Lesson 12: Sequences and Series

Outline

Objectives:
I can determine whether a sequence has a pattern.
I can determine whether a sequence can be generalized to find a formula for the general term in the sequence.
I can determine whether a sequence is arithmetic or geometric.
I can determine the general terms of an arithmetic and geometric sequence.
I can determine the sum of a finite arithmetic or geometric series.
I can determine the sum of certain infinite geometric series.
I can use and interpret summation notation.

Definitions / Vocabulary / Graphical Interpretation:
Sequence characteristics:
- List of numbers written in a definite order
- Can be finite or infinite
- Does not have to have a pattern
- A function whose domain is the set of positive integers
- \( a_n \) is also known as the general term of the sequence

List some examples of sequences:

The Fibonacci Sequence:
How to create the Fibonacci Sequence: \{1, 1, 2, 3, 5, 8, 13, 21….\} by adding the previous two terms

A recursively defined sequence is a sequence where the general term is related to the previous terms.

Converging Sequences:
To determine whether or not a sequence converges we can look at what happens to the general term as \( n \) gets infinitely large.
Examples:

\[ a_n = \frac{2n}{n^2 - n} \] converges to:

\[ a_n = \frac{4n^2 + 2n}{2n^2 - 1} \] converges to:

\[ a_n = \frac{2n^3 - n^2 + 7}{3n^3 - 4n} \] converges to:

\[ a_n = \frac{e^n}{n^{45}} \] converges to:

\[ a_n = e^{-n} - 1 \] converges to:

Bounded Sequences:
If all the terms of a sequence lie between 2 numbers, say \( M \) and \( N \), such that the terms are never greater than \( M \) or less than \( N \), we say the sequence is **bounded**.

- A convergent sequence is bounded.
- However, a bounded sequence is not necessarily convergent. (A bounded sequence does not necessarily have a limit.)
- If a sequence is monotone (always increasing or always decreasing) and bounded, it converges.

Arithmetic Sequences:
A sequence that has a constant amount of increase or decrease between terms (the common difference) is called an arithmetic sequence. **The equation for the general term of an arithmetic sequence** is just a linear function with slope \( d \) (representing the common difference).

\[ a_n = a + (n - 1)d \quad n \geq 1 \]

Geometric Sequences:
A sequence that has a constant percent increase or decrease in terms (common ratio) is called a geometric sequence. **The equation for the general term of a geometric sequence** is an exponential function with base \( r \) (representing the common ratio).

\[ a_n = ar^{n-1} \quad n \geq 1 \]
Examples:

\[ a_n = \frac{(-1)^n 3^{n-1}}{4^{n+1}} \quad n \geq 0 \]

a) Write out the first 5 terms

b) Is the sequence arithmetic, geometric, or neither?

\[ a_1 = 10, \quad a_{n+1} = a_n + 2 \quad \text{for } n \geq 1 \]

a) Write out the first 5 terms

b) Is the sequence arithmetic, geometric, or neither?

c) Find an explicit formula for \( a_n \)

Series and Summation Notation:
Summation notation is a short-hand way of denoting the sum of the terms of a sequence. As with sequences, series can be finite or infinite.

Given a sequence \( \{a_n\}_{n=k}^\infty \) and numbers \( m \) and \( p \) satisfying \( k \leq m \leq p \), the summation from \( m \) to \( p \) of the sequence \( \{a_n\} \) is written:

\[
\sum_{n=m}^{p} a_n = a_m + a_{m+1} + \ldots + a_p
\]

where \( n \) is called the index of summation
\( m \) is the lower limit of summation
\( p \) is the upper limit of summation
Properties of Summation Notation:

Calculating a Finite Arithmetic Series Sum:

\[ S_n = \frac{n}{2} (a_1 + a_n) \]

where \( n \) is the number of terms; \( a_1 \) is the first term, and \( a_n \) is the last term.

Calculating a Finite Geometric Series Sum:
All we need to calculate a finite geometric series is the first term \( a_1 \), the common ratio \( r \), and the number of terms \( n \).

\[ S_n = a + ar + ar^2 + ... + ar^{n-1} = \sum_{i=0}^{n-1} ar^i = \frac{a(1 - r^n)}{1 - r}, \quad r \neq 1 \]

Calculating an Infinite Geometric Series Sum:
When the common ratio is between -1 and 1, we can find the sum of an infinite geometric series with the following:

\[ S_n = a + ar + ar^2 + ... + ar^n + ... = \sum_{i=0}^{\infty} ar^i = \frac{a}{1 - r} \]

Example: Find the sum

1. \( 1 + 4 + 7 + 10 + ... + 76 \)
2. \( \sum_{k=0}^{49} (3k - 5) \)
3. \( \sum_{i=2}^{26} 3 \cdot 2^i \)
4. \( \sum_{i=0}^{\infty} 3 \cdot 2^i \)
Sequences and Series Activity

Objectives:
- Use and interpret notation for sequences and series
- Identify arithmetic and geometric sequences
- Find a formula for the nth term of a sequence (when possible)
- Use and interpret summation notation
- Find the sum of finite arithmetic and geometric sequences
- Find the sum of infinite geometric series (when possible)
Notation for Sequences

1. Write out the first five terms of the following sequences:
   
   a. \( a_n = \frac{n}{n+1} \)
   
   b. \( a_n = 4 \)
   
   c. \( a_n = 2a_{n-1}, \quad a_1 = 3 \)
   
   d. \( a_n = a_{n-2} + a_{n-1}, \quad a_1 = 1, a_2 = 1 \)

2. Identify \( a_4 \) in the following sequences:
   
   a. 2, 7, 12, 17, 22, 27, 32, ..... 
   
   b. 2, 8, 24, 72, 216, 648, ..... 
   
   c. 1, \( b+1, 2b+1, 3b+1, 4b+1 \), .....
**Identifying Sequences**

1. Decide whether each sequence in the table is arithmetic, geometric, or neither. Find the general term, if possible.

<table>
<thead>
<tr>
<th></th>
<th>2, 7, 12, 17, 22, 27, 32, .....</th>
<th>commin difference?</th>
<th>constant ratio?</th>
<th>type of sequence</th>
<th>General Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>2, 8, 24, 72, 216, 648, .....</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b.</td>
<td>5, 1, 5, 1, 5, 1, 5, 1, 5, .....</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c.</td>
<td>1/3, 7/3, 13/3, 19/3, 25/3, .....</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d.</td>
<td>4, -2, 1, -0.5, 0.25, .....</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>e.</td>
<td>1, b+1, 2b+1, 3b+1, 4b+1, .....</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>f.</td>
<td>25, 20, 15, 10, 5, 0, -5, -10, .....</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>g.</td>
<td>1, -1, 1, -1, 1, -1, 1, -1, 1, .....</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h.</td>
<td>1, 2, 1, 2, 1, 2, 1, 2, 1, 2, .....</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Make up your own example of a sequence that is:

- arithmetic:

- geometric:

- neither:
**Summation Notation**

1. Expand and calculate the sum.

   a. \( \sum_{i=1}^{4} 3i + 1 \)

   b. \( \sum_{j=0}^{5} 3j^2 \)

   c. \( \sum_{k=1}^{7} (-1)^k \cdot k \)

   d. Make up your own expression using sigma notation and then expand it.

2. Use sigma notation to express each sum.

   a. \( \frac{1}{3} + \frac{2}{3} + 1 + \frac{4}{3} + \frac{5}{3} + 2 \)

   b. \( 100 + 90 + 80 + 70 + 60 + 50 + 40 + 30 + 20 + 10 \)

   c. \( 2 + 4 + 8 + 16 + 32 + 64 \)

   d. Make up your own sum of terms and then re-write it using sigma notation
**Arithmetic Series**

The sum of \( n \) terms of an arithmetic sequence: \( S_n = \frac{1}{2}n(a_1 + a_n) \)

1. Find the sums of the following finite arithmetic series:
   a. \( 3 + 6 + 9 + \ldots + 210 \)
   
   b. \( \sum_{i=1}^{400} (3i + 1) \)

   c. \( (1 + 2p^2) + 1 + (1 - 2p^2) + (1 - 4p^2) + \ldots + (1 - 96p^2) \)

   d. \( \sum_{n=21}^{72} n - 4 \)

2. A theater is constructed so that each row has 4 more seats than the one in front of it. The first row has 20 seats and there are 50 rows in the theater.
   a. How many seats in the 50\(^{th}\) row?

   b. How many total seats in the theater?
 Finite Geometric Series

The sum of \( n \) terms of a finite geometric sequence: \( S_n = \frac{a_1(1 - r^n)}{1 - r} \). This formula only applies if \( r \neq 1 \)

1. Find the sums of the following finite geometric series:
   a. \( 3 + 6 + 12 + 24 + ... + 3072 \)
   
   b. \( \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + ... + \frac{3}{2^{18}} \)

   c. \( \sum_{i=0}^{20} 5(1.1)^i \)

   d. \( \sum_{i=0}^{20} 5(0.1)^i \)
**Infinite Geometric Series**

The sum of an infinite number of terms of a geometric sequence: \( S = \frac{a_1}{1 - r} \). This formula only applies if \(-1 < r < 1\).

1. Find the sums of the following infinite geometric series:
   a. \( \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + ... \)
   
   b. \( \sum_{x=0}^{\infty} 5(0.1)^x \)
   
   c. \( \sum_{x=1}^{\infty} 5(0.1)^x \)

2. A rubber ball is dropped from a height of 4 feet. Each time it bounces it goes half as high as before (i.e. after it hits it bounces up 2 feet, than 1 foot and so on). What is the total distance covered by the ball once it is done bouncing?
Application of a Geometric Series

1. A clothing outlet found that when they introduced a new shirt, it sold quickly at first, but then as time went on they sold less and less quantities of the design each month. For a particular shirt, they found that they sold 300 units in the first month. They then found their sales of the shirt dropped by roughly 15% each month thereafter.

   a. How many shirts did they sell in the first 3 months (total) after the shirt was introduced?

   b. How many shirts did they sell in the first year after the shirt was introduced?

   c. Based on their estimation, how many shirts should the company expect to sell total? Hint: Look at the summation as time gets infinitely large!