5 cm per second, how fast is the volume changing when $r=5$ cm?

14. A spherical balloon is being inflated at a constant rate of 20 cubic inches per second. How fast is the radius of the balloon changing at the instant the balloon's radius is 6 inches?
3.1.A: Riemann Sums

Learning Objectives
- Approximate area under a graph using rectangles

Definitions: Left-endpoints, right-endpoints, Riemann sum

Introduction

1. Suppose a car travels on a straight road at a constant speed of 65 miles per hour for three hours.
   a. How far has the car gone?

   In this example, the function represents ____________________________ and the area represents the ____________________________.

   b. Where in the graph can we see the total distance traveled?

In chapter 3, we will be investigating how to find the accumulation of a quantity when we are given information about the rate of change.
Determining the Accumulation of a Quantity

How do we approximate the area if the rate of change function is not linear or geometric shapes we are familiar with?

Since we can easily calculate the area of a rectangle, we can use rectangles to get an approximation of the area under the curve.

However, we have a few choices on how to create the rectangles and we need to be consistent. Our rectangles must not overlap, have a base on the x-axis, and a top that intersects with the curve. It is easiest to choose rectangles that (a) have all the same width, and (b) have their heights taken from the same edge of the curve.

The following example reviews this process using a context.
Example: Braking for a Rabbit

2. You are speeding down a road when you spot a rabbit sitting in the middle of the road 400 feet directly ahead of you. You immediately apply the brakes. At the time you apply the brakes you are traveling 100 ft/sec (approximately 68 mph). Your velocity decreases throughout the 10 seconds it takes you to stop the car. Do you hit the rabbit?

Here is some information that may help you:

The speed of the car, in feet per seconds, is given by \( v(t) = (t - 10)^2 \), where \( t \) is the number of seconds since the brakes were applied.

The following table shows the velocity of the car every 2 seconds since you started braking.

<table>
<thead>
<tr>
<th>Time since brakes applied (seconds)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity (feet per second)</td>
<td>100</td>
<td>64</td>
<td>36</td>
<td>16</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ a. \] Estimate the furthest distance the car could have traveled in the 10 seconds after the brakes were applied. Show this accumulation of distance traveled in the graph.
b. Estimate the shortest distance the car could have traveled in the 10 seconds after the brakes were applied. Show this accumulation of distance traveled in the graph.

<table>
<thead>
<tr>
<th>Time since brakes applied (seconds)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity (feet per second)</td>
<td>100</td>
<td>64</td>
<td>36</td>
<td>16</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>


c. What can we conclude about the distance the car traveled?

d. How can we get a better approximation of the distance the car traveled in the 10 seconds?
Riemann Sum

When we approximate the area between the curve and x-axis using rectangles, we are calculating a Riemann Sum.

When computing a Riemann sum, we need to know the width of the rectangles as well as the height of the rectangles. In general, it does not matter where you calculate the function height, it just needs to be the function evaluated at some point in the interval that rectangle is on.

Suppose we want to calculate the area under the graph of a positive function \( f \) and the x-axis on the interval \([a, b]\). We can create rectangles with bases that span the interval \([a, b]\) and sides that reach up to the graph of \( f \).

To approximate the area we are interested in, we can calculate the areas of each rectangle then add up all the areas. The sums of the areas of the rectangles is called a **Riemann Sum**.

\[
\text{Riemann Sum}
\]

If we want \( n \) rectangles over the interval \([a, b]\)

- the width of each rectangle is
- the height of each rectangle is

The area between the curve and x-axis approximation is given by
Left-endpoints

When using left-endpoints, the height of each rectangle comes from the function value at the left edge of the rectangle.

**Approximating with Rectangles using Left-endpoints**

If we want $n$ rectangles over the interval $[a, b]$, the width of each rectangle is $\Delta x = \frac{b-a}{n}$.

If the left-endpoints are given by $x_1, x_2, \ldots, x_{n-1}$, then the area approximation using left-endpoints is given by

Recall Example:

You are speeding down a road when you spot a rabbit sitting in the middle of the road 400 feet directly ahead of you. After you apply the brakes, you are driving at a rate of $v(t) = (t - 10)^2$ feet per second.

Estimate the total distance your traveled over the 10 seconds you were applying the brakes. Use 5 rectangles and left-endpoints for your approximation.
Right-endpoints

When using right-endpoints, the height of each rectangle comes from the function value at the right edge of the rectangle.

**Approximating with Rectangles using Right-endpoints**

If we want \( n \) rectangles over the interval \([a, b]\), the width of each rectangle is \( \Delta x = \frac{b-a}{n} \).

If the right-endpoints are given by \( x_2, x_3, \ldots, x_n \), then the area approximation using right-hand endpoints is given by

**Recall Example:**

You are speeding down a road when you spot a rabbit sitting in the middle of the road 400 feet directly ahead of you. After you apply the brakes, you are driving at a rate of \( v(t) = (t - 10)^2 \) feet per second.

Estimate the total distance your traveled over the 10 seconds you were applying the brakes. Use 5 rectangles and right-endpoints for your approximation.
3. Approximate the area under the function $f(x) = \frac{4}{x}$ on the interval $[1, 5]$ using 4 rectangles and left-endpoints.

4. Approximate the area under the function $f(x) = \frac{4}{x}$ on the interval $[1, 5]$ using 4 rectangles and right-endpoints.

5. Liquid leaked from a damaged tank at a rate of $r(t)$ liters per hour. The rate decreased as time passed and the values of the rate at 2-hour time intervals are shown in the table below. Find the lower and upper estimates for the total amount of liquid that leaked out.

<table>
<thead>
<tr>
<th>$t$ (hours)</th>
<th>$r(t)$ (liters per hour)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8.6</td>
</tr>
<tr>
<td>2</td>
<td>8.3</td>
</tr>
<tr>
<td>4</td>
<td>8.1</td>
</tr>
<tr>
<td>6</td>
<td>7.8</td>
</tr>
<tr>
<td>8</td>
<td>7.6</td>
</tr>
<tr>
<td>10</td>
<td>7.3</td>
</tr>
</tbody>
</table>
3.1.B: The Definite Integral

Learning Objectives

- Evaluate a definite integral graphically
- Describe the meaning of a definite integral using complete sentences.

Definitions: Definite integral, integrand, limits of integration

Introduction

In the last section, we discussed estimating the area under the curve and explored methods to do so. In this section we are going to see how we can determine the exact value when we have the rate of change curve defined by a continuous function. In fact, if the region between the curve and the x-axis creates a geometric shape we are familiar with, we can easily compute the exact area.

Definition of the Definite Integral

So far, we have been able to approximate the area between a curvy function and the x-axis using rectangles. To find a better approximation of the area under the curve, we can take more rectangles. To get an EXACT value for the area under the curve, we will need to take infinitely many rectangles.

Definite Integral

Given a function $f(x)$ that is continuous on the interval $[a, b]$, we divide the interval into $n$ subintervals of width $\Delta x$ and from each interval choose a point, $x_i$.

Then the definite integral of $f(x)$ from $a$ to $b$ is

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x =$$

- $f(x)$ is called the ________________.
- $a$ and $b$ are called the_______________________________.
- $dx$ represents the infinitely small rectangle width that was represented by $\Delta x$ when the rectangle had a measurable width.
1. Evaluate $\int_{0}^{6} \left( \frac{3}{2}x - 3 \right) dx$ by interpreting it in terms of areas.

2. Evaluate $\int_{0}^{6} |x - 4| dx$ by interpreting it in terms of areas.
3. Use the graph of the function $f$ along with the areas of the shaded regions to evaluate the following integrals.

\[
\begin{align*}
\text{area} &= 5 \\
\text{area} &= 2 \\
\text{area} &= 3
\end{align*}
\]

- a. $\int_0^5 f(x)\,dx$
- b. $\int_3^7 f(x)\,dx$
- c. $\int_0^7 f(x)\,dx$

4. Use the graph of the function $f$ to evaluate the following integrals.

\[
\begin{align*}
\text{a. } & \int_0^2 f(x)\,dx \\
\text{b. } & \int_2^7 f(x)\,dx \\
\text{c. } & \int_0^9 f(x)\,dx
\end{align*}
\]
5. An object moves with velocity as given in the graph below (in feet per second). How far did the object travel from $t = 0$ to $t = 20$?

![Graph showing velocity over time](image)

6. For the following examples, the units are given for $x$ and $f(x)$. Give the units of $\int_{a}^{b} f(x)dx$.

   a. $x$ is time measure in hours and $f(x)$ is a rate measured in kilowatts per hour.

   b. $x$ is time measure in hours and $f(x)$ is a velocity measured in miles per hour.

   c. $x$ is time measure in days and $f(x)$ is a rate measured in revenue per day.
7. A city has found that the rate its population changes is given by $p(t)$, measured in people per year, where $t$ is the number of years since 2000. What is the interpretation of $\int_{12}^{17} p(t) \, dt = -3651$?

8. A manufacturing company found that the rate its revenue is changing, measured in hundreds of dollars per unit, can be modeled by the function $r(x)$, where $x$ is the number of units sold. What is the interpretation of $\int_{0}^{450} r(x) \, dx = 4.75$?
Using Technology

For exercises 9 and 10, use a graphing calculator to graph and evaluate the definite integral. There is no need to show work for these problems.

9. Use a graphing calculator to evaluate \( \int_{-2}^{2} (x^3 - 4x + 5) \, dx \)

10. Recall the example: You are speeding down a road when you spot a rabbit sitting in the middle of the road 400 feet directly ahead of you. You immediately apply the brakes. The speed of the car, in feet per seconds, is given by \( v(t) = (t - 10)^2 \), where \( t \) is the number of seconds since the brakes were applied. Do you hit the rabbit?

Use integrals and your calculator to find the exact distance you traveled while breaking.
3.2: The Fundamental Theorem and Antidifferentiation

Learning Objectives
- Evaluate definite integrals using the fundamental theorem of calculus

Definitions: An antiderivative, the antiderivative, indefinite integral

Introduction

In the last section we investigated how to determine the accumulation of a quantity over an input interval when given a rate of change.

To determine the exact amount of accumulation, we can create a definite integral. So far, we have computed the definite integral by calculating the area between the curve and the $x$-axis. In this section, we will learn how to evaluate a definite integral using the relationship between the rate of change function and the original function.

The Fundamental Theorem of Calculus

If $f(x)$ is a continuous function on the interval $[a, b]$ and $f(x) = F'(x)$, then

$$\int_{a}^{b} f(x) \, dx =$$

Example

Can you find a function $F$ such that $F'(x) = 2x$?

Can you find another function $F$ such that $F'(x) = 2x$?

Can you find yet another function $F$ such that $F'(x) = 2x$?
Antiderivatives

____________________________________ of the function $f(x)$ is the function $F(x)$ such that $f(x) = F'(x)$

________________________________________ of a function $f(x)$ is the family of functions, written $F(x) + C$, where $f(x) = F'(x)$ and $C$ represents any constant.

The antiderivative is also called the __________________ as is written as $\int f(x) \, dx$.

Consider the previous example:

Definite Integrals Versus Indefinite Integrals

In the last section, we explored the definite integral. Now, we have been introduced to the indefinite integral.

Definite Integral

- The definite integral has limits of integration.
- The result of a definite integral is a number.
- There is no need for $+C$

Indefinite Integral

- The indefinite integral does not have limits of integration.
- The result of an indefinite integral is a family of functions.
- We must have the $+C$, unless we have information to solve for the constant.
Examples

1. Find an antiderivative of $f(x) = 5x^4$

2. Find the antiderivative of $f(x) = 5x^4$

3. Given $\int_0^1 2x \, dx$ and $F(x) = x^2 + 5$
   a. Verify that $F(x)$ is an antiderivative of the integrand $f(x)$

   b. Use the Fundamental Theorem of Calculus to evaluate the definite integral.

4. Given $\int_0^3 (x^2 + 4x - 3) \, dx$ and $F(x) = \frac{1}{3}x^3 + 2x^2 - 3x$
   a. Verify that $F(x)$ is an antiderivative of the integrand $f(x)$

   b. Use the Fundamental Theorem of Calculus to evaluate the definite integral.
3.3: Antiderivatives of Formulas

Learning Objectives

- Determine indefinite integrals and antiderivatives of functions using integration formulas
  - Constant Multiple Rule, Sum or Difference Rule, Power Rule, Exponential Functions, Natural Logarithm
- Calculate and interpret the antiderivative using various contexts.

Antiderivative Rules

Power Rule:

Constant Multiple Rule:

Sum/Difference Rule:

Exponential Function:

Natural Exponential Function:

Reciprocal Function:
1. $\int x^4 \, dx$

2. $\int \frac{1}{x^3} \, dx$

3. $\int 5x^3 \, dx$

4. $\int 2x \, dx$

5. $\int (x^4 + x^5 - 1) \, dx$

6. $\int (x + 7)(x - 4) \, dx$
7. $\int (3x^2 - 2x + 1)dx$

8. $\int 3e^x dx$

9. $\int \frac{4}{x} dx$

10. $\int \sqrt{x} dx$

11. $\int \left(\frac{6}{x^4} + 2x - 5\right) dx$
Evaluating Definite Integrals

12. \( \int_{-1}^{3} (6x^4 + 2x^3) \, dx \)

13. \( \int_{1}^{5} \frac{2}{x} \, dx \)

14. \( \int_{1}^{4} (2x + 1) \, dx \)
15. $\int_2^3 (2x^3 + 3x^5) \, dx$

16. $\int_2^5 \left( \frac{1}{x^2} \right) \, dx$

17. You are speeding down a road when you spot a rabbit sitting in the middle of the road 400 feet directly ahead of you. You immediately apply the brakes. The speed of the car, in feet per seconds, is given by $v(t) = (t - 10)^2$, where $t$ is the number of seconds since the brakes were applied. Do you hit the rabbit?
18. A deep ocean oil rig has suffered a catastrophic failure, and oil is leaking from the ocean floor wellhead at a rate of \( v(t) = 0.08t^2 - 4t + 60 \) thousand barrels per day, where \( t \) is time in days since the failure. Compute the total volume of oil released during the first 20 days.

19. A company’s marginal cost function is given by \( C'(x) = 10 + x + \sqrt{x} \) where \( x \) is the number of units produced. Find the total cost to produce the first 50 units.
3.4: Substitution

Learning Objectives

- Determine indefinite integration using substitution
- Determine definite integration using substitution

Introduction

Review: Chain Rule

1. Find the derivative of \( f(x) = (x^3 + 4)^7 \)

2. Find the derivative of \( f(x) = e^{x^2+1} \)

3. Find the derivative of \( f(x) = 2\sqrt{x^4 - 1} \)
The Substitution Method for Antiderivatives

The goal is to turn $\int f(g(x)) \, dx$ into $\int f(u) \, du$, where $f(u)$ is an “easier” function to integrate.

1. Write $u$ as a function of $x$ (often, $u$ is the “inside” function).
2. Find the derivative $\frac{du}{dx}$ and solve for $dx$.
3. Substitute $u$ and the expression for $dx$ into the original integral.
   **Note:** all $x$-variables should be replaced or will cancel-out.
4. Evaluate the resulting integral.
5. Put the final expression in terms of $x$.

**Examples: Indefinite Integrals**

4. Evaluate $\int x^2(2x^3 - 1)^7 \, dx$
5. Evaluate $\int 2e^{-3x} \, dx$

6. Evaluate $\int \frac{2}{(x-4)^5} \, dx$
7. Evaluate $\int x^5 \sqrt{5 + x^6} \, dx$

8. Evaluate $\int_0^2 4x(3x^2 + 1)^2 \, dx$
Examples: Definite Integrals

9. Evaluate $\int_1^3 x(3x^2 - 9)^3 \, dx$

10. Evaluate $\int_0^3 x^2 e^{2x^3} \, dx$
11. Evaluate \( \int_0^2 \frac{dx}{3x+4} \)

12. Evaluate \( \int_1^2 \frac{1}{\sqrt{x+2}} \, dx \)
3.5: Integration by Parts

Learning Objectives
- Evaluate the antiderivative using integration by parts

Introduction

In the last section, we learned how to integrate composite functions, such as \( \int xe^{x^2} \, dx \)

Now consider the next example: \( \int x^2 e^x \, dx \)

There is no choice of \( u \) that will simplify this integral. We will need a new technique to solve integrals of this form.

Notice that \( x^2 e^x \) is the product of the two functions \( x^2 \) and \( e^x \). This will help us in identifying when to use this new technique - integration by parts.

Since we are going to use the fact that our integrand is a product between two functions, let's first remind ourselves of the product rule for derivatives.

Recall: Product Rule

\[
\frac{d}{dx} [f(x) \cdot g(x)] =
\]

In the last section, we saw how to undo the chain rule using the method of \( u \)-substitution. Now we will examine how to undo the product rule with the technique known as integration by parts.
Integration by Parts

- $u$ and $dv$ are the integrand functions.
- Once we determine the $u$ and $dv$ functions, we then calculate $du$ (the derivative of $u$) and $v$ (the antiderivative of $dv$) in terms of the input variable.
- Once we have the four functions ($u$, $v$, $du$, and $dv$) calculated, we put them into our formula.

Choosing $u$ and $dv$

Here are some guidelines for choosing $u$ and $dv$. Note that these are not set in stone, but will help as you learn the technique.

- When selecting $u$, select a logarithmic expression if one is present. If not, select an algebraic expression (like $x$ or $dx$).
- Whatever you let $dv$ be, you have to be able to find $v$ using integration.
- It helps if $du$ is simpler than $u$.
- It helps if $v$ is simpler than $dv$.

1. Evaluate $\int x^2 \ln(x) \, dx$
2. Evaluate \( \int \ln(x) \sqrt{x^5} \, dx \)

3. Evaluate \( \int 3xe^x \, dx \)
4. Evaluate $\int 2xe^{5x} \, dx$

5. Evaluate $\int_1^e 2x^2 \ln(x) \, dx$
3.6: Area Between Two Curves

Learning Objectives

- Determine the convergence of improper integrals
- Evaluate improper integrals that are convergent

Definitions: improper integral, convergent, divergent

Introduction

Previously, we have used definite integrals to find the area between the graph of a function and the horizontal axis. In this section, we will see how definite Integrals can also be used to find the area between two curves.

Consider the following problem:

Two objects start from the same location and travel along the same path. The velocity of Object A is given by \( v_A(t) = t + 3 \) m/sec and the velocity of object B is given by \( v_B(t) = t^2 - 4t + 3 \) m/sec.

How far ahead is object A after 3 seconds?
Area Between Two Curves

If \( f(x) \geq g(x) \) for all \( x \) in \([a, b]\), then the area of the region between the graphs of \( f(x) \) and \( g(x) \) and between \( x = a \) and \( x = b \) is given by

1. Find the area enclosed by \( y = 2x \) and \( y = 6x - x^2 \).
2. Find the area enclosed by $y = 2x$ and $y = 3x^2$.

3. Find the area enclosed by $y = x + 1$ and $y = x^2 - 4x + 5$. 
4. Find the area enclosed by $y = 2x^2$ and $y = x^2 + 7$. 
3.7: Applications to Business

Learning Objectives

- Find the equilibrium point
- Find the consumer surplus and producer surplus using area between two curves.
- Find the present value of an investment using integration.

Definitions: supply curve, demand curve, equilibrium point, consumer surplus, producer surplus, future value, present value

Supply Curve, Demand Curve, and Equilibrium Point

Economics tells us that in a free market, the price for an item is related to the quantity that producers will supply and the quantity that consumers will demand. An increase in price tends to increase the quantity supplied and decrease the quantity demanded. The graph below shows a demand curve and a supply curve for a product.

The supply curve indicates how many producers will supply the product (or service) of interest at a particular price. Notice, the supply curve is increasing – lower prices are associated with lower supply, and higher prices are associated with higher quantities supplied.

The demand curve gives the price per unit that the consumer is willing to pay for x units. Notice, the demand curve is decreasing – lower prices are associated with higher quantities demanded, higher prices are associated with lower quantities demanded.

The equilibrium point in the price and quantity for which suppliers are willing to supply and consumers are willing to buy. Equilibrium occurs at the intersection of the two functions, which is where supply equals demand.
Consumer and Producer Surplus

**Consumer Surplus** is the difference between the price that consumers pay and the price that they are willing to pay. On a supply and demand curve (as seen in the image below), it is the area between the demand curve and the equilibrium price.

**Producer Surplus** is the difference between the price suppliers would be willing to accept and the price they can receive by selling the item at the equilibrium price. On a supply and demand curve (as seen in the image below), it is the area between the equilibrium price and the supply curve.

Since consumer and producer surplus are both areas between two curves, we can calculate those values using definite integrals.

Given a demand function $D(x)$, a supply function $S(x)$, and the equilibrium point $(Q, P)$, we have the following:

\[
\text{Consumer Surplus} =
\]

\[
\text{Producer Surplus} =
\]
1. Given the demand function $D(x) = 100 - 0.3x$ and the supply function $S(x) = 0.5x$
   
   a. Find the equilibrium quantity

   b. Find the consumer surplus at the equilibrium point

   c. Find the producer surplus at the equilibrium point
2. Given the demand function \( D(x) = \frac{3200}{\sqrt{x}} \) and the supply function \( S(x) = 2\sqrt{x} \)
   a. Find the equilibrium quantity
   b. Find the consumer surplus at the equilibrium point
   c. Find the producer surplus at the equilibrium point
Introduction: Present Value

In a previous course, you learned about compound interest given that you made a single deposit into an account and let your one-time deposit sit undisturbed, earning interest, for some period of time. Recall:

Let $P$ be the principal amount (initial investment), $r$ be the annual interest rate (as a decimal), and $A$ be the amount in the account at the end of $t$ years.

If we are compounding continuously, we use the formula: $A(t) = Pe^{rt}$

What if instead of investing an amount of money one time, we are able to make continuous deposits into the account that is earning interest continuously.

We can use calculus to solve problems that involve continuous deposits flowing into an account that earns interest. As long as we can model the flow of income with a function, we can use a definite integral to calculate the present value of a continuous income stream.

Present Value

**Present Value**

Suppose money can earn interest at an annual interest rate of $r$, compounded continuously. Let $F(t)$ be a continuous income function (in dollars per year) that applies between year 0 and year $T$.

Then the **present value** of that income stream is given by

The present value is the one-time investment we would need to make today in order to get the same final amount from investing the continuous deposits over time.
3. Find the accumulated present value of an investment over a 10-year period if there is a continuous money flow of $4,800 per year and the interest rate is 2% compounded continuously.

4. A company is considering expanding their production capabilities with a new machine that costs $96,000 and has a projected lifespan of 8 years. They estimate the increased production will provide a constant $12,000 per year of additional income. Money can earn 0.7% per year, compounded continuously. Should the company buy the machine?
3.8: Improper Integrals

Learning Objectives

• Determine the convergence of improper integrals
• Evaluate improper integrals that are convergent

Definitions: improper integral, convergent, divergent

Introduction

When calculating definite integrals using the Fundamental Theorem of Calculus, we assumed
(a) the interval \([a, b]\) is finite and
(b) the function is continuous on the interval.

Consider the following example: Evaluate the integral of \(f(x) = \frac{2}{x^2}\) over the interval \([1, 5]\)

The function \(f(x) = \frac{2}{x^2}\) is shown in the graph below. Shade the region that represents the integral of the function from 1 to 5.
In this section, we will learn how to calculate an integral on an unbounded interval and how to calculate an integral of a function that is not continuous over the interval.

An improper definite integral, or improper integral for short, has one of the two properties:

1. The integral is on an unbounded interval.
2. The integrand function is not bounded at a point in the interval of integration.

1. An integral is on an unbounded interval if the upper or lower limit of integration is infinite.

2. An integrand function is not bounded at a point in the interval of integration if the function is undefined at a point in the interval. This is seen graphically as a vertical asymptote.
In order to evaluate the previous integrals, we will need to write the integral using a limit. Before we learn how to solve problems in this form, let's review some limit properties.

1. \( \lim_{x \to 4} \frac{1}{x} \)

2. \( \lim_{x \to \infty} \frac{1}{x} \)

3. \( \lim_{x \to 0^+} \frac{1}{x} \)

4. \( \lim_{x \to 0^-} \frac{1}{x} \)

5. \( \lim_{x \to 0} e^x \)

6. \( \lim_{x \to \infty} e^x \)

7. \( \lim_{x \to -\infty} e^x \)
Improper Integrals Over Unbounded Intervals

\[ \int_{a}^{\infty} f(x) \, dx = \]

\[ \int_{-\infty}^{b} f(x) \, dx = \]

- If the limit exists (equals a number), we say the improper integral is **convergent**.
- If the limit does not exist or is infinite, we say the improper integral is **divergent**.

8. \[ \int_{1}^{\infty} \frac{2}{x^2} \, dx \]

9. \[ \int_{-\infty}^{-3} e^{x} \, dx \]
10. \( \int_{1}^{\infty} x^2 \, dx \)

11. \( \int_{1}^{\infty} \frac{1}{x} \, dx \)

12. \( \int_{0}^{\infty} 0.5 e^x \, dx \)
Improper Integrals with Discontinuous Integrals

If \( f(x) \) is discontinuous at \( a \), \( \int_a^b f(x) \, dx = \)

If \( f(x) \) is discontinuous at \( b \), \( \int_a^b f(x) \, dx = \)

13. \( \int_0^6 \frac{1}{x^{0.8}} \, dx \)

14. \( \int_0^3 \frac{1}{x^{1.2}} \, dx \)