Lesson 8 – Introduction to Quadratic Functions

We are leaving exponential and logarithmic functions behind and entering an entirely different world. As you work through this lesson, you will learn to identify quadratic functions and their graphs (called parabolas). You will learn the important parts of the parabola including the direction of opening, the vertex, intercepts, and axis of symmetry.

You will use graphs of quadratic functions to solve equations and, finally, you will learn how to recognize all the important characteristics of quadratic functions in the context of a specific application. Even if a problem does not ask you to graph the given quadratic function or equation, doing so is always a good idea so that you can get a visual feel for the problem at hand.

Lesson Topics

Section 8.1: Characteristics of Quadratic Functions
- Identify the Vertical Intercept
- Determine the Vertex
- Domain and Range
- Determine the Horizontal Intercepts (Graphically)

Section 8.2: Solving Quadratic Equations Graphically

Section 8.3: Characteristics of Logarithmic Functions
- Use the Change of Base Formula to graph a logarithmic function and identify important characteristics of the graph.

Section 8.4: Quadratic Regression
## Lesson 8 Checklist

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<th>Due</th>
<th>Score</th>
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Mini-Lesson 8

Section 8.1 – Characteristics of Quadratic Functions

A QUADRATIC FUNCTION is a function of the form

\[ f(x) = ax^2 + bx + c \]

Characteristics Include:
- Three distinct terms each with its own coefficient:
  - An \( x^2 \) term with coefficient \( a \)
  - An \( x \) term with coefficient \( b \)
  - A constant term, \( c \)
  - Note: If any term is missing, the coefficient of that term is 0
- The graph of this function is called a “parabola”, is shaped like a “U”, and opens either up or down
- \( a \) determines which direction the parabola opens (\( a > 0 \) opens up, \( a < 0 \) opens down)
- \( c \) is the vertical intercept with coordinates \((0, c)\)

## Problem 1

WORKED EXAMPLE – GRAPH QUADRATIC FUNCTIONS

Given the Quadratic Function \( f(x) = x^2 + 4x - 2 \), complete the table and generate a graph of the function.

<table>
<thead>
<tr>
<th>Identity the coefficients ( a, b, c )</th>
<th>( a = 1, \ b = 4, \ c = -2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Which direction does the parabola open?</td>
<td>( a = 1 ) which is greater than 0 so parabola opens up</td>
</tr>
<tr>
<td>What is the vertical intercept?</td>
<td>( c = -2 ) so vertical intercept = ((0, -2))</td>
</tr>
</tbody>
</table>

![Graph of the quadratic function](image)
**Problem 2** | **MEDIA EXAMPLE – GRAPH QUADRATIC FUNCTIONS**

Given the Quadratic Function \( f(x) = x^2 - 2x + 3 \), complete the table and generate a graph of the function.

<table>
<thead>
<tr>
<th>Identity the coefficients ( a, b, c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Which direction does the parabola open? Why?</td>
</tr>
<tr>
<td>What is the vertical intercept?</td>
</tr>
</tbody>
</table>

![Graph of Quadratic Function](image)

**Problem 3** | **YOU TRY – GRAPH QUADRATIC FUNCTIONS**

Given the Quadratic Function \( f(x) = 2x^2 - 5 \), complete the table and generate a graph of the function.

<table>
<thead>
<tr>
<th>Identity the coefficients ( a, b, c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Which direction does the parabola open? Why?</td>
</tr>
<tr>
<td>What is the vertical intercept?</td>
</tr>
<tr>
<td>Plot and label on the graph.</td>
</tr>
</tbody>
</table>

![Graph of Quadratic Function](image)
Given a quadratic function, \( f(x) = ax^2 + bx + c \):

The VERTEX is the lowest or highest point (ordered pair) of the parabola

- To find the input value, identify coefficients \( a \) and \( b \) then compute \( -\frac{b}{2a} \)
- Plug this input value into the function to determine the corresponding output value, (i.e. evaluate \( f \left( -\frac{b}{2a} \right) \))
- The Vertex is always written as an ordered pair. \( \text{Vertex} = \left( -\frac{b}{2a}, f \left( -\frac{b}{2a} \right) \right) \)

The AXIS OF SYMMETRY is the vertical line that passes through the Vertex, dividing the parabola in half.

- Equation \( x = -\frac{b}{2a} \)

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**Problem 4 | WORKED EXAMPLE – Quadratic Functions: Vertex/Axis Of Symmetry**

Given the Quadratic Function \( f(x) = x^2 + 4x - 2 \), complete the table below.

<table>
<thead>
<tr>
<th>Identity the coefficients ( a, b, c )</th>
<th>( a = 1, \quad b = 4, \quad c = -2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input Value</strong></td>
<td><strong>Output Value</strong></td>
</tr>
<tr>
<td>( x = -\frac{b}{2a} )</td>
<td>( f(-2) = (-2)^2 + 4(-2) - 2 )</td>
</tr>
<tr>
<td>( = -\frac{4}{2(1)} )</td>
<td>( = 4 - 8 - 2 )</td>
</tr>
<tr>
<td>( = -2 )</td>
<td>( = -6 )</td>
</tr>
</tbody>
</table>

Vertex Ordered Pair: \((-2, -6)\)

<table>
<thead>
<tr>
<th>Identify the Axis of Symmetry Equation</th>
<th>Axis of Symmetry: ( x = -2 )</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Sketch the Graph</th>
<th>axis of symmetry equation ( x = -2 )</th>
</tr>
</thead>
</table>

Vertex \((-2, -6)\)
**Problem 5  | MEDIA EXAMPLE – Quadratic Functions: Vertex/Axis Of Symmetry**

Given the Quadratic Function \( f(x) = x^2 - 2x + 3 \), complete the table, generate a graph of the function, and plot/label the vertex and axis of symmetry on the graph.

<table>
<thead>
<tr>
<th>Identity the coefficients ( a, b, c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Determine the coordinates of the Vertex.</td>
</tr>
<tr>
<td>Identify the Axis of Symmetry Equation.</td>
</tr>
</tbody>
</table>

Graph of the function. Plot/label the vertex and axis of symmetry on the graph.

**Problem 6  | YOU TRY – Quadratic Functions: Vertex/Axis Of Symmetry**

Given the Quadratic Function \( f(x) = 2x^2 - 5 \), complete the table, generate a graph of the function, and plot/label the vertex and axis of symmetry on the graph.

<table>
<thead>
<tr>
<th>Identity the coefficients ( a, b, c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Determine the coordinates of the Vertex.</td>
</tr>
<tr>
<td>Identify the Axis of Symmetry Equation.</td>
</tr>
</tbody>
</table>

Graph of the function. Plot/label the vertex and axis of symmetry on the graph.
Problem 7 | WORKED EXAMPLE – Quadratic Functions: Domain and Range

Determine the Domain and Range of the Quadratic Function \( f(x) = x^2 + 4x - 2 \)

**Domain of \( f(x) \):**
All real numbers. \(-\infty < x < \infty\)  \((-\infty, \infty)\)

**Range of \( f(x) \):**
Since the parabola opens upwards, the vertex \((-2, -6)\) is the lowest point on the graph.
The Range is therefore \(-6 \leq f(x) < \infty\), or \([-6, \infty)\)

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Problem 8 | MEDIA EXAMPLE – Quadratic Functions: Domain and Range

Determine the Domain and Range of \( f(x) = -2x^2 - 6 \).

**Domain of \( f(x) \):**

**Range of \( f(x) \):**
Lesson 8 - Introduction to Quadratic Functions

Problem 9 | YOU TRY – Quadratic Functions: Domain and Range

Determine the Domain and Range of \( f(x) = 2x^2 - 5 \). Sketch the graph and label the vertex.

<table>
<thead>
<tr>
<th>Vertex ordered pair:</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Domain of ( f(x) ):</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Range of ( f(x) ):</th>
</tr>
</thead>
</table>

Finding Horizontal Intercepts of a Quadratic Function

The quadratic function, \( f(x) = ax^2 + bx + c \), will have horizontal intercepts when the graph crosses the \( x \)-axis (i.e. when \( f(x) = 0 \)). These points are marked on the graph above as G and H. To find the coordinates of these points, what we are really doing is solving the equation \( ax^2 + bx + c = 0 \). At this point, we will use the following general calculator process. In the next lesson, we will learn other methods for solving these equations.

Calculator Process to solve \( ax^2 + bx + c = 0 \)
1. Press \( Y= \) then enter \( f(x) \) into \( Y1 \)
2. Enter 0 into \( Y2 \)
3. Use the graphing/intersection method once to determine G and again to determine H.
### Problem 10

**WORKED EXAMPLE – Finding Horizontal Intercepts of a Quadratic Function**

Find the horizontal intercepts of $f(x) = x^2 + 4x - 2$ and plot/label them on the graph.

1. Press Y= then enter $x^2 + 4x - 2$ into Y1
2. Enter 0 into Y2
3. Use the graphing/intersection method once to determine $G$ as $(-4.45, 0)$. You may have to move your cursor close to $G$ during the “First Curve?” part.
4. Use the graphing/intersection method again to determine $H$ as $(0.45, 0)$. You may have to move your cursor close to $H$ during the “First Curve?” part.

![Graph with points G and H labeled](image)

### Problem 11

**MEDIA EXAMPLE – Finding Horizontal Intercepts of a Quadratic Function**

Given the Quadratic Function $f(x) = x^2 - x - 6$, find the horizontal intercepts and plot/label them on the graph. Round to 2 decimals.

![Graph with axes](image)
Problem 12 | YOU TRY – Finding Horizontal Intercepts of a Quadratic Function

Given the Quadratic Function $f(x) = 2x^2 - 5$, find the vertex, vertical intercept, and horizontal intercepts. Plot and label all of these points on the graph. Round your values to two decimals.

| **Vertex:** | (______, ______) |
| **Vertical Intercept:** | (______, ______) |
| **Horizontal Intercepts:** | (______, ______) and (______, ______) |
A quadratic equation of the form \( ax^2 + bx + c = d \) can be solved in the following way using your graphing calculator:

1. Go to Y=
2. Let \( Y1 = ax^2 + bx + c \)
3. Let \( Y2 = d \)
4. Graph the two equations. You may need to adjust your window to be sure the intersection(s) is/are visible.
5. For each intersection, use 2nd>Calc>Intersect. Follow on-screen directions to designate each graph then determine intersection (hitting Enter each time).
6. Solution(s) to the equation are the intersecting \( x \)-values

NOTE: The Intersection method will provide us only with approximate solutions to a quadratic equation when decimal solutions are obtained. To find EXACT solution values, you will need to use the Quadratic Formula. This will be covered in the next lesson.

Problem 13 | WORKED EXAMPLE – Solve Quadratic Equations Graphically

Solve the equation \(-3x^2 - 2x - 4 = -5\) by graphing.

There are two intersection points. Follow the process above to find the intersections \((-1, -5)\) and \((0.33, -5)\). Solutions to the equation are \(x = -1\) and \(x = 0.33\).

Problem 14 | MEDIA EXAMPLE – Solve Quadratic Equations Graphically

Solve \(x^2 - 10x + 1 = 4\). Plot and label the graphs and intersection points that are part of your solution process. Identify the final solutions clearly. Round to 2 decimals.

<table>
<thead>
<tr>
<th>Xmin:</th>
<th>Xmax:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ymin:</td>
<td>Ymax:</td>
</tr>
</tbody>
</table>
Problem 15  YOU TRY – Solve Quadratic Equations Graphically

a) Solve \(2x^2 - 5 = 6\). Plot and label the graphs and intersection points that are part of your solution process. Round your answer to the nearest hundredth. Identify the final solutions clearly.

\[
\begin{align*}
\text{Xmin:} & \quad ______ \\
\text{Xmax:} & \quad ______ \\
\text{Ymin:} & \quad ______ \\
\text{Ymax:} & \quad ______ 
\end{align*}
\]

b) Solve \(x^2 + 9x - 18 = 32\). Plot and label the graphs and intersection points that are part of your solution process. Round your answer to the nearest hundredth. Identify the final solutions clearly.

\[
\begin{align*}
\text{Xmin:} & \quad ______ \\
\text{Xmax:} & \quad ______ \\
\text{Ymin:} & \quad ______ \\
\text{Ymax:} & \quad ______ 
\end{align*}
\]
Section 8.3 – Applications of Quadratic Functions

A large number of quadratic applications involve launching objects into the sky (arrows, baseballs, rockets, etc…) or throwing things off buildings or spanning a distance with an arched shape. While the specifics of each problem are certainly different, the information below will guide you as you decipher the different parts.

HOW TO SOLVE QUADRATIC APPLICATION PROBLEMS

1. Draw an accurate graph of the function using first quadrant values only. Label the x-axis with the input quantity and units. Label the y-axis with the output quantity and units.
2. Identify, plot, and label the vertical intercept.
3. Identify, plot, and label the vertex.
4. Identify, plot, and label the positive horizontal intercept(s) (usually, there is only one horizontal intercept that we care about…if both are needed for some reason, then plot them both and include negative input values in your graph for part 1).
5. Once you have done steps 1–4, THEN read the specific questions you are asked to solve.

Questions that involve the vertical intercept (0, c):
- How high was the object at time t = 0?  c
- What was the starting height of the object?  c

Questions that involve the vertex \((-\frac{b}{2a}, f(-\frac{b}{2a}))\):
- How high was the object at its highest point?  \(f\left(-\frac{b}{2a}\right)\)
- What was the max height of the object?  \(f\left(-\frac{b}{2a}\right)\)
- How long did it take the object to get to its max height?  \(-\frac{b}{2a}\)
- What is the practical range of this function?  \(0 \leq f(x) \leq f\left(-\frac{b}{2a}\right)\)

Questions that involve (usually) the positive horizontal intercept \((x_2, 0)\):
- When did the object hit the ground?  \(x_2\)
- What is the practical domain of this function?  \(0 \leq x \leq x_2\)
- How long did it take the object to hit the ground?  \(x_2\)
- How far was the object from the center?  \(x_2\)
Problem 16 | WORKED EXAMPLE – APPLICATIONS OF QUADRATIC FUNCTIONS

The function \( h(t) = -16t^2 + 80t + 130 \), where \( h(t) \) is height in feet, models the height of an arrow shot into the sky as a function of time (seconds).

Before even LOOKING at the specific questions asked, find the following items and plot/label the graph.

1. Identify the vertical intercept. \((0, 130)\) since \( c = 130 \).

2. Determine the vertex.

   \[
   \text{The input value of the vertex is } x = -\frac{b}{2a} = -\frac{80}{2(-16)} = 2.5 .
   \]

   \[
   \text{The corresponding output value is } f(-\frac{b}{2a}) = f(2.5) = -16(2.5)^2 + 80(2.5) + 130 = 230
   \]

3. Determine the positive horizontal intercept – using the process discussed in earlier examples, we want to solve \(-16t^2 + 80t + 130=0\). Using the intersect method, the positive horizontal intercept is \((6.29, 0)\).

4. Draw an accurate graph of the function using first quadrant values only. Label the horizontal axis with the input quantity and units. Label the vertical axis with the output quantity and units. Label the vertex and intercepts.

QUESTIONS TO ANSWER NOW:

a) After how many seconds does the arrow reach its highest point?
   \[
   \text{The input value of the vertex is } 2.5. \text{ So, the arrow reaches its highest point after } 2.5 \text{ seconds.}
   \]

b) How high is the arrow at its highest point?
   \[
   \text{The output value of the vertex is } 230. \text{ So, the arrow is 230 feet above the ground at its highest point.}
   \]

c) After how many seconds does the arrow hit the ground?
   \[
   \text{The horizontal intercept is } (6.29, 0). \text{ The arrow will hit the ground after } 6.29 \text{ seconds.}
   \]

d) What is the practical domain of this function?
   \[
   \text{Time starts at } 0 \text{ seconds and goes until the arrow hits the ground. So, practical domain is } 0 \leq t \leq 6.29 \text{ seconds.}
   \]

e) What is the practical range of this function?
   \[
   \text{The arrow passes through all height values from } 0 \text{ (when it hits the ground) to its max height of } 230 \text{ ft. So, practical range is } 0 \leq h(t) \leq 230 \text{ feet.}
   \]

f) What does the vertical intercept represent?
   \[
   \text{The vertical intercept represents the height of the arrow at time } t = 0. \text{ Thus, the arrow starts at 130 feet off the ground.}
   \]
Problem 17  | MEDIA EXAMPLE – APPLICATIONS OF QUADRATIC FUNCTIONS

A train tunnel is modeled by the quadratic function \( h(x) = -0.35x^2 + 25 \), where \( x \) is the distance, in feet, from the center of the tracks and \( h(x) \) is the height of the tunnel, also in feet. Assume that the high point of the tunnel is directly in line with the center of the train tracks.

a) Draw a complete diagram of this situation. Find and label each of the following: vertex, horizontal intercept (positive side), and vertical intercept.

b) How wide is the base of the tunnel?

c) A train with a flatbed car 6 feet off the ground is carrying a large object that is 15 feet high. How much room will there be between the top of the object and the top of the tunnel?
Problem 18  YOU TRY – APPLICATIONS OF QUADRATIC FUNCTIONS

A toy rocket is shot straight up into the air. The function \( H(t) = -16t^2 + 128t + 3 \) gives the height (in feet) of a rocket after \( t \) seconds. Round answers to two decimal places as needed. All answers must include appropriate units of measure.

a) Draw a complete diagram of this situation. Find and label each of the following: vertex, horizontal intercept (positive side), and vertical intercept.

b) How long does it take for the rocket to reach its maximum height? Write your answer in a complete sentence.

c) What is the maximum height of the rocket? Write your answer in a complete sentence.

d) How long does it take for the rocket to hit the ground? Write your answer in a complete sentence.

e) Identify the vertical intercept. Write it as an ordered pair and interpret its meaning in a complete sentence.

f) Determine the practical domain of \( H(t) \). Use inequality notation and include units.

g) Determine the practical range of \( H(t) \). Use inequality notation and include units.
Problem 19

### WORKED EXAMPLE – Quadratic Regression

The table below shows the height, $H$ in feet, of an arrow $t$ seconds after being shot.

<table>
<thead>
<tr>
<th>$t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(t)$</td>
<td>95</td>
<td>149</td>
<td>163</td>
<td>153</td>
<td>108</td>
</tr>
</tbody>
</table>

Use the Quadratic Regression feature of your calculator to generate a mathematical model for this situation. Round to three decimals.

- Press STAT>EDIT>ENTER to show data entry area. The STAT button is on the second row, third column.

- Press STAT > CALC > 5:QuadReg

- Thus, your quadratic function (with values rounded as the problem indicates) is $y = -15.857x^2 + 98.143x + 13.6$

Convert this to function notation with the appropriate variables to get $H(t) = -15.857t^2 + 98.143t + 13.6$

- Enter your function into Y1 to obtain a graph of your data and regression line. Use viewing window $x_{\text{min}}=0$ $x_{\text{max}}=7$ $y_{\text{min}}=0$ $y_{\text{max}}=180$
The table below shows the height, \( H \) in feet, of a golf ball \( t \) seconds after being hit.

<table>
<thead>
<tr>
<th>( t )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H(t) )</td>
<td>81</td>
<td>131</td>
<td>148</td>
<td>130</td>
<td>87</td>
</tr>
</tbody>
</table>

a) Use the Quadratic Regression feature of your calculator to generate a mathematical model for this situation. Use function notation with the appropriate variables. Round to three decimals.

b) Use your model to predict the height of the golf ball at 5 seconds. Round your answer to the nearest hundredth. How does this compare to the value in the data table?

c) Using your model to determine the maximum height of the golf ball. Round your answer to the nearest hundredth.

d) Use your model to determine how long it will take the golf ball to hit the ground. Round your answer to the nearest hundredth.

e) Use your model to determine the practical domain and practical range for this situation.

<table>
<thead>
<tr>
<th>Practical Domain</th>
<th>Practical Range</th>
</tr>
</thead>
</table>

f) Use your graphing calculator to create a graph of the data and the function \( H(t) \) on the same screen. Use the practical domain and range to determine an appropriate viewing window. In the space below, sketch what you see on your calculator screen, and indicate the window you used.

<table>
<thead>
<tr>
<th>Xmin:</th>
<th>Xmax:</th>
<th>Ymin:</th>
<th>Ymax:</th>
</tr>
</thead>
</table>