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Attributions: Dr Phillip G Clark, Linda Knop, Daniel Nearing, Dr Ted Coe, Dr Sally Jacobs, Dr James Vicich
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Lesson 1: Functions

Outline

Objectives:
I can distinguish whether or not a relationship is a function from multiple representations: (using the definition of function in words; by solving an equation; using a table of values; using the vertical line test on a graph; from a mapping diagram).
I can translate a table of values to a set of ordered pairs or into function notation.
I can use and interpret function notation.
I can use and interpret set notation.
I can represent or model a real-world relationship with a function.
I can determine the domain and range of a function and write them in interval notation.
I can evaluate a function at a value, solve a function for a value, and interpret the results.
I can add, subtract, multiply, and divide functions, attending to domain restrictions.
I can find and simplify the difference quotient and interpret its meaning.
I can graph and interpret a piece-wise defined function.
I can interpret the zeros of a function.
I can find x-intercepts and y-intercepts if any.
I can test for symmetry.
I can determine whether a function is even, odd, or neither.
I can identify where a function is increasing, decreasing, or constant.

Definitions / Vocabulary / Graphical Interpretation:

Function definition:

The input is sometimes referred to as the ________________________________.

The output is sometimes referred to as the ________________________________.

Note: The output does not have to depend on the input in order to have a function.

4 forms of a function representation:

____________________

____________________

____________________

A table of values represents a function if:

A graph represents a function if:
Domain:

Range:

4 possible cases for a restricted domain:

1)

2)

3)

4)

An example of interval notation:

**Function Notation**

\( f(2) = 7 \) means:

the function name is:

the input is:

the output is:

Example: Given \( h(x) = x^2 + 2x - 4 \)

\( h(4) = \)

\( h(-3x) = \)

\( h(a - 1) = \)

\( h(x + 1) - 3h(x) = \)

Draw an example of a “Mapping” of a function:
Common Parent Functions and Their Graphs

- Constant Function
- Linear Function
- Absolute Value Function
- Quadratic Function
- Cubic Function
- Reciprocal Function
- Reciprocal Squared Function
- Square Root Function
- Cube Root Function
Function Arithmetic

Sum of 2 functions: \((f + g)(x) = \)

Difference of 2 functions: \((f - g)(x) = \)

Product of 2 functions: \((f \cdot g)(x) = \)

Quotient of 2 functions: \(\left( \frac{f}{g} \right)(x) = \)

When performing arithmetic operations on functions it is most important to attend to what aspect of the original and final functions?

The difference quotient formula is given by:

Example: Given \(f(x) = -x^2 + 2x - 3\)

Apply the difference quotient \(\frac{f(x+h) - f(x)}{h}\) to the function above

The “vertical line test” is a way of checking:
**Piecewise Defined Functions**
A piecewise defined function is different from a regular function in that:

An example of a piecewise defined function is:

**Graphical Behavior**
The zeros/roots of a function occur where:

The x-intercept(s) occur where (if):

The y-intercept occurs where (if):

**Tests for symmetry using function notation:**
A function is symmetric about the y-axis if and only if:
for all \( x \) in the domain of \( f \).

A function is symmetric about the origin if and only if:
for all \( x \) in the domain of \( f \).

A function is even when:

A function is odd when:

A function is increasing when it ________________ when reading the graph from left to right.

A function is decreasing when it ________________ when reading the graph from left to right.

Function behavior is constant when:
**Extrema**
A local maximum (or minimum) on a graph occurs at a point where:

An absolute maximum (or minimum) on a graph occurs at a point where:

**Business Applications:**

Fixed costs are costs:

Variable costs are costs:

The formula for total cost is:

The formula for average cost is:

Revenue is:

The formula for profit is:

The formula for profit as a combination of functions is:

The “breakeven point” refers to the point at which:
Functions Activity

Objectives for Functions Activity
- Determine whether or not a given rule is a function
- Use and interpret function notation
- Determine domain and range of a function
- Evaluate a function at an input
- Given an output to a function, find its input
- Combinations of functions
- Piecewise defined function
- Interpret the zeros of a function
- Identify where a function is increasing, decreasing, constant
1. Determine whether or not the following situations are functions. Give an explanation for your answer.

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
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<tbody>
<tr>
<td>1</td>
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<tr>
<td>3</td>
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<td>7</td>
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<td>2</td>
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Yes/No: Explain:

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</thead>
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<tr>
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<td>d</td>
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</tr>
<tr>
<td>e</td>
<td>5</td>
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</table>

Yes/No: Explain:

<table>
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<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
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<tr>
<td>8</td>
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<td>9</td>
<td>7</td>
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<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>

Yes/No: Explain:

<table>
<thead>
<tr>
<th>Input: Social Security Number</th>
<th>Output: Manufacturer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes/No:</td>
<td>Explain:</td>
</tr>
</tbody>
</table>

| Input: Model of a Vehicle     | Output: Person       |
| Input: Phone Number           | Output: Person       |
| Yes/No:                       | Explain:             |

Yes/No: Explain:
Function Notation

\[ y = f(x) \]

output = \( f \) (input)

1. Use function notation to write \( y \) as a function of \( x \).

2. Now use function notation to write \( m \) as a function of \( v \).

3. The number of calories used per minute depends on body weight.
   a. Use function notation to write number of calories, \( c \), as a function of body weight in pounds, \( p \).

   b. Name the independent variable: ______
      Name the dependent variable: ______

4. Consider the hypothetical situation where a car is traveling at a constant speed of 60 mph. The total distance that the car travels depends on the amount of time it travels.
   a. If \( d \) represents distance in miles and \( t \) represents time in hours, write a formula for \( d \) as a function of \( t \). Use function notation.

   b. Which is the input variable? ______
      Which is the output variable? ______

5. Suppose \( T = f(c) \). Which letter represents the independent variable? the dependent variable? input variable? output variable? Which letter represents the name of the function?
Working with Function Notation

1. a. \( f(1) = ? \)
   b. \( f(3) = ? \)
   c. \( 25 = f(x) \). What is the value of \( x \)?
   d. \( 10 = f(x) \). What is the value of \( x \)?

2. \( h(\text{input}) = \text{output} \)

<table>
<thead>
<tr>
<th>Inputs</th>
<th>Outputs</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-8</td>
</tr>
<tr>
<td>-1</td>
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</tr>
<tr>
<td>2</td>
<td>-3</td>
</tr>
<tr>
<td>3</td>
<td>-10</td>
</tr>
</tbody>
</table>

   a. \( h(-1) = ? \)
   b. \( h(0) = ? \)
   c. \( -3 = h(x) \). What is the value of \( x \)?
   d. \( 0 = h(x) \). What is the value of \( x \)?

3. \( g: \{ (1, 2), (3, 4), (5, 6), (7, 8), (9, 6), (11, -3) \} \)
   1. \( g(5) = ? \)
   2. \( g(11) = ? \)
   3. \( 4 = g(x) \). What is the value of \( x \)?
   4. \( 6 = g(x) \). What is the value of \( x \)?

4. The function, \( f \) gives the revenue \( R \), of a company(measured in 1000s of dollars) for producing \( x \) units of a product. Write a sentence to describe the following: \( f(540) = 22.34 \)
What is a Function?

For each statement below, circle True or False. If you think the statement is False, write a statement that makes it True.

1. True  False  A function is a **rule** which takes certain values as inputs and assigns to each input value exactly one output value. The output is a function of the input.

2. True  False  A function is a **relationship** between 2 quantities. If the value of the first quantity determines exactly one value of the second quantity, we say the second quantity is a function of the first.

3. True  False  A function tells the relationship between the **independent** variable (input) and the **dependent** variable (output).

4. True  False  Functions don’t have to be defined by formulas alone. We can use tables, graphs and words to define functions.

5. True  False  Not all relations are functions.

6. True  False  The Vertical Line Test tells you whether a graph is a function.
Practice Using Function Notation

For problems 1-6, use the linear function \( f(x) = 3x + 2 \).

1. Calculate \( \frac{f(4) - f(1)}{4 - 1} \):

2. Find \( f(a) \):

3. Find \( f(a + h) \):

4. Find \( f(a + h) - f(a) \) and simplify:

5. Find \( f(x+1) \) and simplify:

6. Find \( f(x+1) - f(x) \) and simplify:
Domain and Range
The domain of a function is the set of all inputs that yield an output. The range is the set of corresponding outputs.

1. Give the domain and range for the following function:

Use interval notation

Domain:
Range:

2. Find the domains of the following functions:

a. \( f(x) = 3x - 5 \)
   
   b. \( g(x) = \frac{3x - 4}{2x + 1} \)

   c. \( h(x) = \sqrt{3x + 5} \)
1. \( f(x) = x^2 - 4 \)

As the domain values vary from -3 to 0, the range values vary from ____ to ____.
As the domain values vary from 0 to 3, the range values vary from ____ to ____.

2. \( f(x) = \frac{1}{x^2 + 1} \)

As the domain values vary from -3 to 0, the range values vary from ____ to ____.
As the domain values vary from 0 to 3, the range values vary from ____ to ____.

3. \( f(x) = \sqrt{x^2 + 9} \)

As the domain values vary from -3 to 0, the range values vary from ____ to ____.
As the domain values vary from 0 to 1, the range values vary from ____ to ____.
**Combinations of Functions**

Given \( f(x) = x^2 + 2x - 3 \) and \( g(x) = -x^2 + 5x - 5 \), find the following:

1. \( f(x) + g(x) \)

2. \( f(x) - g(x) \)

3. \( 2f(x) + 3g(x) \)

4. \( f(2) - g(4) \)

5. \( f(x) \cdot g(x) \)

6. \( f(x) / g(x) \)
**Piece-Wise Defined Functions**

Given the following piecewise defined function, answer the questions.

\[ f(x) = \begin{cases} 
  x^2 - 1, & x < -1 \\
  0, & -1 \leq x < 2 \\
  4x - 4, & x \geq 2 
\end{cases} \]

Calculate the following values:

1. \( f(-3) = \)  
2. \( f(-1) = \)  
3. \( f(0) = \)  
4. \( f(2) = \)  
5. \( f(3) = \)

Draw a sketch of the graph
**Zeros of a Function**
The zeros of a function are the inputs that make the outputs 0. The following graph displays a company’s profit (in thousands of dollars) as a function of the number of items produced (measured in hundreds of items).

1. Find the zeros and interpret them in the context of the situation.

**Reflection:** Write 2-3 sentences to explain how you can tell from a graph what the zeros or $x$-intercepts are.
Behavior of a Function
A function is **increasing** if its outputs increase as its inputs increase. A function is **decreasing** if the outputs decrease as the inputs increase.

1. Using the following graph, determine where the function is increasing and where it is decreasing. Put your answer in interval notation.

![Graph of a function](image)

Increasing:

Decreasing:

**Reflection:** Write 2-3 sentences to explain how you can tell from a graph whether a function is **increasing** or **decreasing**.
Relative and Absolute Extrema

We have a **relative minimum** if a function has a point that is less than all of the points around it. This often occurs when a function goes from decreasing to increasing. We have a **relative maximum** if a function has a point that is greater than all of the points around it. This often occurs when a function goes from increasing to decreasing.

The maximum value (sometimes referred to as the Absolute Maximum) of a function is the largest output value of a function over its entire domain. The minimum value (sometimes referred to as the Absolute Minimum) of a function is the smallest output value of a function over its entire domain.

1. Identify the relative extrema of the following function:

Relative minimum

Relative maximum

**Reflection:** Write 2-3 sentences to explain whether or not the relative extrema are the absolute extrema.
Lesson 2: Transformations of Functions

Outline

Objectives:
I can identify and classify a function by its parent function.
I can write an equation of a transformed function with the same general characteristics as the parent function given the graph of a function and key points.
I can graph a transformed function with the same general characteristics as the parent function given the algebraic equation of the function.
I can graph a transformed function with the same general characteristics as the parent function given the transformation description in words.
I can identify the domain and range, increasing & decreasing regions, & extrema of a transformation.
I can identify which transformations affect inputs and which affect outputs.

Definitions / Vocabulary / Graphical Interpretation:

Some major types of function transformations include shifts, reflections, and scaling.

Composing two functions means using a __________ as an input to another function.

Shifts:
Vertical shifts are modeled by adding or subtracting a number from an entire function. They occur outside the function notation. Thus, on an x,y graph, they move the function __________ or __________.

Horizontal shifts are modeled by adding or subtracting a number to the input of a function. They occur inside the function notation. Thus, on an x,y graph they move the function __________ or __________.

Reflections:
To reflect about the x-axis, \( y = -f(x) \) multiply the ________________by (-1).

To reflect about the y-axis, \( y = f(-x) \) multiply the ________________by (-1).
Recall: an even function is symmetric about the y-axis, and an odd function is symmetric about the origin.

Scaling:
Vertical (outside) stretch (expansion, dilation) or compression (shrink, contraction):
If \( k > 1 \) then:

If \( 0 < k < 1 \) then:

If \(-1 < k < 0 \) then:

If \( k < -1 \) then:
Horizontal (inside) stretch (expansion, dilation) or shrink (compression, contraction):

If $k > 1$ then:

If $0 < k < 1$ then:

If $-1 < k < 0$ then:

If $k < -1$ then:

**Even Function**
- $f(-x) = f(x)$
- Symmetric about the $y$-axis

**Odd Function**
- $f(-x) = -f(x)$
- Symmetric about the origin

**General Equation Form (parameters):**

$$g(x) = Af(Bx + H) + K$$

To graph:
1) Subtract $H$ from each $x$-coordinate: Horizontal shift
2) Divide the $x$-coordinates in Step 1 by $B$: Horizontal scaling
3) Multiply the $y$-coordinates in Step 2 by $A$: Vertical scaling
4) Add $K$ to the $y$-coordinates from Step 3: Vertical shift

Example: Let $f(x) = x^2$. Find a formula for a function $g$ whose graph is obtained from the graph of $y = f(x)$ after the following sequence of transformations:
- Shift left 3 units
- Reflection across the $y$-axis
- Shift down 1 unit
- Vertical scaling by a factor of 2
- Reflection across the $x$-axis

**Transformation Interpretations:** Let $S = f(p)$ represent the sales of a breakfast sandwich $S$ as a function of price $p$ in dollars. Then:
- $f(p + 1)$ is interpreted as $f(p) + 10$
- $2f(p)$
- $f(2p)$
Transformations Activity

Objectives for Transformations Activity

- Identify horizontal and vertical shifts and represent them using function notation
- Identify reflections and represent them using function notation
- Identify horizontal and vertical stretches and compressions
- Identify whether a function is even, odd or neither
Vertical Shifts

Graph A

Notice that Graph B has been shifted vertically downward by one unit.

1. The equation for Graph A is: \( f(x) = x^2 \)

2. Then the equation for Graph B is: \( g(x) = \) __________

3. Now write a formula that relates \( f(x) \) and \( g(x) \) in a single equation: _______________

Looking for Vertical Shifts in a Table of Data

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<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>g(x)</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>7</td>
</tr>
</tbody>
</table>

4. Compare outputs for corresponding inputs for the \( f \) and \( g \) functions. What do you notice?

5. Using function notation such as \( f(x) = \) ________, write a formula for the function in Table A.

6. Using function notation such as \( g(x) = \) ________, write a formula for the function in Table B.

7. Now write a formula that relates \( f(x) \) and \( g(x) \) in a single equation: _______________
Horizontal Shifts

Graph A

Notice that Graph B has been shifted horizontally to the right by two units.

1. The equation for Graph A is: \( f(x) = x^2 \)

2. Then the equation for Graph B is: \( g(x) = \) __________

3. Now write a formula that relates \( f(x) \) and \( g(x) \) in a single equation: ______________

Looking for Horizontal Shifts in a Table of Data

<table>
<thead>
<tr>
<th>Table A</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>-3</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>( f(x) )</td>
<td>9</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table B</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>9</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
</tr>
</tbody>
</table>

4. Compare inputs for corresponding outputs for the \( f \) and \( g \) functions. What do you notice?

5. Using function notation such as \( f(x) = \) ________, write a formula for the function in Table A.

6. Using function notation such as \( g(x) = \) ________, write a formula for the function in Table B.

7. Now write a formula that relates \( f(x) \) and \( g(x) \) in a single equation: ______________
Reflection across the $x$-axis

Graph A

Graph B

1. Graph A and Graph B are reflections of each other across which axis? _________

2. Use the graphs above to fill in the $y$-values:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td></td>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

3. What pattern do you observe when you compare the 2 columns of $y$-values? When you reflect a graph across the $x$-axis, what happens to the $y$-coordinate values? Explain.

4. The equation for Graph A is: $f(x) = x^2$

Then the equation for Graph B is: $g(x) = _________$

5. When the graph of $y = f(x)$ is reflected across the $x$-axis, how does the equation change?
Reflection across the $y$-axis

1. Graph C and Graph D are reflections of each other across which axis? 
   
   ______________

2. Fill in the missing $x$-values:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>-3</td>
<td>1/8</td>
</tr>
<tr>
<td>-2</td>
<td>1/4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>-2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1/8</td>
</tr>
<tr>
<td>2</td>
<td>1/4</td>
</tr>
<tr>
<td>-2</td>
<td>1</td>
</tr>
<tr>
<td>-3</td>
<td>1/8</td>
</tr>
</tbody>
</table>

3. What pattern do you observe when you compare the 2 columns of $x$-values? When you reflect a graph across the $y$-axis, what happens to the $x$-coordinate values? Explain.

4. The equation for Graph C is $f(x) = 2^x$
   
   Then the equation for Graph D is: $g(x) =$ __________

5. When the graph of $y = f(x)$ is reflected across the $y$-axis, how does the equation change?
Symmetry; Even & Odd Functions

1. On each set of axes, draw a graph that is symmetric about the y-axis.

2. On each set of axes, draw a graph that is symmetric about the origin.

3. A function that is symmetric about the y-axis is called an **EVEN** function.
   a. Graph \( f(x) = x^2 \) and \( g(x) = x^4 + 3 \). Are their graphs symmetric about the y-axis?
   b. Graph \( f(x) = x \) and \( g(x) = x^3 \). Are their graphs symmetric about the y-axis?

4. A function that is symmetric about the origin is called an **ODD** function.
   a. Graph \( f(x) = 2x \) and \( g(x) = x^3 \). Are their graphs symmetric about the origin?
   b. Graph \( f(x) = e^x \) and \( g(x) = 3 \). Are their graphs symmetric about the origin?

**Reflection:** Write 2-3 sentences to explain how you can tell from a graph whether a function is **even** or **odd**. Also, give an example of an equation for an even function; an odd function.
**Vertical Stretch and Compression**

Example of a vertical stretch by a factor of 5 \((k=5)\).

\[
y = f(x)
\]

\[
y = 5f(x)
\]

Example of a vertical compression by a factor of \(1/2\) \((k = 1/2)\).

\[
y = f(x)
\]

\[
y = 0.5f(x)
\]
Horizontal Compressions and Stretches

Example of a horizontal compression by a factor of 1/2 \((k = 2)\).  

\[ y = f(x) \]

\[ y = f(2x) \]

Example of a horizontal stretch by a factor of 3 \((k = 1/3)\).  

\[ y = f(x) \]

\[ y = f\left(\frac{x}{3}\right) \]
Reflect, Shift, Stretch, and Compress (Shrink)

Describe how the function $g$ has been changed from the function $f$. In your description, use phrases such as:

“reflection through the _____ - axis”
“vertical shift up (down) by ______ units”
“horizontal shift to the right (left) by ________ units”
“vertical stretch by a factor of ______”
“vertical compression by a factor of_______”

1. $f(x) = |x| \quad g(x) = |x| + 5$

2. $f(x) = x^2 \quad g(x) = (x + 3)^2$

3. $f(x) = 3x + 1 \quad g(x) = 0.5(3x + 1)$

4. $f(x) = e^x \quad g(x) = e^{-x}$

5. $f(x) = 2^x \quad g(x) = 5 \cdot (2)^x$

6. $f(x) = e^x \quad g(x) = -e^x$

7. $f(x) = \sqrt{x} \quad g(x) = f(4x)$

8. $f(x) = \log x \quad g(x) = \log(0.1x)$
Interpreting Transformations

The function \( f(p) \) gives the number of televisions sold as a function of the price of the television in dollars.

Would you expect \( f \) to be an increasing or decreasing function? Why?

Give a sentence or two to describe the meaning of the following:

1. \( 2f(p) \)

2. \( f(2p) \)

3. \( f(p - 20) \)

4. \( f(p) - 20 \)
Shifting Graphs
Given the graph of $f(x)$, graph the following transformations by moving the points and then redrawing the graph.

1. Use these axes to draw the graph of $y = f(x + 3) + 1$. What kind of shift do you have? What do you expect to happen to the graph?

2. Use these axes to draw the graph of $y = 2f(x + 3)$. What kind of shift do you have? What do you expect to happen to the graph?
3. Use these axes to draw the graph of $y = f(2x)$. What kind of shift do you have? What do you expect to happen to the graph?

4. Use these axes to draw the graph of $y = f(2x) + 1$. What kind of shift do you have? What do you expect to happen to the graph?

5. Use these axes to draw the graph of $y = 2f(2x + 3) + 1$. What kind of shift do you have? What do you expect to happen to the graph?
Lesson 3: Linear Functions

Outline

Objectives:

I can determine the dependent and independent variables in a linear function.
I can read and interpret characteristics of linear functions including x- and y-intercepts.
I can calculate the constant rate of change given information numerically, analytically, or graphically.
I can write the equation of a line given 2 points, a point and a slope, or a slope and a y-intercept.
I can determine whether a line is increasing or decreasing.
I can determine if a slope is undefined.
I can determine if lines are parallel or perpendicular.
I can determine from a table of values whether the data imply a constant rate of change.
I can justify and interpret solutions to application problems and determine whether a linear model will best fit a situation using the proper units.
I can find a linear regression model and interpret the results.

Definitions / Vocabulary / Graphical Interpretation:

Linear functions are primarily characterized by:

Constant rate of change means:

The slope-intercept form of a linear equation is:

To calculate the slope of a line between two points, use the formula:

5 Alternative notations for slope include:

The symbol Δ means:

In general, if the slope

<table>
<thead>
<tr>
<th>is:</th>
<th>the line is:</th>
<th>and looks like:</th>
</tr>
</thead>
<tbody>
<tr>
<td>positive</td>
<td></td>
<td></td>
</tr>
<tr>
<td>negative</td>
<td></td>
<td></td>
</tr>
<tr>
<td>zero</td>
<td></td>
<td></td>
</tr>
<tr>
<td>undefined</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The **point-slope form** of an equation of a line is:

If given two points and asked to find the point-slope form of the line, the first step is to:

The second step is to:

The **horizontal intercept** occurs where the function:

To find the horizontal intercept, plug in _______ for _______ and solve for _______.

The **vertical intercept** occurs where the function:

To find the vertical intercept, plug in _______ for _______ and solve for _______.

**Properties of Lines**

A constant function has

A vertical line has a slope that is

Parallel lines

Perpendicular lines

Horizontal lines

As we build graphs to represent mathematical models, there are 4 important features to attend to. They are:

1) 

2) 

3) 

4)
In order to tell if a table of data represents a linear model, we test to see if:

Line of best fit:

Scatterplot:

Correlation coefficient, $r$, tells the goodness of fit and ranges between _____ and ______.

Regression:

**Objectives, Absolute Value:**

I can graph and interpret absolute value and piecewise defined functions, and determine their domain and range, intervals of increasing and decreasing, and absolute and relative extrema.
I can solve absolute value equations.
I can identify transformations of absolute value equations.
I can determine if a piecewise defined relation is a function.
I can read, evaluate, and interpret information represented as a piecewise defined function.
I can solve and piecewise graph an equation involving multiple absolute values.
I can graph and evaluate transformations of absolute value and piecewise defined functions.

**Definitions / Vocabulary / Graphical Interpretation:**

Absolute value functions come up in situations where:

Absolute value defined using the number line:

Absolute value defined using exponents:

Absolute value defined using set notation in a piecewise-defined function:

Steps to solving an equation with an absolute value:

Example 1: Solve $|3x - 9| = 18$
Linear Functions Activity

Objectives for Linear Activity
- Calculate average rate of change/slope
- Interpret intercepts and slope of linear function
- Linear regression
Average Rate of Change & Slope

On a graph, average rate of change can be visualized as the slope of the line segment that connects two points. On the graph below, mark the two points where \( x_1 = 2 \) and \( x_2 = 4 \). Draw a straight line to connect these two points. Now determine the slope of your line by finding \( \frac{\text{change in } y}{\text{change in } x} \) or \( \frac{y_2 - y_1}{x_2 - x_1} \) or \( \frac{\Delta y}{\Delta x} \).

1. The slope of your line is: ____ The average rate of change in \( y \) when \( 2 \leq x \leq 4 \) is: ____

2. Sometimes we use function notation, \( \frac{f(b) - f(a)}{b - a} \), to express slope or average rate of change. For example, let \( a = 1 \) and \( b = 3 \). Then you can write \( \frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(1)}{3 - 1} \). This expression gives you the average rate of change in \( y \) for \( 1 \leq x \leq 3 \).

Here’s what you do:
   a. Find \( f(1) \): ______
   b. Find \( f(3) \): ______
   c. Calculate: \( \frac{f(3) - f(1)}{3 - 1} = \) ______

3. The average rate of change in \( y \) for values of \( x \) between 1 and 3 is ________. The slope of the line segment between \( (1, f(1)) \) and \( (3, f(3)) \) is ________.

**Reflection:** What is the relationship between average rate of change and slope?
Average Rate of Change (How Fast?)

The number of CDs sold by Musicom Corporation between 1990 and 1996 is shown below.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Sales (millions)</td>
<td>287</td>
<td>408</td>
<td>662</td>
<td>779</td>
</tr>
</tbody>
</table>

1. On average, how fast were Sales increasing between 1990 and 1994? between 1990 and 1996?

Another way to say "how fast (on average)" is to use the phrase "average rate of change."

2. What is the average rate of change in CDs sold between 1990 and 1994? between 1990 and 1996? Be sure to write the word "per" in your answers.

3. Compare your answers with your teammates. Do you agree with each other?

Write a short paragraph to explain carefully to someone else the method to use to calculate average rate of change.
Average Rate of Change

The population for Riverdale City between 1990 and 2010 is shown below.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Population (hundreds)</td>
<td>55</td>
<td>60</td>
<td>68</td>
<td>74</td>
<td>83</td>
<td>88</td>
<td>90</td>
<td></td>
</tr>
</tbody>
</table>

1. Find the average rate of change over each time interval:
   a. 1990 to 1992

   b. 1992 to 1995

   c. 1995 to 1997

   d. 1997 to 2000

   e. 2000 to 2005

   f. 2005 to 2010

2. Now compare the decade of the 90s with this century so far. What do your answers tell you about the population of Riverdale City?

3. Based on the information above, predict what you think the population of Riverdale City will be in 2013.
What's the Story?

In the following problems, determine what story is being told by the given linear function. In your study be sure to attend to the slope, the $y$-intercept and the $x$-intercept.

1. The equation of the line is $C = 4t + 50$, where $t$ = # years since 1960.

2. The equation for the line is $P = 12s$
More Graphs... More Stories

3. The equation of the line is $T = 300 + 200C$, where $C =$ # credits taken.

![Graph of tuition cost vs. number of credits taken](image1)

What’s the story?

4. The equation for the line is $P = -2.4t + 22$

![Graph of pulse count vs. time after exercise](image2)

What’s the story?
Formulas for Linear Functions

Find a linear equation that expresses each of the function representations.

1. Table

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>5</td>
<td>8</td>
<td>11</td>
<td>14</td>
<td>17</td>
</tr>
</tbody>
</table>

2. Graph

3. Words

Suppose there were exactly two runs scored in each inning of a certain baseball game. Express total runs as a function of innings.

4. Set of Points

{ (-3, 35), (-2, 30), (-1, 25), (0, 20), (1, 15), (2, 10), (3, 5) }

Reflection: What do all four examples have in common?
Perpendicular Lines

This graph shows two lines on a set of $x$- and $y$-axes where each tick mark represents one unit on each axis.

1. Work in groups to determine whether the lines are perpendicular. Use mathematics to support your conclusion. Recall: The relationship between perpendicular lines is that their slopes are negative reciprocals.

Reflection: What does this example teach you about analyzing graphs?
Linear Regression
You learned how to perform linear regression in your online lesson. Each group will be assigned a different data set from below and then questions on that data set. Use the regression capabilities of your calculator to answer the questions and be prepared to share your solution with the rest of the class.

1. Marketing Labor Cost for Farm Foods
Statistical Abstract of the United States, 2006, Table 842
Find $L$ as a function of $t$

<table>
<thead>
<tr>
<th>Year</th>
<th>$t$ Year (Since 1990)</th>
<th>$L$ Marketing Labor Cost (billions of dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1990</td>
<td>0.0</td>
<td>154.0</td>
</tr>
<tr>
<td>1994</td>
<td>4.0</td>
<td>186.1</td>
</tr>
<tr>
<td>1995</td>
<td>5.0</td>
<td>196.6</td>
</tr>
<tr>
<td>1996</td>
<td>6.0</td>
<td>204.6</td>
</tr>
<tr>
<td>1997</td>
<td>7.0</td>
<td>216.9</td>
</tr>
<tr>
<td>1998</td>
<td>8.0</td>
<td>229.9</td>
</tr>
<tr>
<td>1999</td>
<td>9.0</td>
<td>241.5</td>
</tr>
<tr>
<td>2000</td>
<td>10.0</td>
<td>252.9</td>
</tr>
<tr>
<td>2001</td>
<td>11.0</td>
<td>263.8</td>
</tr>
<tr>
<td>2002</td>
<td>12.0</td>
<td>273.1</td>
</tr>
<tr>
<td>2003</td>
<td>13.0</td>
<td>285.9</td>
</tr>
</tbody>
</table>

2. Health Services Spending as a function of Physician Services Spending
Source: Statistical Abstract of the United States, 2006, Table 121

Find $T$ as a function of $p$

<table>
<thead>
<tr>
<th>$t$, years since 1990</th>
<th>$p$ Per Capita Spending on Physician and Clinical Services (dollars)</th>
<th>$T$ Total Per Capita Spending on Health Services and Supplies (dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>619</td>
<td>2633</td>
</tr>
<tr>
<td>5</td>
<td>813</td>
<td>3530</td>
</tr>
<tr>
<td>8</td>
<td>914</td>
<td>3962</td>
</tr>
<tr>
<td>9</td>
<td>954</td>
<td>4154</td>
</tr>
<tr>
<td>10</td>
<td>1010</td>
<td>4389</td>
</tr>
<tr>
<td>11</td>
<td>1085</td>
<td>4733</td>
</tr>
<tr>
<td>12</td>
<td>1162</td>
<td>5115</td>
</tr>
<tr>
<td>13</td>
<td>1249</td>
<td>5452</td>
</tr>
</tbody>
</table>
3. Carbon Monoxide Pollutant Concentrations
Source: Statistical Abstract of the United States, 2006, Table 359
Find \( P \) as a function of \( t \)

<table>
<thead>
<tr>
<th>( t ), Year (Since 1990)</th>
<th>( P ) Carbon Monoxide Pollutant Concentration (parts per million)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>4.7</td>
</tr>
<tr>
<td>9</td>
<td>3.9</td>
</tr>
<tr>
<td>10</td>
<td>3.4</td>
</tr>
<tr>
<td>11</td>
<td>3.2</td>
</tr>
<tr>
<td>12</td>
<td>3</td>
</tr>
<tr>
<td>13</td>
<td>2.8</td>
</tr>
</tbody>
</table>

4. Private College Enrollment as a Function of Public College Enrollment
Source: Statistical Abstract of the United States, 2006, Table 204
Find \( P \) as a function of \( x \)

<table>
<thead>
<tr>
<th>Year</th>
<th>( x ) Public College Enrollment (thousands)</th>
<th>( P ) Private College Enrollment (thousands)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>11,753</td>
<td>3,560</td>
</tr>
<tr>
<td>2001</td>
<td>12,233</td>
<td>3,695</td>
</tr>
<tr>
<td>2002</td>
<td>12,752</td>
<td>3,860</td>
</tr>
<tr>
<td>2003, proj</td>
<td>12,952</td>
<td>3,958</td>
</tr>
<tr>
<td>2004, proj</td>
<td>13,092</td>
<td>4,003</td>
</tr>
<tr>
<td>2005, proj</td>
<td>13,283</td>
<td>4,068</td>
</tr>
<tr>
<td>2006, proj</td>
<td>13,518</td>
<td>4,146</td>
</tr>
<tr>
<td>2007, proj</td>
<td>13,752</td>
<td>4,223</td>
</tr>
<tr>
<td>2008, proj</td>
<td>14,034</td>
<td>4,316</td>
</tr>
<tr>
<td>2009, proj</td>
<td>14,251</td>
<td>4,389</td>
</tr>
<tr>
<td>2010, proj</td>
<td>14,380</td>
<td>4,436</td>
</tr>
<tr>
<td>2011, proj</td>
<td>14,494</td>
<td>4,478</td>
</tr>
<tr>
<td>2012, proj</td>
<td>14,612</td>
<td>4,520</td>
</tr>
<tr>
<td>2013, proj</td>
<td>14,730</td>
<td>4,560</td>
</tr>
</tbody>
</table>
5. Death Rate Due to Heart Disease
Statistical Abstract of the United States, 2006, Table 106

Find $R$ as a function of $t$

<table>
<thead>
<tr>
<th>Years after 1980, $t$</th>
<th>Age-Adjusted Death Rate Due to Heart Disease (deaths/100,000 people) $r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>412.1</td>
</tr>
<tr>
<td>1</td>
<td>397.0</td>
</tr>
<tr>
<td>2</td>
<td>389.0</td>
</tr>
<tr>
<td>3</td>
<td>388.9</td>
</tr>
<tr>
<td>4</td>
<td>378.8</td>
</tr>
<tr>
<td>5</td>
<td>375.0</td>
</tr>
<tr>
<td>6</td>
<td>365.1</td>
</tr>
<tr>
<td>7</td>
<td>355.9</td>
</tr>
<tr>
<td>8</td>
<td>352.5</td>
</tr>
<tr>
<td>9</td>
<td>332.0</td>
</tr>
<tr>
<td>10</td>
<td>321.8</td>
</tr>
<tr>
<td>11</td>
<td>313.8</td>
</tr>
<tr>
<td>12</td>
<td>306.1</td>
</tr>
<tr>
<td>13</td>
<td>309.9</td>
</tr>
<tr>
<td>14</td>
<td>299.7</td>
</tr>
<tr>
<td>15</td>
<td>296.3</td>
</tr>
<tr>
<td>16</td>
<td>288.3</td>
</tr>
<tr>
<td>17</td>
<td>280.4</td>
</tr>
<tr>
<td>18</td>
<td>272.4</td>
</tr>
<tr>
<td>19</td>
<td>267.8</td>
</tr>
<tr>
<td>20</td>
<td>257.6</td>
</tr>
<tr>
<td>21</td>
<td>247.8</td>
</tr>
<tr>
<td>22</td>
<td>240.8</td>
</tr>
<tr>
<td>23</td>
<td>232.1</td>
</tr>
</tbody>
</table>
6. Cigarettes and Heart Disease
Abstract of the United States, 2006, Table 106

Find $D$ as a function of $p$

<table>
<thead>
<tr>
<th>Year</th>
<th>People Who Smoke Cigarettes (percent) $P$</th>
<th>Heart Disease Death Rate (deaths per 100,000 people) $D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1974</td>
<td>36.9</td>
<td>458.8</td>
</tr>
<tr>
<td>1979</td>
<td>33.1</td>
<td>401.6</td>
</tr>
<tr>
<td>1983</td>
<td>31.6</td>
<td>388.9</td>
</tr>
<tr>
<td>1985</td>
<td>30</td>
<td>375</td>
</tr>
<tr>
<td>1987</td>
<td>28.8</td>
<td>355.9</td>
</tr>
<tr>
<td>1988</td>
<td>28.1</td>
<td>352.5</td>
</tr>
<tr>
<td>1990</td>
<td>25.4</td>
<td>321.8</td>
</tr>
<tr>
<td>1991</td>
<td>25.8</td>
<td>313.8</td>
</tr>
<tr>
<td>1992</td>
<td>26.3</td>
<td>306.1</td>
</tr>
<tr>
<td>1993</td>
<td>24.7</td>
<td>309.9</td>
</tr>
<tr>
<td>1994</td>
<td>24.9</td>
<td>299.7</td>
</tr>
<tr>
<td>1995</td>
<td>24.5</td>
<td>296.3</td>
</tr>
<tr>
<td>1997</td>
<td>24</td>
<td>280.4</td>
</tr>
<tr>
<td>1998</td>
<td>23.4</td>
<td>272.4</td>
</tr>
<tr>
<td>1999</td>
<td>22.7</td>
<td>267.8</td>
</tr>
<tr>
<td>2000</td>
<td>22.6</td>
<td>257.6</td>
</tr>
<tr>
<td>2001</td>
<td>22</td>
<td>247.8</td>
</tr>
<tr>
<td>2002</td>
<td>21.4</td>
<td>240.8</td>
</tr>
<tr>
<td>2003</td>
<td>21.1</td>
<td>232.1</td>
</tr>
</tbody>
</table>

7. US Resident Population as a Function of US Population
Statistical Abstract of the United States, 2006, Table 2

Find $r$ as a function of $p$

<table>
<thead>
<tr>
<th>Years since 1990, $t$</th>
<th>US Population (thousands) $p$</th>
<th>Resident Population (thousands) $r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>250,132</td>
<td>249,623</td>
</tr>
<tr>
<td>1</td>
<td>253,493</td>
<td>252,981</td>
</tr>
<tr>
<td>2</td>
<td>256,894</td>
<td>256,514</td>
</tr>
<tr>
<td>3</td>
<td>260,255</td>
<td>259,919</td>
</tr>
<tr>
<td>4</td>
<td>263,436</td>
<td>263,126</td>
</tr>
<tr>
<td>5</td>
<td>266,557</td>
<td>266,278</td>
</tr>
<tr>
<td>6</td>
<td>269,667</td>
<td>269,394</td>
</tr>
<tr>
<td>7</td>
<td>272,912</td>
<td>272,647</td>
</tr>
<tr>
<td>8</td>
<td>276,115</td>
<td>275,854</td>
</tr>
<tr>
<td>9</td>
<td>279,295</td>
<td>279,040</td>
</tr>
<tr>
<td>10</td>
<td>282,402</td>
<td>282,192</td>
</tr>
<tr>
<td>11</td>
<td>285,329</td>
<td>285,102</td>
</tr>
<tr>
<td>12</td>
<td>288,173</td>
<td>287,941</td>
</tr>
<tr>
<td>13</td>
<td>291,028</td>
<td>290,789</td>
</tr>
<tr>
<td>14</td>
<td>293,907</td>
<td>293,655</td>
</tr>
</tbody>
</table>
8. Consumer Spending on Farm Foods  
Statistical Abstract of the United States, 2007, Table 818

Find \( a \) as a function of \( h \)

<table>
<thead>
<tr>
<th>Year, ( t )</th>
<th>At-home spending on farm foods (billion dollars) ( h )</th>
<th>Away-from-home spending on farm foods (billion dollars) ( a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1995</td>
<td>316.9</td>
<td>212.6</td>
</tr>
<tr>
<td>1996</td>
<td>328.0</td>
<td>218.7</td>
</tr>
<tr>
<td>1997</td>
<td>339.2</td>
<td>227.3</td>
</tr>
<tr>
<td>1998</td>
<td>346.8</td>
<td>238.2</td>
</tr>
<tr>
<td>1999</td>
<td>370.7</td>
<td>254.6</td>
</tr>
<tr>
<td>2000</td>
<td>390.2</td>
<td>270.9</td>
</tr>
<tr>
<td>2001</td>
<td>403.9</td>
<td>283.6</td>
</tr>
<tr>
<td>2002</td>
<td>416.8</td>
<td>292.6</td>
</tr>
<tr>
<td>2003</td>
<td>437.2</td>
<td>307.0</td>
</tr>
<tr>
<td>2004</td>
<td>463.5</td>
<td>325.4</td>
</tr>
</tbody>
</table>

9. Late Airline Departures and Arrivals  
Source: Statistical Abstract of the United States 2007, Table 1054

Find \( a \) as a function of \( d \)

<table>
<thead>
<tr>
<th>Year, ( t )</th>
<th>Late Airline Departures ( d )</th>
<th>Late Airline Arrivals ( a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1995</td>
<td>827.9</td>
<td>1039.3</td>
</tr>
<tr>
<td>1997</td>
<td>846.9</td>
<td>1083.8</td>
</tr>
<tr>
<td>1998</td>
<td>870.4</td>
<td>1070.1</td>
</tr>
<tr>
<td>1999</td>
<td>937.3</td>
<td>1152.7</td>
</tr>
<tr>
<td>2000</td>
<td>1131.7</td>
<td>1356.0</td>
</tr>
<tr>
<td>2001</td>
<td>953.8</td>
<td>1104.4</td>
</tr>
<tr>
<td>2002</td>
<td>717.4</td>
<td>868.2</td>
</tr>
<tr>
<td>2003</td>
<td>834.4</td>
<td>1057.8</td>
</tr>
<tr>
<td>2004</td>
<td>1187.6</td>
<td>1421.4</td>
</tr>
<tr>
<td>2005</td>
<td>1279.4</td>
<td>1466.1</td>
</tr>
</tbody>
</table>
Lesson 4: Quadratic Functions

Outline

Objectives:

I can find the zeros of a quadratic equation by factoring and using the quadratic equation.
I can find the vertex and axis of symmetry of a quadratic equation.
I can determine where a quadratic function is increasing & decreasing.
I can find the domain & range.
I can determine and interpret the concavity of a function.
I can recognize and use the multiple forms of a quadratic equation.
I can determine the equation of a quadratic function given the zeros and a point or given the vertex and a point.
I can perform quadratic regression on a set of data and interpret the results.

Definitions / Graphical Interpretation / Characteristics:

Concavity

Draw concave up:

Draw concave down:

General form of a quadratic:

If the general form’s lead coefficient $a$ is $> 0$, then the parabola is concave__________.
If the general form’s lead coefficient $a$ is $< 0$, then the parabola is concave__________.

Vertex

Formula for the vertex from the general form (as an ordered pair):

If the quadratic is concave up, then the vertex is an absolute ____________________.
If the quadratic is concave down, then the vertex is an absolute ____________________.

The equation for the axis of symmetry from the vertex:
Zeros, Roots, $x$ – intercepts

Three possibilities for the “zeros” or “roots” of a quadratic:

a) No real roots:

b) 1 Double root:

c) 2 Distinct roots:

Three methods of finding the “zeros” or “roots” of a quadratic are

1) 

2) 

3) Quadratic Formula from the general form: 

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

End Behavior and Limit Notation

If $a > 0$ then 

$$\lim_{x \to \infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = \infty$$

If $a < 0$ then 

$$\lim_{x \to \infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = -\infty$$

Quadratic Equation Forms

General form of a quadratic equation:

Vertex form of a quadratic equation:

Factored form of a quadratic equation:

Calculator steps to doing quadratic regression
Quadratic Functions Activity

Objectives for Quadratic Functions Activity
- Find and interpret intercepts
- Find and interpret the vertex
- Interpreting concavity
- Domain and range
- Forms of a quadratic function
- Applications of Quadratics
- Quadratic Regression
**Quadratic Model**

A baseball is “popped” straight up by a batter. The ball’s height (in feet) above the ground $t$ seconds later is given by $h(t) = -16t^2 + 64t + 3$.

1. Graph the height function on your graphing calculator. This graph shows that the ball goes up relatively fast at first. And then because of gravity, it slows down as it continues upward. The ball eventually reaches a maximum height and then begins its descent downward.

Notice that this graph is not the picture of the path of the ball popped straight up vertically. Instead, it is a graph of the height as a function of time. By hand, draw 2 sketches: Draw a graph of the ball’s height over time and then draw a picture of the ball’s actual path straight up and back down. Do you see the difference?

<table>
<thead>
<tr>
<th>Height of ball as a function of time</th>
<th>Actual path of ball</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Using the height function given, locate the intercepts. That is, find $h(0)$ and find $t$ when $h(t) = 0$. Interpret the meaning of each intercept. In the context of the baseball problem, what is happening at each intercept on the graph?
3. What is the maximum height attained by the ball? How many seconds does it take for the ball to reach this height?

4. Evaluate and interpret $h(1)$. What does it mean?

5. State the function’s practical domain and range. What can you say about the concavity of the graph? The practical domain and range are the domain and range that make sense for this situation.
Forms of the Quadratic Equation

A quadratic equation can be written in more than one format. Strive to be flexible in moving from one format to another depending on what information you need.

General Form \( y = ax^2 + bx + c \)
This form allows you to find (or tells you):

1. the zeros from the quadratic formula

2.

3.

Vertex Form \( y = a(x - h)^2 + k \)
This form allows you to find (or tells you):

1.

2.

3.

Factored Form \( y = a(x - r)(x - s) \)
(when it's possible to factor)
This form allows you to find (or tells you):

1.

2.
Using Forms of the Quadratic Equation

1. Find the zeros of the function: \( f(x) = 3(x + 2)(x - 1) \)

2. Find the vertex of the graph of: \( y = -(x + 1)^2 + 3 \)

3. Find the y-intercept of the graph of: \( f(x) = 3x^2 - 4x - 1 \)

4. Write an equation for the line of symmetry for the graph of \( y = -(x + 1)^2 + 3 \).

5. Determine the concavity and find the zeros of: \( f(x) = x^2 + x - 6 \)

6. Does the graph of \( f \) open upward or downward? \( f(x) = -(5 + 3x - x^2) \)

7. Write a quadratic equation for a function with zeros \( x = 3 \) and \( x = -1 \) and a y-intercept at \( (0, -6) \)

8. Determine the domain and range of: \( g(x) = 3x^2 - 4x + 2 \)
Mystery Graphs!

Write a quadratic equation for each of the parabolas graphed below. Use your graphing calculator to check your answers. HINT: Think of the graph as a transformation of $y = x^2$.

1. Equation ________________

2. Equation ________________

3. Equation ________________

4. Equation ________________
Applications of Quadratic Functions

1. A rectangle has an area 154 square inches. The length of a rectangle is 8 more than twice the width. What are the dimensions of the rectangle?

2. A hotel finds that if they charge $300 per room, they can book 100 rooms. They also find that for every $50 increase, they lose 5 bookings. What rate should they charge to maximize their revenue? What is their maximum revenue at this rate?

3. A farmer wants to build a pen against a side of a mountain. He has 200 feet of fencing. What is the maximum area he can enclose? What are the dimensions of his pen of maximum area?
Lesson 5: Polynomial Functions

Outline

Objectives:
I can identify power functions and when they have symmetry.
I can analyze and interpret the behavior of polynomial functions, including end behavior, increasing and decreasing intervals, extrema, and symmetry.
I can solve polynomial equations analytically and graphically.
I can find the real zeros of a polynomial analytically and graphically.
I can graph polynomial functions and determine their domain and range.
I can build a polynomial function given a set of roots or characteristics and a point.
I can perform arithmetic operations on polynomials and state the domain of the resulting function.
I can classify polynomials by name when represented numerically, graphically, or analytically and determine appropriate regression models.
I can interpret the results of an appropriate regression model.

Definitions / Vocabulary / Graphical Interpretation:

Power Functions: are represented by the form \( f(x) = x^n \)

A scale factor (stretch or shrink) or a reflection of a power function is represented by the variable \( a \), when \( y = ax^n \)

Note: \( y = ax^n \) scaling of the parameter \( a \) is allowed; however, vertical and horizontal shifts (moving left or right, up or down) do not preserve the power function form.

Long-run behavior of a power function

Power functions that are “even” exhibit end behavior such that in the long run, the outer ends of the function extend in the same direction. That is, in their parent form,
\[
\lim_{x \to \pm \infty} f(x) = \infty \quad \text{or} \quad \lim_{x \to \pm \infty} f(x) = -\infty .
\]

Power functions that are “odd” exhibit end behavior such that in the long run, the outer ends of their function extend in opposite directions. That is, in their parent form,
\[
\lim_{x \to \infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = -\infty \quad \text{or} \quad \lim_{x \to \infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = \infty .
\]

Direct proportionality is represented by the form:

Inverse proportionality is represented by the form:

Power regression on the calculator uses the PwrReg command.


**Polynomials:**

A polynomial is a function that is the sum of terms of the form $ax^n$ where $a$ is a real-number coefficient and $n$ is a non-negative integer. The degree is determined by the largest exponent.

A first degree polynomial yields a line.

A second degree polynomial yields a

A third degree polynomial yields a

Concavity:

Inflection Points tell where

Multiplicity tells how many times a linear factor

Roots/Zeros/x-intercepts tell

The relationship between the number of inflection points and the degree of the polynomial is:

The degree of a polynomial and its end behavior are related:
   If the polynomial’s leading degree is even, the end behavior:

   If the polynomial’s leading degree is odd, the end behavior:
Example 1: Given factored form $P(x) = -2(x + 3)(2x - 1)^2$

$y$-intercept:

Degree:

End Behavior:

Linear Factors:

Roots/Zeros/x-intercepts:

Repeated Roots:

Multiplicity:

Ex 2: Find a polynomial $f(x)$ of degree 3 with zeros $x = -1, x = 2, x = 4$, and $f(1) = 8.$
Polynomial Functions Activity

Objectives:
- Create power functions
- Find and interpret intercepts
- Find intervals of increasing and decreasing function values
- Identify end behavior
- Find and interpret extrema
- Identify and interpret concavity
- Find and interpret inflection points
- Find polynomial functions given intercepts
- Polynomial regression
**Power Functions**

1. $y$ is directly proportional to the square of $x$ and when $x = 3$, $y = 36$. Create a power function relating $x$ and $y$ and use it to find $y$ when $x$ is 7.

2. $a$ is inversely proportional to the cube root of $b$ and when $b = 27$, $a = 8$. Create a power function relating $a$ and $b$ and use it to find $b$ when $a = 16$.

3. Driving to Tucson from Scottsdale at 65mph would take approximately two hours. Is the time the drive takes directly or inversely proportional to the speed? Write a formula for the proportion and use it to figure out what speed you would need to average in order to get there in 1 and a half hours.

4. Use the regression capabilities of your calculator to find the equation of the power function through the points (2, 24) and (-4, -192).
Behavior of Polynomials
1. Given the polynomial \( f(x) = x^4 + 3x^3 - 28x^2 - 60x \) answer the following questions:
   a. What is the degree of \( f \)?

   b. Use your graphing calculator to draw a graph of \( f \). Be sure to choose a window that allows you to see the whole graph.

   ![Graph of f(x)](attachment:image)

   c. Identify all of the intercepts of \( f \). Write your answers as points \((a,b)\).

   d. Rewrite the polynomial in factored form.

   e. Identify all the intervals where \( f \) is increasing and decreasing. Write your answers in interval notation.

   f. Estimate the location and value of any relative extrema (e.g. local minimums and local maximums).

   g. Estimate the location and value of any inflections points.

   h. Identify the end behavior of the graph using limit notation.
Degrees and Zeros
1. Determine the degree and zeros of the following polynomials

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Degree</th>
<th>Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = (x + 3)(x - 5)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$g(x) = (x + 2)(x - 1)(x - 4)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k(x) = (3x + 2)(x - 3)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f(x) = x(x + 3)(3x - 5)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m(x) = x(x - 3)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n(x) = x^2(x - 3)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p(x) = x^2(x - 3)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q(x) = x^2(x - 3)^3$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Take a look at the zeros of the last four functions from the previous problem:

$m(x) = x(x - 3) \quad n(x) = x^2(x - 3) \quad p(x) = x^2(x - 3)^2 \quad q(x) = x^2(x - 3)^3$

What do you notice at the zeros? What kind of generalizations can you make?

When a zero is repeated an even number of times …

When a zero is repeated an odd number of times…
End Behavior
1. Using limit notation, describe the end behavior of the following polynomials. See if you can complete this activity without using your calculator.

a. \( f(x) = x^7 - 3x^6 + 45x^4 - 321x + 729 \)

b. \( g(x) = x^2 + 3x^4 - 17x^7 + 3x^3 + 17 - 4x \)

c. \( h(x) = x(x - 2)(3x - 1)(4x - 2)(4x + 2)^2 \)

d. \( k(x) = -2(x - 3)^2(2x - 5)(x^2 + 4) \)
Finding Polynomials

1. Find polynomials for the following situations. You may leave your answers in factored form.
   a. Roots at -3, 2 and 7.

   b. Roots at -3, 2, and 3 and goes through the point (4,7)

   c. Roots at -2, 0 and a double root at 1.

   d. Roots at -2, 0, a double root at 1, a triple root at 2 and goes through the point (3, 2).

   e. For the graph
Lesson 6: Rational Functions

Outline

Objectives:
I can analyze and interpret the behavior of rational functions, including end behavior, increasing and decreasing intervals, asymptotes, extrema, and symmetry using proper set or interval notation.
I can solve rational equations analytically and graphically.
I can graph rational functions, identifying asymptotes, holes, and end behavior.
I can build a rational function given a set of roots or characteristics.
I can perform transformations on rational functions.
I can perform arithmetic operations (specifically, division) on functions to create a rational function and state the domain of the resulting function.
I can analyze and interpret a rational function in context.

Definitions / Vocabulary / Graphical Interpretation:

Rational function definition:
The domain of rational function is restricted when______________________________

Vertical asymptotes appear where:

Using \( \frac{x - 3}{x + 2} \) complete the following table on your calculator:

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2.1</th>
<th>-2.01</th>
<th>-2.001</th>
<th>-2</th>
<th>-1.999</th>
<th>-1.99</th>
<th>-1.9</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

One-sided limit notation:

Discontinuities occur when there are:
1) Holes, which occur when:

2) Vertical asymptotes, which occur when:

Rational functions do not have any vertical asymptotes when:
1)
2)
End Behavior and Horizontal Asymptotes

Example: Fill in the table: \( f(x) = \frac{x-3}{x+2} \quad g(x) = \frac{3x^2}{(x-1)(x-3)} \)
\[ h(x) = \frac{x^2+1}{x-2} \quad k(x) = \frac{5x}{x^2-4} \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>10,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( g(x) )</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>( h(x) )</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>( k(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Project the outputs for \( x = 100,000 \):
- \( f(100,000) \)
- \( g(100,000) \)
- \( h(100,000) \)
- \( k(100,000) \)

RULES FOR DETERMINING HORIZONTAL ASYMPTOTES:
Recall, the end behavior of a **regular** polynomial is determined by its:

The end behavior of a **rational** polynomial function is determined by looking at:

If the degree of the numerator is **less than** the degree of the denominator, then the horizontal asymptote is at \( y = \)___________.

If the degree of the numerator is **greater than** the degree of the denominator, then there is NO horizontal asymptote.

If the degree of the numerator is **equal to** the degree of the denominator, then the horizontal asymptote is: _________________________________
Example: Given \( r(x) = \frac{4x - x^2}{x^4 - 16x^2} \)

1) Find where the domain of \( r(x) \) is restricted (set denom = 0 and solve for \( x \)).

2) To find the vertical asymptotes, we must first distinguish which are VA’s and which points are holes. Thus, we need to find the reduced form of \( r(x) \).

3) Next, test the values that restricted the domain of the original function in the denominator of the reduced function. If the number forces the denominator of the reduced function to zero, then there is a vertical asymptote at that number. If the number forces the denominator to some number other than zero, then there is a hole at the point on the function that is generated by that \( x \)-value.

4) To determine whether there is a horizontal asymptote, compare the degrees of the leading terms.
   - If deg num > deg denom, then there is NO H.A.
   - If deg num < deg denom, then H.A. at \( y = 0 \).
   - If deg num = deg denom, then there is a HA at ratio of the coefficients of the leading terms.

5) To find the zeros / roots / \( x \)-intercepts of a rational function: Set \( y = 0 \).
   Note: it is possible that there are no \( x \)-intercepts, example:______________.

6) To find the \( y \)-intercept of a rational function: \( x = 0 \)
   Note: it is possible that there are no \( x \)-intercepts, example:______________.
6 STEPS TO GRAPHING A RATIONAL FUNCTION:

1) Find the domain
2) Reduce $r(x)$ to lowest terms
3) Find $x$- and $y$- intercepts, if they exist
4) Determine vertical asymptotes and holes, if they exist
5) Analyze end behavior
6) Use a sign diagram and plot points to sketch graph (see video solution to the example below for sign diagram instruction).

Example: Graph $\frac{2x^3 - x^2 - 3x}{x^3 - 4x}$
Rational Functions Activity

Objectives for Rational Functions Activity

- Find and interpret horizontal asymptotes
- Find and interpret vertical asymptotes
- Find and interpret intercepts
- Identify holes when present
Long Run Behavior

1. You can use any method you like to identify the long run behavior of these rational functions. Use limit notation to identify the long run behavior of the following rational functions:

   a. \( f(x) = \frac{3x^2}{4x^3 + 5x - 21} \)

   b. \( g(x) = \frac{17x^3 - 4x^2 + 1}{34x^3 + 3x} \)

   c. \( h(x) = \frac{2x^5 - 35x^2 + 21}{60x^4 + 12x^3 + 129} \)

   d. \( k(x) = \frac{2x^2 + 3x^4 - 1}{3 + 2x - 4x^4} \)
Analysis of Rational Functions
What is a rational function? How do you recognize it algebraically? Graphically?

1. Explain how you can determine the long-run behavior (horizontal asymptotes) of a rational function:
   algebraically:
   
   graphically:
   
   by a table:

2. Explain how you can find the zeros of a rational function:
   algebraically:
   
   graphically:
   
   by a table:

3. Explain how you can determine the vertical asymptote(s) of a rational function:
   algebraically:
   
   graphically:
   
   by a table:
Analyzing Rational Functions Further

1. Given the function \( y = \frac{2x + 3}{(x + 7)(x - 2)} \),
   
a. find any zero’s of the function
   
b. find the equation for any vertical asymptote(s)
   
c. find the equation for any horizontal asymptote

2. Given the rational function \( y = \frac{2x - 1}{x - 3} \),
   
a. Find the Vertical Asymptote(s):
   
b. Find the Horizontal Asymptote:
   
c. Find the zero(s):

3. Given the rational function \( y = \frac{5x - 1}{(x - 3)(x + 7)} \)
   
a. Find the Vertical Asymptote(s):
   
b. Find the Horizontal Asymptote:
   
c. Find the zero(s):
For each of the following rational functions,

a. Find the zero’s of the function

b. Find the equation for the vertical asymptote(s)

c. Find the equation for the horizontal asymptote

d. Using your answers to the above questions, and your graphing calculator, draw the graph of the function.

1. \[ f(x) = \frac{3(x-9)}{(x+1)(x-5)} \]  

2. \[ g(x) = \frac{x-2}{2x+3} \]
For each of the following rational functions,

a. Find the zero’s of the function

b. Find the equation for the vertical asymptote(s)

c. Find the equation for the horizontal asymptote

d. Using your answers to the above questions, and your graphing calculator, draw
the graph of the function.

3. \( h(x) = \frac{3x+6}{2x-7} \)

a. 

b. 

c. 

d. 

4. \( k(x) = \frac{7(x+6)}{(x-2)(x+6)} \)

a. 

b. 

c. 

d. 

\[ \begin{align*}
36(x+2) & = 27(x+3) & 4. & & k(x) = \frac{7(x+6)}{(x-2)(x+6)} \\
7(x+7) & = (2)(x-1)(x+7) \\
\end{align*} \]
Application of a Rational Function

1. Each month it costs a gas company $1000 in fees plus an additional $2 per gallon to produce gas.
   a. Create an equation that gives the companies monthly cost as a function of number of gallons produced.
   
   b. Create a function that gives the companies average cost per gallon as a function of the number of gallons produced (hint: \( A(x) = \frac{C(x)}{x} \)).

   c. Find and interpret the horizontal asymptote of your average cost function.

   d. Find and interpret the vertical asymptote of your average cost function.
Lesson 7: Function Composition

Outline

Objectives:
I can perform compositions of functions and state the domain of the resulting function.
I can perform function compositions given a table of values, a graph, or an algebraic formula.
I can decompose functions.
I can justify and interpret solutions to application problems involving a composition of functions.

Definitions / Vocabulary / Graphical Interpretation:

Function composition is a way of combining functions.
The composition of $g$ with $f$, denoted $(g \circ f)(x)$ can also be rewritten as:
(provided $x$ is an element of the domain of $f$, and $f(x)$ is an element of the domain of $g$).

Function Composition

Composition of a function as a mapping between sets (draw a picture):

Composition of Functions Using Tables:

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$g(x)$</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$f(g(x))$</td>
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<td></td>
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<tr>
<td>$g(f(x))$</td>
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<tr>
<td>$f(f(x))$</td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>
Composition of Functions Using Graphs:

Given that \( f(x) = x^3 \) and \( g(x) = x^2 + 4x + 1 \)

\[
f(g(x))
\]

Identify the inside function of \( f(g(x)) \):

Identify the outside function of \( f(g(x)) \):

**Function Decomposition:**

Consider \( h(x) = \sqrt{x^2 + 3} \)

Determine an \( f(x) \) and a \( g(x) \) such that \( h(x) = f(g(x)) \)

\[
f(x) =
\]

\[
g(x) =
\]

**Note: Domain Issues:** The values of the function being input (inside function) must be in the domain of the outside function. Thus, we must exclude

Practical application: An oil slick is growing in a circular fashion. The radius of the oil slick is growing at a rate of 5 ft/sec. Create a composite function that gives the area of the oil slick as a function of time.

Step 1) The equation that models the radius slick as a function of time

Step 2) The equation of the area as a function of radius is:

Step 3) The equation of area as a function of time is:
Functions Composition Activity

Objectives:
- Compose two functions give algebraically, by a table, by a graph, in words
- Decompose a function into two (or more) functions
- Compose more than two functions
- Determine the domain of the composition of two functions
Function Composition Algebraically

1. Given \( f(x) = 3x + 4 \), \( g(x) = x^2 + 1 \), and \( h(x) = \frac{2}{x - 5} \) find:

a. \( f(g(0)) = \)

b. \( g(f(0)) = \)

c. \( f(g(2)) = \)

d. \( g(f(1)) = \)

e. \( f(g(x)) = \)

f. \( g(f(x)) = \)

g. \( h(f(x)) = \)

h. \( f(h(x)) = \)
Function Composition from a Table

1. Given that \( h(x) = f(g(x)) \), fill out the table of values for \( h(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( g(x) )</th>
<th>( x )</th>
<th>( h(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
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<td>1</td>
<td>4</td>
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<td>1</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

2. Given that \( h(x) = f(g(x)) \), fill in the missing values

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( g(x) )</th>
<th>( x )</th>
<th>( h(x) )</th>
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</thead>
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<tr>
<td>1</td>
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<td></td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>
Function Composition from a Graph

1. Use the graphs below to evaluate:

a) \( f(g(6)) = \) __________________________

b) \( g(f(2)) = \) __________________________

c) \( g(f(0)) = \) __________________________

2. Use the Graph of \( f \) and the table for \( g \) to evaluate the following:

\[
\begin{array}{|c|c|}
\hline
x & g(x) \\
\hline
0 & 3 \\
2 & 0 \\
3 & 4 \\
4 & 2 \\
\hline
\end{array}
\]

a) \( f(g(4)) = \) ______________

b) \( g(f(2)) = \) ______________

c) \( f(g(2)) = \) ______________
More Algebraic Compositions

1. Let \( f(x) = x^2 + 1 \) and \( g(x) = 2x + 3 \).
   
   a. \( f(7) = \) ________  
   b. \( g(3) = \) ________  
   c. \( f(g(3)) = \) ________  
   
   d. \( f(g(x)) = \) ____________________  
   e. \( g(f(x)) = \) ____________________  

2. Use the words input and output, as appropriate, to fill in the missing blanks:

   The function \( f(g(t)) \) uses the ____________ of the function \( g \) as the _____________ to the function \( f \). The function \( g(f(t)) \) uses the ____________ of the function \( f \) as the _____________ to the function \( g \).

3. Let \( u(x) = p(q(x)) \) and \( v(x) = q(p(x)) \) where \( p(x) = 3x - 4 \) and \( q(x) = x^2 + 5 \).

   a. Calculate \( u(4) \) and \( v(4) \). Are they the same?
      \( u(4) = \) ________  
      \( v(4) = \) ________  

   b. Find formulas for \( u(x) \) and \( v(x) \) in terms of \( x \). What can you conclude about the order of functions in doing a composition?

      \( u(x) = \) ____________________  

      \( v(x) = \) ____________________
4. Let \( f(x) = x^2 + 3 \) and \( g(x) = 2x + 1 \).

a. \( f(7) = \)

b. \( g(3) = \)

c. \( f(g(3)) = \)

d. \( f(f(3)) = \)

e. \( f(g(x)) = \)

f. \( g(f(x)) = \)

g. \( g(g(x)) = \)
**Function Decomposition**

Just as we can compose two functions to create a new function, we can decompose a function into two separate functions, one being the input of the other.

**Warm-up:** Use the words *input* and *output*, as appropriate, to fill in the missing blanks:
The function \( f(g(t)) \) uses the __________ of the function \( g \) as the __________ to the function \( f \). The function \( g(f(t)) \) uses the __________ of the function \( f \) as the __________ to the function \( g \).

1. Let \( g(x) = \frac{1}{x+1} \). Decompose \( g \) into functions, \( f \) and \( h \), such that \( g(x) = f(h(x)) \). [Do not use \( f(x) = x \) or \( g(x) = x \)]

\[
h(x) = ____________ 
\]

\[
f(x) = ____________
\]

2. Consider the composite function \( w(x) = \sqrt{1 + x^2} \).

Find two functions (\( f \) and \( g \)) such that \( w(x) = f(g(x)) \). [Do not use \( f(x) = x \) or \( g(x) = x \)]

\[
g(x) = ____________
\]

\[
f(x) = ____________
\]
3. Consider the composite function \( w(x) = \sqrt{1 + x^2} \).

Find three functions \( f, g, \) and \( h \) such that \( w(x) = f(g(h(x))) \).  [Do not use \( f(x) = x \), \( g(x) = x \), or \( h(x) = x \)]

\[ h(x) = \] ________________

\[ g(x) = \] ________________

\[ f(x) = \] ________________

4. Now consider the composite function \( f(x) = 3(x - 1)^2 + 5 \). Decompose \( f \) into three functions, \( u, v, \) and \( w \), such that \( f(x) = u(v(w(x))) \).  [Do not use \( u(x) = x \), \( v(x) = x \), or \( w(x) = x \)]

\[ w(x) = \] ________________

\[ v(x) = \] ________________

\[ u(x) = \] ________________

Can you see a way to decompose \( f \) into four functions?  Demonstrate how to do it:
**Domain of a Composition**

When finding the domain of a composition we have to take into consideration the domains of the ‘inside’ and ‘outside’ functions as well as the domain of our composition.

**For example:**

Given $f(x) = x^2$ and $g(x) = \sqrt{x}$, then our two composed functions are

$h(x) = f(g(x)) = (\sqrt{x})^2 = x$ and $k(x) = g(f(x)) = \sqrt{x^2} = x$

They both simplify down to $x$, but are not the exact same function as they have different domains. The domain of $h(x)$ is $[0, \infty)$ since we cannot input negative numbers into $\sqrt{x}$ which is the inside function of the composition. On the other hand, the domain of $k(x)$ is $(-\infty, \infty)$ because our input goes into $x^2$ first which has a domain of all real numbers.

**Example 2**

Given $f(x) = x^2 - 4$ and $g(x) = \sqrt{x}$. For our composition $h(x) = g(f(x)) = \sqrt{x^2 - 4}$ we have an inside function that has a domain of all real numbers, but an outside function that has a domain of $[0, \infty)$ so we must make sure that we only get non-negative numbers from our inside function. Now $f(x) = x^2 - 4$ is negatives for inputs between -2 and 2, so we must exclude those from the domain of $h(x)$. Thus we get $(-\infty, -2) \cup (2, \infty)$ for our domain.

Find the following compositions, $f(g(x))$ and $g(f(x))$ and their domains.

1. $f(x) = x^2$, $g(x) = \frac{1}{x - 4}$

2. $f(x) = \sqrt{x - 4}$, $g(x) = \frac{1}{x}$
Lesson 8: Inverse Functions

Outline

Inverse Function Objectives:
I can determine whether a function is one-to-one when represented numerically, graphically, or algebraically.
I can determine the inverse of a relation when represented numerically, analytically, or graphically.
I can use and interpret inverse notation in a context.

Definitions / Vocabulary / Graphical Interpretation:
An inverse function, \( f^{-1} \), is a function that maps the outputs of \( f \) to the inputs of \( f \).
A practical example of an inverse function is decoding a spy message. Explain:

Given the function \( f(x) = 3x + 2 \)
Explain the mathematical processes of the function, and then reverse the process to explain the inverse.

Draw an example of a mapping of an inverse function:

Inverse notation: if \( f(x) = x^2 \), then \( f^{-1}(x) = \sqrt{x} \) with domain restricted to:

Composing inverse functions gives back the original input. If \( f(g(x)) = x \), AND \( g(f(x)) = x \) then \( f \) and \( g \) are inverse functions and they are said to be invertible.

List three properties of inverses that are true if \( f \) and \( g \) are inverse functions:

1) 

2) 

3)
Key idea: If a function passes the horizontal line test, it is invertible. Why is that?

Is $f(x) = x^2$ invertible? Why or why not? Is it one-to-one? Why not? Can we make it invertible? How?

In order to find an inverse function you need to solve for the input variable.

Example: Let $f(x) = \frac{3x}{x + 1}$

a) Assuming $f$ is one-to-one, find a formula for $f^{-1}(x)$

b) Check your answer to (a) using function composition.

c) Find the range of $f$. 
Finding inverses using a table:

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
</tr>
</tbody>
</table>

Find $f^{-1}(9)$

Finding inverses using a graph:

Find $f^{-1}(1)$

Interpreting functions:

Suppose $R = f(x)$ gives the revenue, in hundreds of dollars, when a company sells $x$ pairs of shoes. Interpret:

$f(10) = 5$

$f^{-1}(20) = 40$
Inverse Functions Activity

Objectives:

- Find the input of a function given an output
- Find the inverse function
- Determine the domain and range of function and its inverse
- Determine whether or not an inverse function exists
- Use and interpret inverse function notation
Inverse Function Notation
1. Explain the difference in meaning of the notation $f(2) = 5$ versus the notation $f^{-1}(5) = 2$.

2. Suppose the point (10, -5) lies on the graph of a function $f$. What point lies on the graph of $f^{-1}$?

3. The number of people (in thousands) in a city is given by the function $f(t) = 20 + 0.4t$, where $t$ is the number of years since 1970.
   a. In the context of this problem, explain what $f(25)$ and $f^{-1}(25)$ mean (no calculations required). What is the unit of measure (number of people or number of years) for $f(25)$ and $f^{-1}(25)$?
   b. Now calculate $f^{-1}(25)$.

4. The graph of $f$ from problem 3 is shown below. Estimate $f^{-1}(25)$ by reading the graph below.
5. Suppose we have the function $w = j(x)$ where $w$ represents the average daily quantity of water (in gallons) required by an oak tree of height $x$ feet.

   a. What does the expression $j(25)$ represent? What are its units of measure?

   b. What does the expression $j^{-1}(25)$ represent? What are its units of measure?

   c. What does the following equation tell you about $v$: $j(v) = 50$

   d. Re-write the statement $j(v) = 50$ in terms of $j^{-1}$.

   e. On a certain acreage, oak trees on average measure $z$ feet high and an oak tree of average height requires $p$ gallons of water. Represent this statement first in terms of $j$ and then in terms of $j^{-1}$.

6. The total cost, $C$, in dollars for a clothing factory to make ‘$j$’ jackets is given by the function $C = f(j)$. Interpret the meaning of the following notation within the context of the story just given.

   a. $f(30)=678$

   b. $f^{-1}(30) = 678$
Calculating Inverses Numerically

1. Using the chart, find
   a. $h(0)$
   b. $h^{-1}(-1)$
   c. $h(-2)$
   d. $h^{-1}(-2)$
   e. $(h(2))^{-1}$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$h(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>3</td>
</tr>
<tr>
<td>-1</td>
<td>-2</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
</tr>
</tbody>
</table>

2. Using the graph estimate:
   a. $f(0)$
   b. $f^{-1}(0)$
   c. $f(-1)$
   d. $f^{-1}(-1)$
Calculating Inverse Functions
For the following functions find:
   a. The inverse function
   b. Write the inverse function using inverse function notation
   c. State the domain and range of the original function
   d. State the domain and range of the inverse function

1. \( f(x) = 3x - 2 \)

2. \( g(x) = \frac{1}{x} - 2 \)

3. \( h(x) = \sqrt{1 + x} \)
4. The formula $F = f(C) = 1.8C + 32$ converts temperatures in degrees Celsius, $C$, to degrees Fahrenheit, $F$.

a. What is the input to the function $f$? What is the output?

b. Find a formula for the inverse function giving Celsius as a function of Fahrenheit.

c. Use inverse function notation to write your formula.

$$f^{-1}(\underline{\text{____}}) = \underline{\text{____}}$$

d. What is the input to the function $f^{-1}$? the output?

e. Interpret the meaning of the notation: $f(50) = 122$

f. Interpret the meaning of the notation: $f^{-1}(200) = 93.3$
5. The formula $V = f(r) = \frac{4}{3} \pi r^3$ gives the volume of a sphere of radius $r$.

a. What is the input to the function $f$? What is the output?

b. Find a formula for the inverse function giving radius as a function of volume.

c. Use inverse function notation to write your formula found in #2 above.

$$f^{-1}(\_\_) = \_\_\_\_$$

d. What is the input to the function $f^{-1}$? The output?

e. Suppose you already know the radius of the sphere. Which function gives you the volume?

f. Now suppose you already know the volume. Which function gives you the radius?

g. Explain the meaning of $f^{-1}(V) = 5$. 

Verifying Inverse Functions

1. Suppose \( f(x) = 2x - 4 \) and \( g(x) = \frac{x + 4}{2} \). Are \( f \) and \( g \) inverse functions?

   a. Use algebraic methods to verify. That is, find \( f(g(x)) \) and then find \( g(f(x)) \).

First find \( f(g(x)) \):

Now find \( g(f(x)) \):

What do you conclude?

b. Demonstrate the inverse relationship by means of a graph:

c. Explain verbally:

<table>
<thead>
<tr>
<th>Describe in words what ( f ) “does to its input.”</th>
<th>Describe in words what ( g ) “does to its input.”</th>
</tr>
</thead>
<tbody>
<tr>
<td>i.</td>
<td>i.</td>
</tr>
<tr>
<td>ii.</td>
<td>ii.</td>
</tr>
</tbody>
</table>

d) Fill in the cells for the output and then explain the inverse relationship:

<table>
<thead>
<tr>
<th>( f(x) = 2x - 4 )</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>input</td>
<td>2</td>
</tr>
<tr>
<td>output</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( g(x) = \frac{x + 4}{2} )</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>input</td>
<td>0</td>
</tr>
<tr>
<td>output</td>
<td></td>
</tr>
</tbody>
</table>
Lesson 9: Exponential Functions

Outline

Objectives:
I can analyze and interpret the behavior of exponential functions.
I can solve exponential equations analytically and graphically.
I can determine the domain and range of exponential functions.
I can classify a function as exponential when represented numerically, analytically, or graphically.
I can determine regression models from data using appropriate technology and interpret the results.
I can justify and interpret solutions to application problems.

Definitions / Vocabulary / Graphical Interpretation:

Doubling Pennies Problem:

<table>
<thead>
<tr>
<th>Day</th>
<th>Pennies</th>
<th>Average Rate of Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The units on the average rate of change in this situation are______________________.

Since our rate of change is doubling for each unit increase, we have an increasing rate of change, and we would expect the graph to be__________________________.

Exponential Functions have the form $f(x) = ab^x$ where $a \neq 0$ and $b > 0$.

The input variable is located in the exponent. The output corresponding to an input of zero is called the______________________________ and is represented by $a$.

The base $b$ represents repeated multiplication of the______________________________.
With linear functions we have a constant change in amount, but in exponential functions we have a constant ________________. In the doubling pennies problem from page 1, since we were repeatedly multiplying by 2, our base would be __________, and our initial value was ___________. Therefore, the exponential model of this function $f(x) = ab^x$ would be ____________________.

**Graphs of Exponential Functions:**

1) For a positive initial value $a$, if the **base $b > 1$**, then the graph of the function looks like:

It is always increasing and concave up.

The domain is ________________ and the range is_________________________.

The exponential function has a horizontal asymptote at _________________.

2) For a positive initial value $a$, if the **base $0 < b < 1$**, then the graph of the function looks like:

It is always decreasing and concave up.

Its domain is ________________ and its range is_________________________.

It also has a horizontal asymptote at _______________________.

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Finding an exponential function given 2 points: We have two methods: elimination and substitution.

Elimination Method:
Given the points (2, 6) and (5,48) and recalling that \( y = ab^x \) find the exponential function between the two points.

Step 1: Plug in the \((x,y)\) coordinates of the given points into the exponential equation.

\[
6 = ab^2 \quad \text{and} \quad 48 = ab^5
\]

Step 2: Divide the equations to ‘eliminate’ \( a \) and solve for \( b \).

\[
\frac{6}{48} = \frac{ab^2}{ab^5} = b^{-3}
\]

Step 3: Plug your answer for \( b \) into either of your equations from Step 1 and solve for \( a \).

Substitution Method:
Given the points (2, 12) and (-1, 3/2) and recalling that \( y = ab^x \) find the exponential function between the two points.

Step 1: Plug in the \((x,y)\) coordinates of the given points into the exponential equation:

\[
12 = ab^2 \quad \text{and} \quad \frac{3}{2} = ab^{-1}
\]

Step 2: Solve for \( a \) or \( b \) in one equation and substitute into the other equation to solve for the other variable.

Step 3: Rewrite the equation using the factors \( a \) and \( b \) in \( y = ab^x \) form.
End Behavior: The parent graph of an exponential has the following properties:

If \( b > 1 \), then \( \lim_{x \to \infty} f(x) = \) and \( \lim_{x \to -\infty} f(x) = \)

If \( 0 < b < 1 \) then \( \lim_{x \to \infty} f(x) = \) and \( \lim_{x \to -\infty} f(x) = \)

Thus, graphs of exponential functions have horizontal asymptotes.
Note: if the function has been vertically shifted, these limits will change!

Key idea: Growth factor \((b) = (1 + \text{growth rate})\)

Growth factor \((b) = (1 + \text{constant percent change})\)

Note: If the function is decaying, the growth rate is negative, thus the growth factor \(b\) is between zero and one. \(0 < b < 1\).

Example of exponential growth:
Given an initial salary \(a = \$50,000\) and a guaranteed raise (constant percent change) of 5\% per year, determine a function \(S(t)\) that models your annual income each year.

New amount = old amount + 5\% of old amount
New amount = (100\% + 5\%) of old amount
New amount = (1 + 0.05) of old amount
New amount \((S) = 1.05\) of old amount
Let \(a\) = the initial salary and \(t\) = time in years

\(S(t) = \)_____________________________

Example of exponential decay:
Given an initial population \(a\) of bacteria and a decay rate of 24\% per hour, give a model that tells the population \(P(h)\) of bacteria after \(h\) hours.

New amount = old amount – 24\% of old amount
New amount = 100\% – 24\% of old amount
New amount = (1 – .24) of old amount
New amount \((P) = 0.76\) of old amount
Let \(a\) = the initial population and \(h\) = time in hours

\(P(h) = \)_____________________________
Financial applications:

Annual Compounding
We can write a generalized formula for an interest rate \( r \) compounded annually:

\[
B = P(1 + r)^t
\]

Multiple Compoundings
We can also write a generalized formula for interest compounded multiple times during the year:

\[
B = P\left(1 + \frac{r}{n}\right)^{nt}
\]

Where
- \( B \) = final balance in the account
- \( P \) = Initial amount deposited in the account
- \( r \) = annual % rate
- \( n \) = number of compounding periods per year
- \( t \) = number of years of compounding

Continuous Compounding
Finally, as the number of compounding times increases per year, we can represent continuous compounding using the constant \( e \) (also called the “natural base”) as the growth factor, giving the formula:

\[
B = Pe^{rt}
\]

Where
- \( B \) = final balance in the account
- \( P \) = Initial amount deposited in the account
- \( r \) = annual % rate
- \( t \) = number of years of compounding

\( e \) = growth factor for continuous compounding where \( e = \lim_{n \to \infty} (1 + \frac{1}{n})^n \)

Changing between forms: Notice that there are two formulas to represent exponential growth, one for periodic compounding and one for continuous compounding.

\[
y = ab^t \quad \text{for periodic compounding} \quad \quad \quad \quad \quad \quad \quad y = ae^{kt} \quad \text{for continuous compounding.}
\]

The difference between the two forms is the base growth factor where \( b = e^k \). We use this fact to convert between the two forms of exponential equations.
Effective Yield
Effective yield is the annual rate of return on an investment, and converts a nominal interest rate to an annual effective yield rate, based on the interest rate and the number of times of compounding. Thus, you may have 2 different accounts with the same nominal rate, but they may have different effective yields if they have different compounding periods. Effective yield allows you to compare the nominal interest rates.

The key to finding the effective yield is to first calculate the growth factor based on the nominal rate and number of compounding periods, then subtract 1 and turn it into a percent.

*Example 1:* An account pays interest at the rate of 5% per year compounded monthly.

Nominal Rate = 5% (the advertised rate; does not account for compounding)

Effective Rate = 5.12% (what you actually earn because of compounding):

\[
\left(1 + \frac{.05}{12}\right)^{12} = 1.05116
\]

*Example 2:* Say you have $500 to invest at 6%. Set up a formula to calculate the amount in the account after \(t\) years if interest is compounded:

- Annually

- Quarterly

- Continuously
**Determining if data are exponential:** Given a table of inputs and outputs, how does one determine what algebraic model best fits the data? We begin by looking at the first differences between the first outputs to determine whether a constant rate of change is shown implying a linear model. Second, we look at the second differences to see if a quadratic model might be a good fit. Third, we can look at the common ratio of successive terms to determine if an exponential model might be appropriate.

**Note:** Inputs in this table are equally spaced. Does that matter? Why or why not?

**Stock Price Example:**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>1st Diffs</th>
<th>2nd Diffs</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1990</td>
<td>6.75</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1992</td>
<td>9.12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1994</td>
<td>12.30</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1996</td>
<td>16.61</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1998</td>
<td>22.42</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>30.27</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Exponential Functions Activity

Objectives:

- Recognize situations having a constant percent change as exponential
- Create an exponential model given two points
- Create and interpret an exponential model in a context
- Compound interest problems
- Perform exponential regression
Exponential vs Linear Growth

1. How is exponential growth different from linear growth?

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>13</td>
<td>15</td>
</tr>
<tr>
<td>$g(x)$</td>
<td>5</td>
<td>10</td>
<td>20</td>
<td>40</td>
<td>80</td>
<td>160</td>
</tr>
</tbody>
</table>

Explain the difference:

2. Name the following attributes as linear or exponential, as appropriate.

_________ constant percent change
_________ constant amount of change
_________ an initial starting amount is given
_________ multiply the value of $y$ by the same factor for each unit increase in $x$
_________ "the longer it goes, the faster it grows"
_________ add (subtract) to the value of $y$ by the same amount for each unit increase in $x$

3. Suppose you are hired in a new job with a starting salary of $30,000. Fill in the table with your annual salary increases depending on whether your boss is using a linear model or an exponential model.

Linear Model: After each year, you will get a salary raise of $1500. That amount of increase is constant every year.

Exponential Model: After each year, you will always get a salary raise of 5% from the previous year.

<table>
<thead>
<tr>
<th>Year</th>
<th>$\text{Amount of Raise}$</th>
<th>$\text{Salary}$</th>
<th>$\text{Amount of Raise}$</th>
<th>$\text{Salary}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Linear Model</td>
<td>Linear Model</td>
<td>Exponential Model</td>
<td>Exponential Model</td>
</tr>
<tr>
<td>0</td>
<td>$0.00$</td>
<td>$30,000$</td>
<td>$0.00$</td>
<td>$30,000$</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Which model would you rather be paid by? ☺
4. Read each sentence about a town's population growth (decline) and determine the formula that matches the statement.

____ i. A town starts with 500 people and grows by 50 people per year.

____ ii. A town starts with 500 people and grows by 50% per year.

____ iii. A town starts with 500 people and declines by 50% per year.

____ iv. A town starts with 500 people and declines by 50 people per year.

a. \( y = 500(50)^t \) 

b. \( y = 500(0.50)^t \) 

c. \( y = (500 \cdot 50)^t \) 

d. \( y = 500(50t) \) 

e. \( y = 500 + 50t \) 

f. \( y = 500(−0.50)^t \) 

g. \( y = 500(t)^{50} \) 

h. \( y = 500 − 50t \) 

i. \( y = 500(1.5)^t \) 

5. Each formula below describes the population trend for a fictitious town. Write a sentence that describes the town according to each formula, where \( P \) represents population and \( t \) represents number of years.

a. \( P = 1000(1.1)^t \)

b. \( P = 4000 − 50t \)

c. \( P = 3000(0.75)^t \)

d. \( P = 2000 + 80t \)
The Form of an Exponential Function, \( P = ab^t \)

1. The populations, \( P \), of six towns with time \( t \) in years are given by:
   
   i. \( P = 1000(1.08)^t \)
   ii. \( P = 600(1.12)^t \)
   iii. \( P = 2500(0.9)^t \)
   iv. \( P = 1200(1.185)^t \)
   v. \( P = 800(0.78)^t \)
   vi. \( P = 2000(0.99)^t \)

   a. Which towns are growing in size? Which are shrinking?

   b. Which town is growing the fastest? What is the annual percent growth rate for that town?

   c. Which town is shrinking the fastest? What is the annual percent decay rate for that town?

   d. Which town has the largest initial population (at \( t = 0 \))? Which town has the smallest?

2. A town has population 3000 people at year \( t = 0 \). Write a formula for the population, \( P \), in year \( t \) if the town:

   a. Grows by 200 people per year.

   b. Grows by 6% per year.

   c. Shrinks by 50 people per year.

   d. Shrinks by 4% per year.
An Exponential Model

On August 2, 1988, a U.S. District Court judge imposed a fine on the city of Yonkers, New York, for defying a federal court order involving housing desegregation. The fine started at $100 for the first day and was to double daily until the city chose to obey the court order.

1. What was the daily percent growth rate of the fine?

2. Find a formula for the fine as a function of \( t \), the number of days since August 2, 1988.

3. If the city of Yonkers had waited 30 days before obeying the court order, what would the fine have been?

4. In 1988, the annual budget of the city was $337 million. If the city had chosen to disobey the court order, at what point would the fine have wiped out the entire annual budget?
Distance Traveled

1. Suppose you travel a distance of \( m \) miles in \( h \) hours. The table gives certain values for \( m \) and \( h \).

<table>
<thead>
<tr>
<th>time elapsed in hours, ( h )</th>
<th>distance in miles, ( m )</th>
<th>Average rate of change (miles per hour)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>- - -</td>
</tr>
<tr>
<td>0.5</td>
<td>26</td>
<td></td>
</tr>
<tr>
<td>1.25</td>
<td>65</td>
<td></td>
</tr>
<tr>
<td>2.75</td>
<td>143</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>260</td>
<td></td>
</tr>
</tbody>
</table>

a. What kind of function is this? linear or exponential?

b. What do you notice about the average rate of change? Is it constant?

c. Give a formula for distance traveled as a function of time.

d. In the context of this problem, interpret the slope in your formula.

Mockingbird Population

2. Suppose you count the number of mockingbirds in your backyard and find that there are 10. The next day you count again, and find that the number is now 20. On the third day, you count again, and find 40 mockingbirds! Assuming that this pattern of growth continues, fill in the table.

<table>
<thead>
<tr>
<th>Day, ( d )</th>
<th>Number of birds, ( N )</th>
<th>Average rate of change (birds per day)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
<td>- - -</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. What kind of function is this? linear or exponential?

b. What do you notice about the average rate of change? Is it constant?

c. Give a formula for the number of mockingbirds as a function of time.

d. What growth factor did you use in your formula? In the context of this problem, what does it mean?
Exponential Growth & Decay

1. Given the percent change, find the growth (or decay) factor:
   a. Growth of 10% per year
   b. Growth of 1% per year
   c. Growth of 90% per year
   d. Growth of 25% per year
   e. Growth of 100% per year
   f. Decline of 10% per year
   g. Decline of 1% per year
   h. Decline of 90% per year
   i. Decline of 25% per year

2. Given the growth (or decay) factor, find the percent change. Write your answer as a growth or decay rate.
   a. 1.12
   b. 0.99
   c. 2.41
   d. 0.75
   e. 0.01
   f. 4
1. A colony of bacteria starts with 300 organisms and doubles every week.

<table>
<thead>
<tr>
<th>Weeks</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bacteria</td>
<td>300</td>
<td>600</td>
<td>1200</td>
<td>2400</td>
<td>4800</td>
</tr>
</tbody>
</table>

a. Write an exponential equation to represent the data in the table.

b. How many bacteria will there be after 8 weeks? _____________

2. The population of a small town is 7,000, and is growing at a rate of 12% per year.

a. Write an exponential equation to represent the population growth.

b. What will the population of the town be in 15 years? _____________

3. 150 grams of a radioactive element decays at rate of 9.5% per day.

a. Write an exponential equation to represent this scenario. __________________

b. How much radioactive material will be left in 2 weeks? _____________

4. \(Y = 15000(1.17)^x\) describes the growth of Happy College. What information does this formula give us about the college?

5. There are 950 students enrolled in Math 150 at the beginning of the semester. If students drop at the rate of 1% per week, how many students will be enrolled during the 15th week of the semester?

6. Imagine that 2000 people catch a cold, all at the same time. Half of those who are sick get well each day.

a. Write an equation to represent the number of people who are sick on any given day.

b. How many people will be well on day 7?
Finding Exponential Formulas

For each of these exercises, use the method of common ratios to find each exponential formula:

Common Ratio Method
\[
\frac{ab^{y_2}}{ab^{y_1}} = \frac{y_2}{y_1}
\]

1. Find a formula for the exponential function that passes through the two points (0, 1) and (2, 100).

2. Find a formula for the exponential function that passes through the two points (0, 1) and (4, 1/16).

3. Find a formula for the exponential function that passes through the points (7, 3.21) and (13, 3.75).
**Compound Interest**

A formula used for compounding is \( B = P \left(1 + \frac{r}{n}\right)^{nt} \), where \( B \) is the balance, \( P \) is the starting amount, \( r \) is the annual growth rate, \( n \) is the number of compounding periods in one year, and \( t \) is the number of years.

1. What does the fraction \( \frac{r}{n} \) represent?

2. What does the product \( nt \) represent?

3. Suppose you invest $1000 in fund A which pays 5% compounded daily. After 10 years, what is the balance in your account?

4. Suppose you can invest the $1000 in fund B which pays 5% compounded every second for a fee of $100. Would it be worth it to invest in account B if you plan to remove the money after 10 years? Make a conjecture with your group and then prove or disprove your conjecture.

**Nominal versus Effective Rates**

*Example:* An account pays interest at the rate of 5% per year compounded monthly.

Nominal Rate = 5% (the advertised rate; does not account for compounding)

Effective Rate = 5.12% (what you actually earn because of compounding):

\[
\left(1 + \frac{.05}{12}\right)^{12} = 1.05116
\]

4. Another investment earns 4.5% compounded daily. What is its effective rate?
Continuous Compounding

A formula used for continuous compounding is $B = Pe^{rt}$, where $B$ is the balance, $P$ is the starting amount, $r$ is the annual interest rate and $t$ is the number of years. The irrational number $e$ is called the natural number and is used for the base in this formula. Recall that $e = 2.71828....$

1. For continuous growth, how many compounding periods per year are there?

2. Suppose an account pays interest at the rate of 5% compounded continuously.
   a. If you invest $1000 in this account, what will your balance be after one year?
   b. In this case, what is your effective growth rate?

3. If you put $1000 in an account that pays 6% compounded daily, isn’t the balance growing continuously? Why not use $B = Pe^{rt}$? Explain.
Lesson 10: Logarithmic Functions

Outline

Objectives:
I can analyze and interpret the behavior of logarithmic functions, including end behavior and asymptotes.
I can solve logarithmic equations analytically and graphically.
I can graph logarithmic functions.
I can determine the domain and range of logarithmic functions.
I can determine the inverse function of a logarithmic function.
I can determine regression models from data using appropriate technology and interpret the results.
I can justify and interpret solutions to application problems.

Definitions / Vocabulary / Graphical Interpretation:

Two ways to solve for a variable in the exponent are 1) graphically using the calculator intersect function; 2) algebraically using logarithms. A LOG IS AN EXPONENT.

The logarithmic function \( \log_b y = x \) is the __________________ of the exponential function \( b^x = y \).

Thus, the output of the logarithmic function is the exponent (input) of the exponential function. The base of the exponential function is the base of the logarithmic function. The input (argument) to a logarithmic function cannot be zero or negative.

Ex 1: rewrite \( 2^{1/2} \) in log form:

Ex 2: rewrite \( 5^x \) in log form:

Ex 3: rewrite \( 3^{3x} \) in log form:

Ex 4: rewrite \( \log_3 25 = 2 \) in exponential form:

Ex 5: rewrite \( \log_2 x = y \) in exponential form:

Ex 6: Solving for a variable in the exponent using logarithms \( 200 = 100(1.045)^t \)
Recall the general form of an exponential function:  \( y = ab^t \)

Where:
- \( y \) = final amount
- \( a \) = initial amount
- \( b \) = growth factor = \((1 + \text{growth rate})\)
- \( t \) = time

Doubling time example: Find the time it takes $100 to double at an annual interest rate of 4.5%

Set up: \( 200 = 100(1+.045)^t \) and solve for \( t \)

Half-life example: Find the half-life of a substance that is decaying at a rate of 10% per day.

Set up: \( \frac{1}{2} a = a(1-0.1)^t \) and solve for \( t \)

NOTE: We do not need to know the initial amount to find doubling time or half-life, we just need to know the rate the initial amount is growing or decaying at.

**Bases and Properties of Logarithms**

The two most common bases for logarithms are the common base 10, and the natural base \( e \).

**Properties of the Common Logarithm:**
- Calculator LOG key calculates \( \log_{10} \)
- \( \log_{10} y = x \) means \( 10^x = y \)
- \( \log 1 = 0 \) and \( \log 10 = 1 \)
- The functions \( 10^x \) and log \( x \) are inverses:
  \( \log(10^x) = x \) for all \( x \) and \( 10^{\log x} = x \) for all \( x > 0 \) (all log arguments must be positive)
- For \( a \) and \( b \) both positive and all \( t \):
  \( \log(ab) = \log a + \log b \)
  \( \log\left(\frac{a}{b}\right) = \log a - \log b \)
  \( \log(b^t) = t \cdot \log b \)
Properties of the Natural Logarithm:

- \( y = \ln x \) means \( e^y = x \)
- \( \ln 1 = 0 \) and \( \ln e = 1 \)
- The functions \( e^x \) and \( \ln x \) are inverses:
  \( \ln(e^x) = x \) for all \( x \) and \( e^{\ln x} = x \) for all \( x > 0 \)
- For \( a \) and \( b \) both positive and all \( t \):
  \( \ln(ab) = \ln a + \ln b \)
  \( \ln\left(\frac{a}{b}\right) = \ln a - \ln b \)
  \( \ln(b^t) = t \cdot \ln b \)

Change of Base Formula

To change an uncommon base logarithm to one which is calculator friendly so that it may be evaluated, we typically use a common base for ‘\( c \)’ such as base 10 or base \( e \). Note: the formula below works for all bases, so long as the bases on the right side of the equation are the same.

\[
\log_b a = \frac{\log_c a}{\log_c b}
\]

Converting between a Periodic Growth Rate and a Continuous Growth Rate

Any exponential function can be written as \( y = ab^t \) or \( y = ae^{kt} \)

So we can equate \( b = e^k \iff \ln b = \ln e^k = k \)

Ex: Convert \( Q = 5(1.2)^t \) into the form \( Q = ae^{kt} \)

Annual Growth Rate: \hspace{1cm} Continuous Growth Rate:

Graphs of Exponential and Logarithmic Functions, Asymptotes, and End Behavior

Exponential functions and logarithmic functions are inverses of each other. Thus, the output of the logarithmic function is the input to the exponential function; and the input to the logarithmic function is the output of the exponential function. The graph of the logarithmic function is the graph of the exponential function reflected about the line \( y = x \). The exponential graph has a horizontal asymptote and the logarithmic graph has a vertical asymptote.
Below is a graph of both \( f(x) = \ln x \) and \( f(x) = e^x \)

The parent function \( f(x) = \ln x \) has a vertical asymptote at \( x = 0 \)
End behavior of: \( f(x) = \ln x \)

\[
\lim_{x \to \infty} \ln x = \infty
\]
\[
\lim_{x \to 0^+} \ln x = -\infty
\]

The parent function \( f(x) = e^x \) has a horizontal asymptote at \( y = 0 \)
End behavior: \( f(x) = e^x \)

\[
\lim_{x \to \infty} e^x = \infty
\]
\[
\lim_{x \to -\infty} e^x = 0
\]

### Domain and Range of Logarithmic Functions

Note: if a transformation is applied to the logarithmic function, the domain may change. For example, if the function is shifted horizontally, the limits above will shift accordingly. The \( x \)-value that makes the argument of the log 0 becomes the vertical asymptote. The logarithmic function does not have a horizontal asymptote.

Ex: The vertical asymptote of the function \( f(x) = 2 + \log(3 - x) \) is \( x = \)

General behavior:

\[ f(x) = a + b \ln x, \quad b \neq 0 \]

If \( b > 0 \) then the function is increasing (slowly) and concave down.
If \( b < 0 \) then the function is decreasing (slowly) and concave up.
Logarithmic Models and Scales

pH Scale (Chemical Acidity)  \[ \text{pH} = -\log [H^+] \]
where \( H^+ \) is the hydrogen ion concentration given in moles per liter. The greater the hydrogen ion concentration, the more acidic the solution.

Richter Scale (Seismic Activity)  \[ \log \frac{I_c}{I_n} = R \]
where \( I_c \) is the intensity of the earthquake, and \( I_n \) is how much the earth moves on a normal day (1 micron = 10^{-4} \text{ cm})

Decibels dB (Sound Intensity)  \[ dB = 10 \log \left( \frac{I}{I_o} \right) \]
where \( I \) is the intensity of the sound measured in \( \frac{\text{watts}}{\text{meter}^2} \), and \( I_o \) is the softest audible sound \( 10^{-2} \frac{\text{watts}}{\text{meter}^2} \).
Logarithmic Functions Activity

Objectives

- Understand and use definition of log
- Rewrite exponential equations as logarithmic equations
- Rewrite logarithmic equations as exponential equations
- Identify domain, range and shape of graph of logarithmic function
- Use properties of logs
- Use logarithms to solve exponential equations
- Solve logarithmic equations
- Applications of logarithms
**Definition of Log**

Log is defined as follows: \( y = \log_b(x) \iff x = b^y \)

1. Use the definition of log to convert each of the following to exponential form:

   a. \( \log_2(16) = 4 \)

   b. \( \log_5(125) = 3 \)

   c. \( \log(100) = 2 \)

   d. \( \ln(e) = 1 \)

   e. \( \log_a(c) = d \)

2. Use the definition of log to convert each of the following to logarithmic form:

   a. \( 3^2 = 9 \)

   b. \( 10^3 = 1000 \)

   c. \( 100^{\frac{1}{2}} = 10 \)

   d. \( t^r = s \)
Graph of a Logarithmic Function
What does the common log function look like?

*Draw its graph here.*

1. Use function notation to write the equation for the common log function:

2. Name the base of the common log function: __________

3. Describe the “behavior” of the graph of the common log function.
   a. Is the graph concave up or concave down? ______________________
   b. As \( x \to \infty \), \( y \to \) _________________
   c. As \( x \to 0^+ \) (read: “as \( x \) approaches zero from the right side”), \( y \to \) ______
   d. As \( x \) increases, does \( y \) increase or decrease? ________________
   e. The log function is a ______________ function. <increasing or decreasing?>
   f. The graph has an asymptote. Is it vertical or horizontal? ______________
   g. The equation for the asymptote is: ______________
   h. Does the log function have a \( y \)-intercept? If so, identify it: ______________
   i. Does the log function have a \( x \)-intercept? If so, identify it: ______________
Properties of Logs

1. Use the properties of logarithms to write an equivalent expression for each of the following:

   a. \( \log(10^4) = \)

   b. \( 10^{\log x} = \)

   c. \( \log(xy) = \)

   d. \( \log \left( \frac{x}{y} \right) = \)

   e. \( \log \left( \frac{xy}{z} \right) = \)

   f. \( \log x^2 + \log y^2 = \)

   g. \( \log \sqrt{x} = \)

2. Use log properties to write as a single logarithm

   a. \( 5 \log_3 x + 2 \log_3 5 \)

   b. \( 3 \ln x - \ln 4 \)

   c. \( 2 \log x + \log(x + 2) \)

   d. \( \ln(x) + \ln(6) - \ln(y) \)
3. Tell whether each statement involving logarithms is true or false (Assume $x$, $y$ and $z$ are positive.) If it is false, change it so it is true.

a. \( \log \sqrt{x} = \frac{1}{2} \log (x) \)

b. \( \ln(xy) = \ln(x) + \ln(y) \)

c. \( \log (100) = 2 \)

d. \( \log \left( \frac{1}{x^3} \right) = -3 \log(x) \)

e. \( \log(x^2y) = 2 \log(x) + 2 \log(y) \)

f. \( \log \left( \frac{x}{yz} \right) = \log(x) - \log(y) + \log(z) \)
4. If \( \log a = 2 \), \( \log b = 3 \) and \( \log c = 5 \) evaluate the following:

a. \( \log\left(\frac{a^{-2}}{b^3 c}\right) \)

b. \( \log \sqrt{b^2 c^{-3} a^4} \)

c. \( \frac{\log(ab^{-4})}{\log(ab)^4} \)

d. \( (\log c)\left(\log\left(\frac{a}{b^{-2}}\right)^3\right) \)
Converting Between Exponential Forms

1. Convert each of the following to the form \( y = ab^x \) and state the initial value, the annual rate, and the continuous annual rate.

   a. \( y = 56e^{0.1x} \)

   b. \( y = 77e^{-1x} \)

   c. \( y = 32e^{-6x} \)

2. Convert each of the following to the form \( y = ae^{kx} \) and state the initial value, the annual rate, and the continuous annual rate.

   a. \( y = 59(1.07)^x \)

   b. \( y = 67(.72)^x \)

   c. \( y = 599(.6)^x \)
Solving Exponential and Logarithmic Equations

1. For the following problems give both the exact answer and the decimal approximation.
   a. Solve for $x$: $\log x = 5$
   
   b. Solve for $x$: $\ln x = 3$
   
   c. Solve for $a$: $\log_a 25 = 2$
   
   d. Solve for $t$: $\log (2t + 1) + 3 = 0$
   
   e. Solve for $x$: $3 \log (2x + 6) = 6$
f. Solve for $x$: $\log_{10} 2 = 2$

g. Solve for $x$: $100 \cdot 2^x = 337,000,000$

h. Solve for $x$: $5(1.031)^x = 8$

i. Solve for $x$: $e^{0.044x} = 6$

j. Solve for $t$: $20e^{4t+1} = 60$

k. Solve for $x$: $\log_2 (x) + \log_2 (x - 2) = 3$
Applications of Logarithms
Decibels

Noise level (in decibels) = $10 \cdot \log \left( \frac{I}{I_0} \right)$

$I = \text{sound intensity of object in watts/cm}^2$
$I_0 = \text{sound intensity of benchmark object in watts/cm}^2$

Sound intensity is measured in watts per square centimeter (watts/cm$^2$)

1. The sound intensity of a refrigerator motor is $10^{-11}$ watts/cm$^2$. A typical school cafeteria has sound intensity of $10^{-8}$ watts/cm$^2$. How many orders of magnitude more intense is the sound of the cafeteria?

2. If a sound doubles in intensity, by how many units does its decibel rating increase?

3. Loud music can measure 110 dB whereas normal conversation measures 50 dB. How many times more intense is loud music than normal conversation?
Richter Scale

In 1935 Charles Richter defined the magnitude of an earthquake to be

$$\log \left( \frac{I_c}{I_n} \right) = R$$

where $I_c$ is the intensity of the earthquake (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake) and $I_n$ is the intensity of a "standard earthquake" (whose amplitude is 1 micron = $10^{-4}$ cm). Basically it is a measure of how much the earth moved during the earthquake ($I_c$) versus how much the earth moves on a normal day ($I_n$).

The magnitude of a standard earthquake is

$$R = \log \left( \frac{I_c}{I_n} \right) = \log(1) = 0$$

Richter studied many earthquakes that occurred between 1900 and 1950. The largest had magnitude of 8.9 on the Richter scale, and the smallest had magnitude 0. This corresponds to a ratio of intensities of 800,000,000, so the Richter scale provides more manageable numbers to work with.

Each number increase on the Richter scale indicates an intensity ten times stronger. For example, an earthquake of magnitude 6 is ten times stronger than an earthquake of magnitude 5. An earthquake of magnitude 7 is $10 \times 10 = 100$ times strong than an earthquake of magnitude 5. An earthquake of magnitude 8 is $10 \times 10 \times 10 = 1000$ times stronger than an earthquake of magnitude 5.

Questions

1. Early in the century the earthquake in San Francisco registered 8.3 on the Richter scale. In the same year, another earthquake was recorded in South America that was four time stronger (in other words, its amplitude was four times as large). What was the magnitude of the earthquake in South American?

2. A recent earthquake in San Francisco measured 7.1 on the Richter scale. How many times more intense was the amplitude of the San Francisco earthquake described in Example 1?
3. If one earthquake’s amplitude is 25 times larger than another, how much larger is its magnitude on the Richter scale?

4. How much more intense is (or how many times larger is the amplitude of) an earthquake of magnitude 6.5 on the Richter scale as one with a magnitude of 4.9?

5. The 1985 Mexico City earthquake had a magnitude of 8.1 on the Richter scale and the 1976 Tangshan earthquake was 1.26 as intense. What was the magnitude of the Tangshan earthquake?

6. If the intensity of earthquake A is 50 microns and the intensity of earthquake B is 6500 microns, what is the difference in their magnitudes as measured by the Richter scale?
More Exponential Functions and Logarithms

Objectives

- Build models using exponential functions and logarithmic functions
- Analyze models using characteristics of exponential and logarithmic functions
1. You place $1000 into an account paying a nominal rate of 5.5% compounded quarterly (4 times per year).

   a. Find an equation for the balance $B$, after $t$ years.

   b. What is the annual growth rate (to four decimal places).

   c. How much money will be in the account after 10 years?

   d. How long will it take for the amount of money to double (round your answer to two decimal places)?
2. Find a formula for an exponential function, $f$, with $f(1) = 10$ and $f(3) = 14.4$. Do this problem algebraically and check your answer using regression. Write your exponential function in both forms (i.e. $y = ab^t$ and $y = ae^{kt}$)

3. The half-life of carbon-14 is approximately 5728 years. If a fossil is found with 20% of its initial amount of carbon-14 remaining, how old is it?
4. A population of bacteria decays at a continuous rate of 10% per hour.
   a. What is the half-life of these bacteria?

   b. If the population starts out with 100,000 bacteria, create a function to represent
      the number of bacteria, \( N \), after \( t \) hours.

   c. Use your function found in part b. to find out how many bacteria would remain
      after 1 day (24 hours).

   d. What is the decay rate of the bacteria (i.e. by what percentage does the bacteria
      decrease each hour)?
5. A population of bacteria is measured to be at 1,000 after 10 minutes since it appeared. 25 minutes after it appeared, it is measured to be 10,000.
   a. What is the initial size of the population?

   b. What is the doubling time of the population?

   c. When will the population reach 1,000,000?
6. The population of a town is given by the following table:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Population in thousands</td>
<td>100</td>
<td>108</td>
<td>117</td>
<td>127</td>
<td>138</td>
<td>149</td>
<td>162</td>
<td>175</td>
<td>190</td>
<td>205</td>
</tr>
</tbody>
</table>

a. Use your calculator to find an exponential model to fit the data.

b. What is the annual growth rate of the city? What is the continuous growth rate?

c. What is the doubling time of the city?

d. According to the model, when will the population of the city be 1,000,000?
The table to the left gives the US population between 1966 and 1998.

a. Find an exponential model to fit this data.

b. According to your model, what should the US population have been on July 1, 2011?

c. In July of 2011, the US census bureau estimated the population at 313,232,044 (from http://www.indexmundi.com/united_states/population.html). According to the model from part a, when should the US population have reached 313,232,044? To what do you attribute the difference in your answers?
8. Using the population clock (http://www.census.gov/main/www/popclock.html) record the population for the US and the World at 4-5 intervals (say every half hour). Use this data to build exponential functions to model the US and World Population. Then use the models to predict what the population will be at the start of class next week.
Lesson 11: Systems of Equations and Matrices

Outline

Objectives:
I can solve systems of three linear equations in three variables.
I can solve systems of linear inequalities
I can model and solve real-world problems.

Example 1:

<table>
<thead>
<tr>
<th></th>
<th>Model A</th>
<th>Model B</th>
<th>Total Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gizmos</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dodads</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

System:

The goal of solving a system is to find the values of $x$ and $y$ that satisfy both equations simultaneously. Three methods of solving a system we will investigate include graphing, solving algebraically, and by using matrices.

I. Graphical Solution

When solving systems of two linear equations, there are three possible outcomes:

1. 
2. 
3.

In an attempt to solve the system in the calculator, we must first solve each equation for $y$ and then enter the equations in the calculator into $Y_1$ and $Y_2$ to find the intersection, if it exists.

II. Algebraic Solution

Elimination Method Example:

Substitution Method Example:
III. Matrices

An individual element in a matrix is identified by its location in the matrix.

Coefficient matrix

Column vector

Solution vector

An identity matrix is a matrix that has the following characteristics:

Back to Example 1:

\[
\begin{bmatrix}
4 & 5 & x \\
9 & 14 & y
\end{bmatrix}
= 
\begin{bmatrix}
335 \\
850
\end{bmatrix}
\]

or

\[AX = Y\]

Inverse Matrix Method: Just as an inverse function “undoes what a function does”, the inverse matrix undoes what the matrix does.

\[AX = Y\]

\[AA^{-1} = I\]

\[A^{-1} \cdot AX = A^{-1}Y\]

\[X = A^{-1}Y\]

We can get a solution to a system of equations by multiplying the inverse of the coefficient matrix by the (solution) column vector in the proper order (and order is critical!) on the graphing calculator.

To set up the matrices on the calculator: Access the MATRIX command (over the $x^{-1}$ key) by using the keystrokes $2^{\text{nd}} \rightarrow x^{-1} \rightarrow$ EDIT $\rightarrow$ (enter the size of the coefficient matrix A by # rows x #columns) $\rightarrow$ ENTER $\rightarrow$ (enter individual entries from the coefficient matrix) $\rightarrow$ (2nd quit). Then, MATRIX $\rightarrow$ EDIT $\rightarrow$ (arrow down to matrix B) $\rightarrow$ (enter size of column vector) $\rightarrow$ (enter individual entries from the column vector) $\rightarrow$ (2nd quit).

Now, to do the calculation, enter: MATRIX $\rightarrow$ (select matrix [A] if not automatically selected) $\rightarrow$ ENTER $\rightarrow$ $x^{-1}$ $\rightarrow$ (times) $\rightarrow$ MATRIX $\rightarrow$ (select matrix [B]) $\rightarrow$ ENTER $\rightarrow$ ENTER and read the result of the variable column vector.

An augmented matrix is the coefficient matrix combined with the solution column vector. In order to solve a system in reduced row echelon form on the calculator, we must input the augmented matrix.
Reduced Row Echelon Form Method:
A matrix is said to be in **row echelon form** provided all of the following conditions hold:
1. 
2. 
3. 

A matrix is said to be in **reduced row echelon form** if both the following conditions hold:
1. 
2. 

Calculator keystrokes for reduced row echelon form:
\[\text{MATRIX} \rightarrow \text{EDIT} \rightarrow \text{(enter augmented matrix size)} \rightarrow \text{ENTER} \rightarrow \text{(enter augmented matrix individual entries)} \rightarrow 2^{\text{nd}} \rightarrow \text{QUIT} \rightarrow \text{MATRIX} \rightarrow \text{MATH} \rightarrow \text{(scroll down to rref)} \rightarrow \text{ENTER} \rightarrow \text{MATRIX} \rightarrow \text{(select the augmented matrix and close parentheses)} \rightarrow \text{ENTER} \text{ and read the solution.}\]

Example:

<table>
<thead>
<tr>
<th></th>
<th>Brand X</th>
<th>Brand Y</th>
<th>Brand Z</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vitamin A</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vitamin D</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vitamin E</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Potential Solutions:** There are three possibilities for solutions:
**A Consistent System with One Solution:**
How can I tell?

Geometric Interpretation:

**A Consistent Solution with Infinite Solutions:**
How can I tell?

Geometric Interpretation:

**An Inconsistent System:**
How can I tell?

Geometric Interpretation:
Systems of Equations and Matrices Activity

Objectives:
- Solve 2x2 system using substitution, elimination and graphing
- Solve 2x2 system using matrices
- Solve 3x3 system using matrices
- Set up system of equations and solve for applications
Solving 2x2 Systems

1. Each person in your group should use one of the following methods. Make sure you all get the same result.

Solve the following system of equations: \[
\begin{align*}
5x + 2y &= 12 \\
3x - 4y &= 2 \\
\end{align*}
\]

a. Using elimination
b. Using substitution
c. Graphically
d. Using matrices

2. Solve the following system of equations: \[
\begin{align*}
\frac{x}{4} + \frac{y}{2} &= -5 \\
\frac{3x}{2} - \frac{y}{2} &= 1 \\
\end{align*}
\]

a. Using elimination
b. Using substitution
c. Graphically
d. Using matrices
Solving a 2x2 System Using an Inverse Matrix:

Example:
Consider:
\[
\begin{align*}
3x + 2y &= 8 \\
2x - 3y &= -25
\end{align*}
\]

Create matrix: \([A]\) to be \(\begin{bmatrix} 3 & 2 \\ 2 & -3 \end{bmatrix}\) and \([B]\) to be \(\begin{bmatrix} 8 \\ -25 \end{bmatrix}\). \([A]^{-1}[B]=\begin{bmatrix} x \\ y \end{bmatrix}\). Find the inverse matrix using your calculator and then perform the multiplication to get the solution.

1. Now try the method on the following system:
\[
\begin{align*}
2x - 3y &= -18 \\
3x + 2y &= -1
\end{align*}
\]
Applications Involving 2x2 Systems

1. A company develops two different types of snack mix. Type A requires 4 ounces of peanuts and 9 ounces of chex mix while type B requires 5 ounces of peanuts and 13 ounces of chex mix. There is a total of 335 ounces of peanuts available and 850 ounces of chex mix. How much of each type can the company produce?

   Set up a system for the problem and solve it by any method you like.

2. A person flies from Phoenix to Tucson and back (about 200 miles each way). He finds that it takes him 2 hours to fly to Tucson against a headwind and only 1 hour to fly back with the wind. What was the airspeed of the plane (speed if there was no wind) and the speed of the wind?
3. Two numbers when added together are 960 and when subtracted are 94. Find the two numbers. Set up a system for the situation and solve it by any method you like.

4. A man has 19 coins in his pocket, all of which are dimes and quarters. If the total value of his change is $3.55, how many dimes and how many quarters does he have?
Solving 3x3 Systems
1. Solve the following 3x3 systems using your graphing calculator. Determine if the system is consistent (one solution or infinitely many solutions) or inconsistent (no solution). If it has infinitely many solutions, write the dependent variable(s) in terms of the independent variable(s).

\[
\begin{align*}
2x + y + z &= 3 \\
-x + 2y + 2z &= 1 \\
-x + y + 3z &= 6
\end{align*}
\]

a.

\[
\begin{align*}
2x - 4y &= 8 \\
2x - 3y + 2z &= -2 \\
-4x + 7y - 2 &= -6
\end{align*}
\]

b.

\[
\begin{align*}
2x - 2y + 4z &= 2 \\
x + y + 3z &= -2 \\
x + y + 3z &= 3
\end{align*}
\]

c.
d. \[
\begin{bmatrix}
5 & -7 & 4 & 2 \\
3 & 2 & -2 & 3 \\
2 & -1 & 3 & 4
\end{bmatrix}
\]

e. \[
\begin{bmatrix}
1 & 3 & 2 & 5 \\
1 & 5 & 3 & 9 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

f. \[
\begin{align*}
x + y + z &= 2 \\
-2x - y + z &= -2 \\
x + 3y + 2z &= 1
\end{align*}
\]

g. \[
\begin{align*}
x + 4y - 2z &= 0 \\
-2x - 6y + 5z &= 5 \\
3x + 14y - 4z &= 8
\end{align*}
\]
Finding a Polynomial Given Points

Example: Set up a system of equations to find a parabola that passes through (0,96), (2, 192), and (5, 96).

For the first point we know that \( y = ax^2 + bx + c \) becomes
\[ 96 = a \cdot 0^2 + b \cdot 0 + c \] so \( 0a + 0b + c = 96 \)

For the second point we know that \( y = ax^2 + bx + c \) becomes
\[ 192 = a \cdot 2^2 + b \cdot 2 + c \] so \( 4a + 2b + c = 192 \)

For the second point we know that \( y = ax^2 + bx + c \) becomes
\[ 96 = a \cdot 5^2 + b \cdot 5 + c \] so \( 25a + 5b + c = 96 \)

So the system becomes:
\[
\begin{align*}
0a + 0b + c &= 96 \\
4a + 2b + c &= 192 \\
25a + 5b + c &= 96
\end{align*}
\]

\[
\begin{bmatrix}
0 & 0 & 1 & 96 \\
4 & 2 & 1 & 192 \\
25 & 5 & 1 & 96
\end{bmatrix} \Rightarrow
\begin{bmatrix}
1 & 0 & 0 & -16 \\
0 & 1 & 0 & 80 \\
0 & 0 & 1 & 96
\end{bmatrix}
\]

So our quadratic would be \( y = ax^2 + bx + c \)

1. Find a parabola that passes through the points: (-3,2), (0,5), and (-6,5)

2. Find a cubic function, \( y = ax^3 + bx^2 + cx + d \), that passes through the points: (-3,2), (0,5), (3,0), and (-5,0). Note: In this case you will have 4 equations with 4 variables so set up your system and use your graphing calculator to solve.
Applications Involving 3x3 Systems

1. In a particular factory, skilled workers are paid $15 per hour, unskilled workers are paid $9 per hour and shipping clerks are paid $10 per hour. Recently the company has received an increase in orders and will need to hire a total of 70 workers. The company has budgeted a total of $880 per hour for these new hires. Due to union requirements, they must hire twice as many skilled employees as unskilled. How many of each type of worker should the company hire?

2. I have 27 coins (some are pennies, some are dimes, and some are quarters). I have twice as many pennies as dimes. I have $2.97 in coins. How many of each type of coin do I have?
3. You have $3000 to invest in 3 stocks. Stock A is predicted to yield 10% per year. Stock B is the safest and is predicted to yield 5% per year. Stock C is risky but is expected to yield 27% per year. You decide to spend twice as much on stock B than on stock C. You hope to make 11% each year on your stock. Use a matrix (and your calculator) to find the amount of each type of stock that you should buy.

4. An inheritance of $9,600 is to be split among 3 children. To pay back money owed to the two oldest children, it is written that the oldest child gets $2,000 more than the youngest and the middle child get $1,000 more than the youngest. How much should each child get?
5. There are 3 candidates to choose for president and 8,000 people are expected to vote. Candidate A is expected to receive twice as many votes as candidate C. Candidate C is expected to receive 3 times the votes as candidate B. Predict how many votes each candidate will receive.

6. Mary invested $1,000 in three separate investments. At the end of the year they earned 4%, 5% and 6%, respectively. She invested twice as much in the account earning 5% as she did in the one paying 4%. She made a total of $54 in interest during the year. How much did she allocate to each investment?
Lesson 12: Sequences and Series

Outline

Objectives:
I can determine whether a sequence has a pattern.
I can determine whether a sequence can be generalized to find a formula for the general term in the sequence.
I can determine whether a sequence is arithmetic or geometric.
I can determine the general terms of an arithmetic and geometric sequence.
I can determine the sum of a finite arithmetic or geometric series.
I can determine the sum of certain infinite geometric series.
I can use and interpret summation notation.

Definitions / Vocabulary / Graphical Interpretation:

Sequence characteristics:
- List of numbers written in a definite order
- Can be finite or infinite
- Does not have to have a pattern
- A function whose domain is the set of positive integers
- $a_n$ is also known as the general term of the sequence

List some examples of sequences:

The Fibonacci Sequence:
How to create the Fibonacci Sequence: $\{1, 1, 2, 3, 5, 8, 13, 21\ldots\}$ by adding the previous two terms

A recursively defined sequence is a sequence where the general term is related to the previous terms.

Converging Sequences:
To determine whether or not a sequence converges we can look at what happens to the general term as $n$ gets infinitely large.
Examples:

\[ a_n = \frac{2n}{n^2 - n} \] converges to:

\[ a_n = \frac{4n^2 + 2n}{2n^2 - 1} \] converges to:

\[ a_n = \frac{2n^3 - n^2 + 7}{3n^3 - 4n} \] converges to:

\[ a_n = \frac{e^n}{n^{45}} \] converges to:

\[ a_n = e^{-n} - 1 \] converges to:

**Bounded Sequences:**
If all the terms of a sequence lie between 2 numbers, say \( M \) and \( N \), such that the terms are never greater than \( M \) or less than \( N \), we say the sequence is **bounded**.

- A convergent sequence is bounded.
- However, a bounded sequence is not necessarily convergent. (A bounded sequence does not necessarily have a limit.)
- If a sequence is monotone (always increasing or always decreasing) and bounded, it converges.

**Arithmetic Sequences:**
A sequence that has a constant amount of increase or decrease between terms (the common difference) is called an arithmetic sequence. **The equation for the general term of an arithmetic sequence** is just a linear function with slope \( d \) (representing the common difference).

\[ a_n = a + (n - 1)d \quad n \geq 1 \]

**Geometric Sequences:**
A sequence that has a constant percent increase or decrease in terms (common ratio) is called a geometric sequence. **The equation for the general term of a geometric sequence** is an exponential function with base \( r \) (representing the common ratio).

\[ a_n = ar^{n-1} \quad n \geq 1 \]
Examples:

\[ a_n = \frac{(-1)^n 3^{n-1}}{4^{n+1}} \quad n \geq 0 \]

a) Write out the first 5 terms

b) Is the sequence arithmetic, geometric, or neither?

\[ a_1 = 10, \quad a_{n+1} = a_n + 2 \quad \text{for } n \geq 1 \]

a) Write out the first 5 terms

b) Is the sequence arithmetic, geometric, or neither?

c) Find an explicit formula for \( a_n \)

Series and Summation Notation:
Summation notation is a short-hand way of denoting the sum of the terms of a sequence. As with sequences, series can be finite or infinite.

Given a sequence \( \{a_n\}_{n=k}^{\infty} \) and numbers \( m \) and \( p \) satisfying \( k \leq m \leq p \), the summation from \( m \) to \( p \) of the sequence \( \{a_n\} \) is written:

\[
\sum_{n=m}^{p} a_n = a_m + a_{m+1} + \ldots + a_p
\]

where

- \( n \) is called the index of summation
- \( m \) is the lower limit of summation
- \( p \) is the upper limit of summation
Properties of Summation Notation:

- 
- 
- 
- 
- 

Calculating a Finite Arithmetic Series Sum:

\[ S_n = \frac{n}{2} (a_1 + a_n) \]

where \( n \) is the number of terms; \( a_1 \) is the first term, and \( a_n \) is the last term.

Calculating a Finite Geometric Series Sum:
All we need to calculate a finite geometric series is the first term \( a_1 \), the common ratio \( r \), and the number of terms \( n \).

\[ S_n = a_1 + ar + ar^2 + \ldots + ar^{n-1} = \sum_{i=0}^{n-1} ar^i = \frac{a(1 - r^n)}{1 - r}, \quad r \neq 1 \]

Calculating an Infinite Geometric Series Sum:
When the common ratio is between -1 and 1, we can find the sum of an infinite geometric series with the following:

\[ S_\infty = a + ar + ar^2 + \ldots = \sum_{i=0}^{\infty} ar^i = \frac{a}{1 - r} \]

Example: Find the sum

1. \[ 1 + 4 + 7 + 10 + \ldots + 76 \]

2. \[ \sum_{k=0}^{49} (3k - 5) \]

3. \[ \sum_{i=2}^{26} 3 \cdot 2^i \]

4. \[ \sum_{i=0}^{\infty} 3 \cdot 2^i \]
Sequences and Series Activity

Objectives:
- Use and interpret notation for sequences and series
- Identify arithmetic and geometric sequences
- Find a formula for the nth term of a sequence (when possible)
- Use and interpret summation notation
- Find the sum of finite arithmetic and geometric sequences
- Find the sum of infinite geometric series (when possible)
**Notation for Sequences**

1. Write out the first five terms of the following sequences:
   
   a. \( a_n = \frac{n}{n + 1} \)
   
   b. \( a_n = 4 \)
   
   c. \( a_n = 2a_{n-1}, \ a_1 = 3 \)
   
   d. \( a_n = a_{n-2} + a_{n-1}, \ a_1 = 1, a_2 = 1 \)

2. Identify \( a_4 \) in the following sequences:
   
   a. 2, 7, 12, 17, 22, 27, 32, ..... 
   
   b. 2, 8, 24, 72, 216, 648, ..... 
   
   c. 1, \( b+1 \), 2\( b+1 \), 3\( b+1 \), 4\( b+1 \), .....
**Identifying Sequences**

1. Decide whether each sequence in the table is arithmetic, geometric, or neither. Find the general term, if possible.

<table>
<thead>
<tr>
<th></th>
<th>commin difference?</th>
<th>constant ratio?</th>
<th>type of sequence</th>
<th>General Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>2, 7, 12, 17, 22, 27, 32, .....</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b.</td>
<td>2, 8, 24, 72, 216, 648, .....</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c.</td>
<td>5, 1, 5, 1, 5, 1, 5, 1, 5, .....</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d.</td>
<td>1/3, 7/3, 13/3, 19/3, 25/3, .....</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>e.</td>
<td>4, -2, 1, -0.5, 0.25, .....</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>f.</td>
<td>1, b+1, 2b+1, 3b+1, 4b+1, .....</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>g.</td>
<td>25, 20, 15, 10, 5, 0, -5, -10, .....</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h.</td>
<td>1, -1, 1, -1, 1, -1, 1, -1, 1, .....</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>i.</td>
<td>1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, ...</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Make up your own example of a sequence that is:

   * arithmetic: *

   * geometric: *

   * neither: *
**Summation Notation**

1. Expand and calculate the sum.

   a. \[ \sum_{i=1}^{4} 3i + 1 \]

   b. \[ \sum_{j=0}^{5} 3j^2 \]

   c. \[ \sum_{k=1}^{7} (-1)^k \cdot k \]

   d. Make up your own expression using sigma notation and then expand it.

2. Use sigma notation to express each sum.

   a. \[ \frac{1}{3} + \frac{2}{3} + 1 + \frac{4}{3} + \frac{5}{3} + 2 \]

   b. \[ 100 + 90 + 80 + 70 + 60 + 50 + 40 + 30 + 20 + 10 \]

   c. \[ 2 + 4 + 8 + 16 + 32 + 64 \]

   d. Make up your own sum of terms and then re-write it using sigma notation


**Arithmetic Series**

The sum of \( n \) terms of an arithmetic sequence:  \( S_n = \frac{1}{2} n(a_1 + a_n) \)

1. Find the sums of the following finite arithmetic series:
   a. \( 3 + 6 + 9 + \ldots + 210 \)

   b. \( \sum_{i=1}^{400} (3i + 1) \)

   c. \( (1 + 2p^2) + 1 + (1 - 2p^2) + (1 - 4p^2) + \ldots + (1 - 96p^2) \)

   d. \( \sum_{n=21}^{72} n - 4 \)

2. A theater is constructed so that each row has 4 more seats than the one in front of it. The first row has 20 seats and there are 50 rows in the theater.
   a. How many seats in the 50th row?

   b. How many total seats in the theater?
Finite Geometric Series

The sum of \( n \) terms of a finite geometric sequence: \( S_n = \frac{a_1(1 - r^n)}{1 - r} \). This formula only applies if \( r \neq 1 \)

1. Find the sums of the following finite geometric series:
   a. \( 3 + 6 + 12 + 24 + \ldots + 3072 \)
   
   \[
   S = \frac{3(1 - 2^{11})}{1 - 2} = \frac{3(2^{11} - 1)}{1 - 2} = 3(2^{11} - 1)
   \]

   b. \( \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + \ldots + \frac{3}{2^{18}} \)
   
   \[
   S = \frac{\frac{3}{8}(1 - \frac{1}{2^{18}})}{1 - \frac{1}{2}} = \frac{\frac{3}{8}(2^{18} - 1)}{1 - 2} = \frac{3(2^{18} - 1)}{8}
   \]

   c. \( \sum_{i=0}^{20} 5(1.1)^i \)
   
   \[
   S = \frac{5(1.1)(1 - 1.1^{20})}{1 - 1.1} = \frac{5(1.1)(1 - 1.1^{20})}{-0.1}
   \]

   d. \( \sum_{i=0}^{20} 5(0.1)^i \)
   
   \[
   S = \frac{5(0.1)(1 - 0.1^{20})}{1 - 0.1} = \frac{5(0.1)(1 - 0.1^{20})}{0.9}
   \]
Infinite Geometric Series

The sum of an infinite number of terms of a geometric sequence: \( S = \frac{a_1}{1 - r} \). This formula only applies if \(-1 < r < 1\).

1. Find the sums of the following infinite geometric series:
   a. \( \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + ... \)
   
   b. \( \sum_{x=0}^{\infty} 5(0.1)^x \)

   c. \( \sum_{x=1}^{\infty} 5(0.1)^x \)

2. A rubber ball is dropped from a height of 4 feet. Each time it bounces it goes half as high as before (i.e. after it hits it bounces up 2 feet, than 1 foot and so on). What is the total distance covered by the ball once it is done bouncing?


Application of a Geometric Series

1. A clothing outlet found that when they introduced a new shirt, it sold quickly at first, but then as time went on they sold less and less quantities of the design each month. For a particular shirt, they found that they sold 300 units in the first month. They then found their sales of the shirt dropped by roughly 15% each month there after.

   a. How many shirts did they sell in the first 3 months (total) after the shirt was introduced?

   b. How many shirts did they sell in the first year after the shirt was introduced?

   c. Based on their estimation, how many shirts should the company expect to sell total? Hint: Look at the summation as time gets infinitely large!