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Gloag **Gloag** **Kramer** **Landers**

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Authors
Andrew Gloag, Anne Gloag, Melissa Kramer, Mara Landers

Contributor
William Meacham

Editor
Annamaria Farbizio

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Chapter 1

Introduction to Functions

1.1 Introduction to Functions

The study of expressions, equations, and functions is the basis of mathematics. Each mathematical subject requires knowledge of manipulating equations to solve for a variable. Careers such as automobile accident investigators, quality control engineers, and insurance originators use equations to determine the value of variables.

Functions are methods of explaining relationships and can be represented as a rule, a graph, a table, or in words. The amount of money in a savings account, how many miles run in a year, or the number of trout in a pond are all described using functions.

Throughout this chapter, you will learn how to choose the best variables to describe a situation, simplify an expression using the Order of Operations, describe functions in various ways, write equations, and solve problems using a systematic approach.

Functions as Rules and Tables

Instead of purchasing a one-day ticket to the theme park, Joseph decided to pay by ride. Each ride costs \$2.00. To describe the amount of money Joseph will spend, several mathematical concepts can be used.

First, an expression can be written to describe the relationship between the cost per ride and the number of rides, r. An equation can also be written if the total amount he wants to spend is known. An inequality can be used if Joseph wanted to spend less than a certain amount.

Example 1: Using Joseph's situation, write the following:

a. An expression representing his total amount spent

b. An equation that shows Joseph wants to spend exactly $22.00 on rides

c. An inequality that describes the fact that Joseph will not spend more than $26.00 on rides

Solution: The variable in this situation is the number of rides Joseph will pay for. Call this r.

a. $2(r)$

b. $2(r) = 22$

c. $2(r) \leq 26$

In addition to an expression, equation, or inequality, Joseph's situation can be expressed in the form of a function or a table.

Definition: A **function** is a relationship between two variables such that the input value has ONLY one output value.

Writing Equations as Functions

A function is a set of ordered pairs in which the first coordinate, usually x, matches with exactly one second coordinate, y. Equations that follow this definition can be written in function notation. The y coordinate represents the **dependent variable**, meaning the values of this variable depend upon what is substituted for the other variable.

Consider Joseph's equation $m = 2r$. Using function notation, the value of the equation (the money spent m) is replaced with $f(r)$. f represents the function name and (r) represents the variable. In this case the parentheses do not mean multiplication; they separate the function name from the **independent variable.**

$$\begin{array}{c} input \\ \downarrow \\ \underbrace{f(x)}_{\substack{function \\ box}} = y \leftarrow output \end{array}$$

Example 2: Rewrite the following equations in function notation.

a. $y = 7x - 3$

b. $d = 65t$

c. $F = 1.8C + 32$

Solution:

a. According to the definition of a function, $y = f(x)$, so $f(x) = 7x - 3$.

b. This time the dependent variable is d. Function notation replaces the dependent variable, so $d = f(t) = 65t$.

c. $F = f(C) = 1.8C + 32$

Why Use Function Notation?

Why is it necessary to use function notation? The necessity stems from using multiple equations. Function notation allows one to easily decipher between the equations. Suppose Joseph, Lacy, Kevin, and Alfred all went to the theme park together and chose to pay \$2.00 for each ride. Each person would have the same equation $m = 2r$. Without asking each friend, we could not tell which equation belonged to whom. By substituting function notation for the dependent variable, it is easy to tell which function belongs to whom. By using function notation, it will be much easier to graph multiple lines (Chapter 4).

Example 3: Write functions to represent the total each friend spent at the park.

Solution: $J(r) = 2r$ represents Joseph's total, $L(r) = 2r$ represents Lacy's total, $K(r) = 2r$ represents Kevin's total, and $A(r) = 2r$ represents Alfred's total.

Using a Function to Generate a Table

A function really is an equation. Therefore, a table of values can be created by choosing values to represent the **independent variable.** The answers to each substitution represent $f(x)$.

Use Joseph's function to generate a table of values. Because the variable represents the number of rides Joseph will pay for, negative values do not make sense and are not included in the value of the independent variable.

Table 1.1:

R	$J(r) = 2r$
0	$2(0) = 0$
1	$2(1) = 2$
2	$2(2) = 4$
3	$2(3) = 6$
4	$2(4) = 8$
5	$2(5) = 10$
6	$2(6) = 12$

As you can see, the list cannot include every possibility. A table allows for precise organization of data. It also provides an easy reference for looking up data and offers a set of coordinate points that can be plotted to create a graphical representation of the function. A table does have limitations; namely it cannot represent infinite amounts of data and it does not always show the possibility of fractional values for the independent variable.

Domain and Range of a Function

The set of all possible input values for the independent variable is called the **domain.** The domain can be expressed in words, as a set, or as an inequality. The values resulting from the substitution of the domain represent the **range** of a function.

The domain of Joseph's situation will not include negative numbers because it does not make sense to ride negative rides. He also cannot ride a fraction of a ride, so decimals and fractional values do not make sense as input values. Therefore, the values of the independent variable r will be whole numbers beginning at zero.

Domain: All whole numbers

The values resulting from the substitution of whole numbers are whole numbers times two. Therefore, the **range** of Joseph's situation is still whole numbers just twice as large.

Range: All even whole numbers

Example 4: A tennis ball is bounced from a height and bounces back to 75% of its previous height. Write its function and determine its domain and range.

Solution: The function of this situation is $h(b) = 0.75b$, where b represents the previous bounce height.

Domain: The previous bounce height can be any positive number, so $b \geq 0$.

Range: The new height is 75% of the previous height, and therefore will also be any positive number (decimal or whole number), so the range is **all positive real numbers.**

Multimedia Link For another look at the domain of a function, see the following video where the narrator solves a sample problem from the California Standards Test about finding the domain of an unusual function. Khan Academy CA Algebra I Functions (6:34)

Figure 1.1: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/88

Write a Function Rule

In many situations, data is collected by conducting a survey or an experiment. To visualize the data, it is arranged into a table. Most often, a function rule is needed to predict additional values of the independent variable.

Example 5: *Write a function rule for the table.*

Number of CDs	2	4	6	8	10
Cost ($)	24	48	72	96	120

Solution: You pay \$24 for 2 CDs, \$48 for 4 CDs, and \$120 for 10 CDs. That means that each CD costs \$12.

We can write the function rule.

Cost $=$ \$12 $\times$ number of CDs or $f(x) = 12x$

Example 6: *Write a function rule for the table.*

-3	-2	-1	0	1	2	3
3	2	1	0	1	2	3

Solution: The values of the dependent variable are always the positive outcomes of the input values. This relationship has a special name, the absolute value. The function rule looks like this: $f(x) = |x|$.

Represent a Real-World Situation with a Function

Let's look at a real-world situation that can be represented by a function.

Example 7: *Maya has an internet service that currently has a monthly access fee of \$11.95 and a connection fee of \$0.50 per hour. Represent her monthly cost as a function of connection time.*

Solution: Let $x =$ the number of hours Maya spends on the internet in one month and let $y =$ Maya's monthly cost. The monthly fee is \$11.95 with an hourly charge of \$0.50.

The total cost $=$ flat fee $+$ hourly fee $\times$ number of hours. The function is $y = f(x) = 11.95 + 0.50x$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra: Domain and Range of a Function (12:52)

1. Rewrite using function notation: $y = \frac{5}{6}x - 2$.
2. What is one benefit of using function notation?
3. Define *domain*.
4. *True or false?* Range is the set of all possible inputs for the independent variable.
5. Generate a table from $-5 \le x \le 5$ for $f(x) = -(x)^2 - 2$
6. Use the following situation for question 6: *Sheri is saving for her first car. She currently has \$515.85 and is savings \$62 each week.*

Figure 1.2: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/454

(a) Write a function rule for the situation.
(b) Can the domain be ";all real numbers"? Explain your thinking.
(c) How many weeks would it take Sheri to save $1,795.00?

In 7 - 11, identify the domain and range of the function.

7. Dustin charges $10 per hour for mowing lawns.
8. Maria charges $25 per hour for math tutoring, with a minimum charge of $15.
9. $f(x) = 15x - 12$
10. $f(x) = 2x^2 + 5$
11. $f(x) = \frac{1}{x}$
12. What is the range of the function $y = x^2 - 5$ when the domain is $-2, -1, 0, 1, 2$?
13. What is the range of the function $y = 2x - \frac{3}{4}$ when the domain is $-2.5, 1.5, 5$?
14. Angie makes $6.50 per hour working as a cashier at the grocery store. Make a table of values that shows her earning for the input values 5, 10, 15, 20, 25, 30.
15. The area of a triangle is given by: $A = \frac{1}{2}bh$. If the base of the triangle is 8 centimeters, make a table of values that shows the area of the triangle for heights 1, 2, 3, 4, 5, and 6 centimeters.
16. Make a table of values for the function $f(x) = \sqrt{2x+3}$ for the input values $-1, 0, 1, 2, 3, 4, 5$.
17. Write a function rule for the table.

3	4	5	6
9	16	25	36

18. Write a function rule for the table.

hours	0	1	2	3
cost	15	20	25	30

19. Write a function rule for the table.

0	1	2	3
24	12	6	3

20. Write a function that represents the number of cuts you need to cut a ribbon in x number of pieces.
21. Solomon charges a $40 flat rate and $25 per hour to repair a leaky pipe. Write a function that represents the total fee charged as a function of hours worked. How much does Solomon earn for a three-hour job?
22. Rochelle has invested $2500 in a jewelry making kit. She makes bracelets that she can sell for $12.50 each. How many bracelets does Rochelle need to make before she breaks even?
23. Make up a situation in which the domain is all real numbers but the range is all whole numbers.

Quick Quiz

1. Write a function rule to describe the following table:

# of Books	1	2	3	4	5	6
Cost	4.75	5.25	5.75	6.25	6.75	7.25

2. Simplify: $84 \div [(18 - 16) \times 3]$.

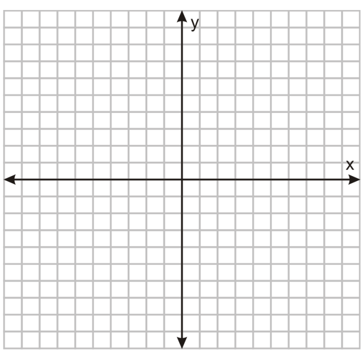
3. Evaluate the expression $\frac{2}{3}(y + 6)$ when $y = 3$.

4. Rewrite using function notation: $y = \frac{1}{4}x^2$.

5. You purchased six video games for $29.99 each and three DVD movies for $22.99. What is the total amount of money you spent?

Functions as Graphs

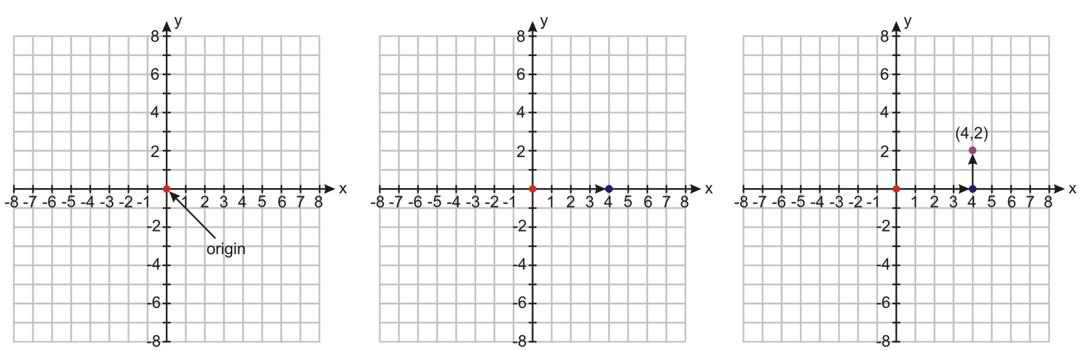
Once a table has been created for a function, the next step is to visualize the relationship by graphing the coordinates *(independent value, dependent value)*. In previous courses, you have learned how to plot ordered pairs on a coordinate plane. The first coordinate represents the horizontal distance from the origin (the point where the axes intersect). The second coordinate represents the vertical distance from the origin.



To graph a coordinate point such as (4,2) we start at the origin.

Because the first coordinate is positive four, we move 4 units to the right.

From this location, since the second coordinate is positive two, we move 2 units up.



Example 1: *Plot the following coordinate points on the Cartesian plane.*

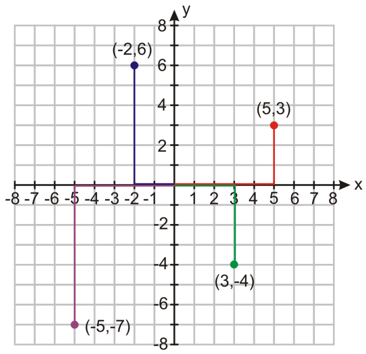
(a) (5, 3)

(b) (–2, 6)

(c) $(3, -4)$

(d) $(-5, -7)$

Solution: We show all the coordinate points on the same plot.



Notice that:

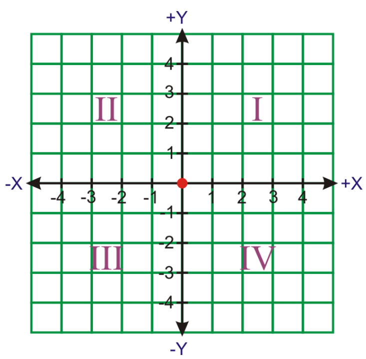
For a positive x value we move to the right.

For a negative x value we move to the left.

For a positive y value we move up.

For a negative y value we move down.

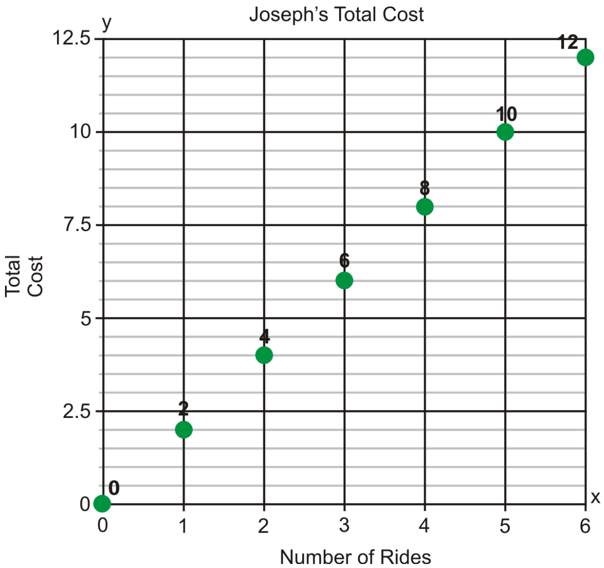
When referring to a coordinate plane, also called a Cartesian plane, the four sections are called **quadrants.** The first quadrant is the upper right section, the second quadrant is the upper left, the third quadrant is the lower left and the fourth quadrant is the lower right.



Suppose we wanted to visualize Joseph's total cost of riding at the amusement park. Using the table generated in Lesson 1.5, the graph can be constructed as (number of rides, total cost).

Table 1.2:

r	$J(r) = 2r$
0	$2(0) = 0$
1	$2(1) = 2$
2	$2(2) = 4$
3	$2(3) = 6$
4	$2(4) = 8$
5	$2(5) = 10$
6	$2(6) = 12$



The green dots represent the combination of $(r, J(r))$. The dots are not connected because the domain of this function is all whole numbers. By connecting the points we are indicating that all values between the ordered pairs are also solutions to this function. Can Joseph ride $2\frac{1}{2}$ rides? Of course not! Therefore, we leave this situation as a **scatter plot.**

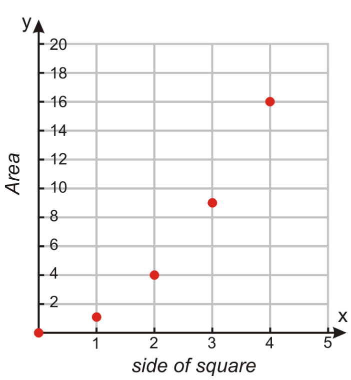
Example 2: Graph the function that has the following table of values.

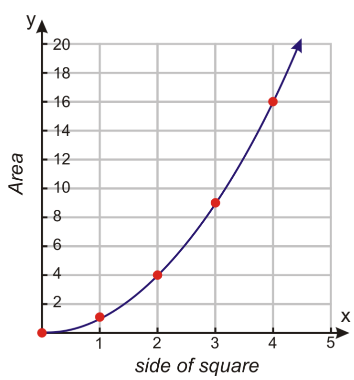
Side of the Square	0	1	2	3	4
Area of the Square	0	1	4	9	16

Solution: The table gives us five sets of coordinate points:

(0, 0), (1, 1), (2, 4), (3, 9), (4, 16).

To graph the function, we plot all the coordinate points. Because the length of a square can be fractional values, but not negative, the domain of this function is all positive real numbers, or $x \geq 0$. This means the ordered pairs can be connected with a smooth curve. This curve will continue forever in the positive direction, shown by an arrow.



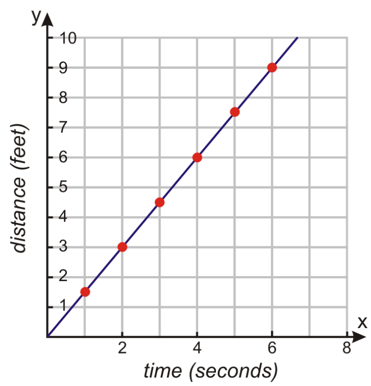


Writing a Function Rule Using a Graph

In many cases, you are given a graph and asked to determine its function. From a graph, you can read pairs of coordinate points that are on the curve of the function. The coordinate points give values of dependent and independent variables. These variables are related to each other by a rule. It is important we make sure this rule works for all the points on the curve.

In this course, you will learn to recognize different kinds of functions. There will be specific methods that you can use for each type of function that will help you find the function rule. For now, we will look at some basic examples and find patterns that will help us figure out the relationship between the dependent and independent variables.

Example 3: The graph below shows the distance that an inchworm covers over time. Find the function rule that shows how distance and time are related to each other.



Solution: Make table of values of several coordinate points to identify a pattern.

Time	0	1	2	3	4	5	6
Distance	0	1.5	3	4.5	6	7.5	9

We can see that for every minute the distance increases by 1.5 feet. We can write the function rule as:

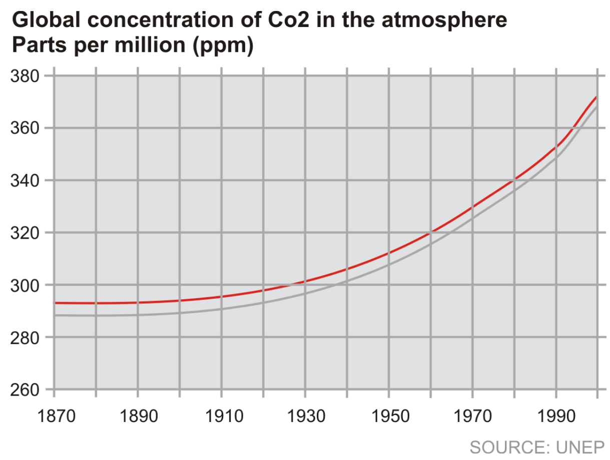
Distance $= 1.5 \times$ time

The equation of the function is $f(x) = 1.5x$

Analyze the Graph of a Real-World Situation

Graphs are used to represent data in all areas of life. You can find graphs in newspapers, political campaigns, science journals, and business presentations.

Here is an example of a graph you might see reported in the news. Most mainstream scientists believe that increased emissions of greenhouse gases, particularly carbon dioxide, are contributing to the warming of the planet. The graph below illustrates how carbon dioxide levels have increased as the world has industrialized.



From this graph, we can find the concentration of carbon dioxide found in the atmosphere in different years.

1900 - 285 parts per million

1930 - 300 parts per million

1950 - 310 parts per million

1990 - 350 parts per million

In Chapter 9, you will learn how to approximate an equation to fit this data using a graphing calculator.

Determining Whether a Relation Is a Function

You saw that a function is a **relation** between the independent and the dependent variables. It is a rule that uses the values of the independent variable to give the values of the dependent variable. A function rule can be expressed in words, as an equation, as a table of values, and as a graph. All representations are useful and necessary in understanding the relation between the variables.

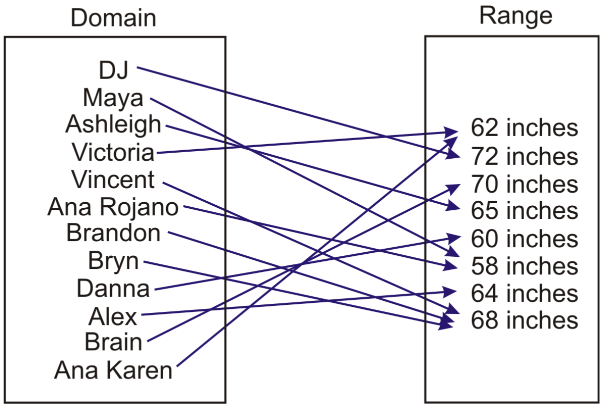
Definition: A **relation** is a set of ordered pairs.

Mathematically, a function is a special kind of relation.

Definition: A **function** is a relation between two variables such that the independent value has EXACTLY one dependent value.

This usually means that each $x-$value has only one $y-$value assigned to it. But, not all functions involve x and y.

Consider the relation that shows the heights of all students in a class. The domain is the set of people in the class and the range is the set of heights. Each person in the class cannot be more than one height at the same time. This relation is a function because for each person there is exactly one height that belongs to him or her.



Notice that in a function, a value in the range can belong to more than one element in the domain, so more than one person in the class can have the same height. The opposite is not possible, one person cannot have multiple heights.

Example 4: *Determine if the relation is a function.*

a) (1, 3), (–1, –2), (3, 5), (2, 5), (3, 4)

b) (–3, 20), (–5, 25), (–1, 5), (7, 12), (9, 2)

Solution:

a) To determine whether this relation is a function, we must follow the definition of a function. Each $x-$coordinate can have ONLY one $y-$coordinate. However, since the $x-$coordinate of 3 has two $y-$coordinates, 4 and 5, this relation is NOT a function.

b) Applying the definition of a function, each $x-$coordinate has only one $y-$coordinate. Therefore, this relation is a function.

Determining Whether a Graph Is a Function

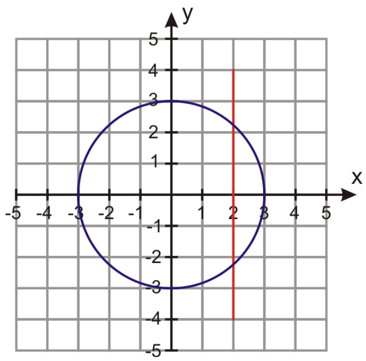
One way to determine whether a relation is a function is to construct a **flow chart** linking each dependent value to its matching independent value. Suppose, however, all you are given is the graph of the relation. How can you determine whether it is a function?

You could organize the ordered pairs into a table or a flow chart, similar to the student and height situation. This could be a lengthy process, but it is one possible way. A second way is to use the **Vertical Line Test.** Applying this test gives a quick and effective visual to decide if the graph is a function.

Theorem: Part A) A relation is a function if there are no vertical lines that intersect the graphed relation in more than one point.

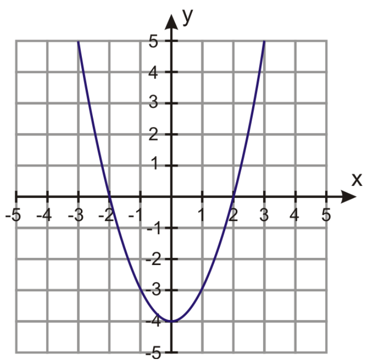
Part B) If a graphed relation does not intersect a vertical line in more than one point, then that relation is a function.

Is this graphed relation a function?



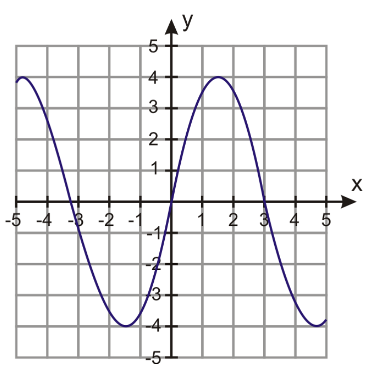
By drawing a vertical line (the red line) through the graph, we can see that the vertical line intersects the circle more than once. Therefore, this graph is NOT a function.

Here is a second example:



No matter where a vertical line is drawn through the graph, there will be only one intersection. Therefore, this graph is a function.

Example 4: *Determine if the relation is a function.*



Solution: Using the Vertical Line Test, we can conclude the relation is a function.

For more information:

Watch this YouTube video giving step-by-step instructions of the Vertical Line Test. CK-12 Basic Algebra: Vertical Line Test (3:11)

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the

Figure 1.3: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/715

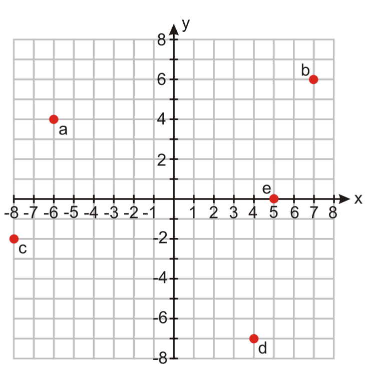
number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra: Functions as Graphs (9:34)

Figure 1.4: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/716

In 1 – 5, plot the coordinate points on the Cartesian plane.

1. (4, –4)
2. (2, 7)
3. (–3, –5)
4. (6, 3)
5. (–4, 3)

Using the coordinate plane below, give the coordinates for a – e.

6. 

In 7 – 9, graph the relation on a coordinate plane. According to the situation, determine whether to connect the ordered pairs with a smooth curve or leave as a scatter plot.

7.

X	–10	–5	0	5	10
Y	–3	–0.5	2	4.5	7

Table 1.3:

Side of cube (in inches)	**Volume of cube (in inches^3)**
0	0
1	1
2	8
3	27
4	64

Table 1.4:

Time (in hours)	**Distance (in miles)**
–2	–50
–1	25
0	0
1	5
2	50

In 10 – 12, graph the function.

10. Brandon is a member of a movie club. He pays a $50 annual membership and $8 per movie.
11. $f(x) = (x-2)^2$
12. $f(x) = 3.2^x$

In 13 – 16, determine if the relation is a function.

13. (1, 7), (2, 7), (3, 8), (4, 8), (5, 9)
14. (1, 1), (1, –1), (4, 2), (4, –2), (9, 3), (9, –3)
15.

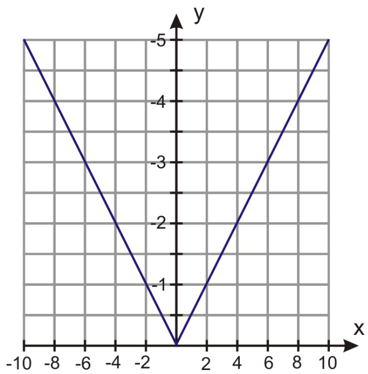
Age	20	25	25	30	35
Number of jobs by that age	3	4	7	4	2

16.

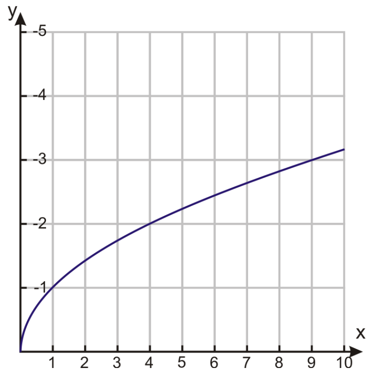
-4	-3	-2	-1	0
16	9	4	1	0

In 17 and 18, write a function rule for the graphed relation.

17.



18.



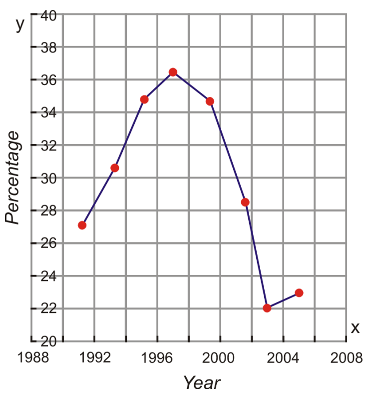
19. The students at a local high school took the Youth Risk Behavior Survey. The graph below shows the percentage of high school students who reported that they were current smokers. A person qualifies as a current smoker if he/she has smoked one or more cigarettes in the past 30 days. What percentage of high school students were current smokers in the following years?

(a) 1991

(b) 1996

(c) 2004

(d) 2005



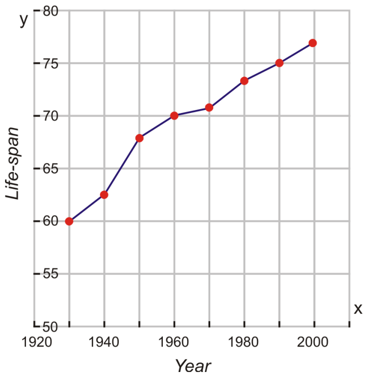
20. The graph below shows the average lifespan of people based on the year in which they were born. This information comes from the National Vital Statistics Report from the Center for Disease Control. What is the average lifespan of a person born in the following years?

(a) 1940

(b) 1955

(c) 1980

(d) 1995



21. The graph below shows the median income of an individual based on his/her number of years of education. The top curve shows the median income for males and the bottom curve shows the median income for females (Source: US Census, 2003). What is the median income of a male who has the following years of education?

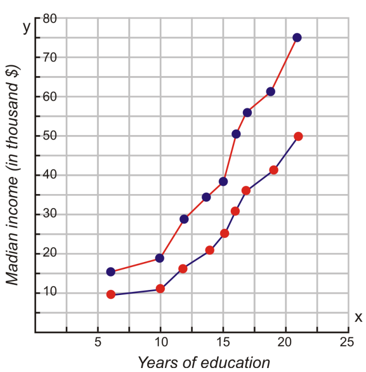
(a) 10 years of education

(b) 17 years of education

What is the median income of a female who has the same years of education?

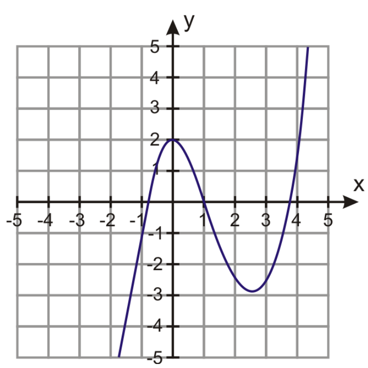
(c) 10 years of education

(d) 17 years of education

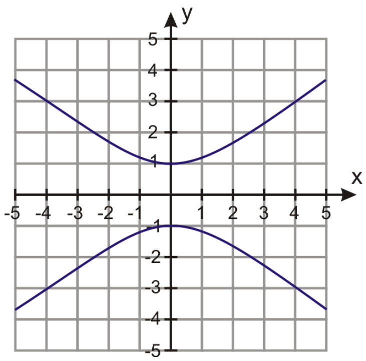


In 22 – 23, determine whether the graphed relation is a function.

22.



23.



You saw in the last chapter that linear graphs and equations are used to describe a variety of real-life situations. In mathematics, the goal is to find an equation that explains a situation as presented in a problem. In this way, we can determine the rule that describes the relationship. Knowing the equation or rule is very important since it allows us to find the values for the variables. There are different ways to find the best equation to represent a problem. The methods are based on the information you can gather from the problem.

This chapter focuses on several formulas used to help write equations of linear situations, such as slope-intercept form, standard form, and point-slope form. This chapter also teaches you how to fit a line to data and how to use a fitted line to predict data.

1.2 Linear Equations

Linear Equations in Slope-Intercept Form

Previously, you learned how to graph solutions to two-variable equations in slope-intercept form. This lesson focuses on how to write an equation for a graphed line. There are two things you will need from the graph to write the equation in slope-intercept form:

1. The $y-$intercept of the graph
2. The slope of the line

Having these two pieces of information will allow you to make the appropriate substitutions in the slope-intercept formula. Recall from the last chapter,

Slope-intercept form: $y = (slope)x + (y - intercept)$ or $y = mx + b$

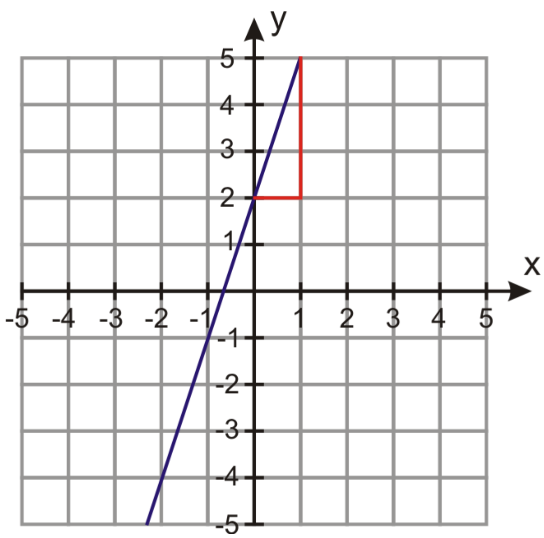
Example 1: Write the equation for a line with a slope of 4 and a $y-$intercept (0, –3).

Solution: Slope-intercept form requires two things: the slope and $y-$intercept. To write the equation, you substitute the values into the formula.

$$y = (slope)x + (y - intercept)$$
$$y = 4x + (-3)$$
$$y = 4x - 3$$

You can also use a graphed line to determine the slope and $y-$intercept.

Example 2: *Use the graph below to write its equation in slope-intercept form.*



Solution: The $y-$intercept is (0, 2). Using the slope triangle, you can determine the slope is $\frac{rise}{run} = \frac{-3}{-1} = \frac{3}{1}$. Substituting the value 2 for b and the value 3 for m, the equation for this line is $y = 3x + 2$.

Writing an Equation Given the Slope and a Point

Sometimes it may be difficult to determine the $y-$intercept. Perhaps the $y-$intercept is rational instead of an integer. Maybe you don't know the $y-$intercept. All you have is the slope and an ordered pair. You can use this information to write the equation in slope-intercept form. To do so, you will need to follow several steps.

Step 1: Begin by writing the formula for slope-intercept form $y = mx + b$.

Step 2: Substitute the given slope for m.

Step 3: Use the ordered pair you are given (x, y) and substitute these values for the variables x and y in the equation.

Step 4: Solve for b (the $y-$intercept of the graph).

Step 5: Rewrite the original equation in Step 1, substituting the slope for m and the $y-$intercept for b.

Example 3: *Write an equation for a line with slope of 4 that contains the ordered pair (–1, 5).*

Solution:

Step 1: Begin by writing the formula for slope-intercept form.

$$y = mx + b$$

Step 2: Substitute the given slope for m.

$$y = 4x + b$$

Step 3: Use the ordered pair you are given (–1, 5) and substitute these values for the variables x and y in the equation.

$$5 = (4)(-1) + b$$

Step 4: Solve for b (the $y-$intercept of the graph).

$$\begin{aligned}5 &= -4 + b\\ 5+4 &= -4+4+b\\ 9 &= b\end{aligned}$$

Step 5: Rewrite $y = mx + b$, substituting the slope for m and the $y-$intercept for b.

$$y = 4x + 9$$

Example 4: Write the equation for a line with a slope of –3 containing the point (3, –5).

Solution: Using the five-steps from above:

$$\begin{aligned}y &= (slope)x + (y - intercept)\\ y &= -3x + b\\ -5 &= -3(3) + b\\ -5 &= -9 + b\\ 4 &= b\\ y &= -3x + 4\end{aligned}$$

Writing an Equation Given Two Points

In many cases, especially real-world situations, you are given neither the slope nor the $y-$intercept. You might have only two points to use to determine the equation of the line.

To find an equation for a line between two points, you need two things:

1. The $y-$intercept of the graph
2. The slope of the line

Previously, you learned how to determine the slope between two points. Let's repeat the formula here.

The slope between any two points (x_1, y_1) and (x_2, y_2) is: $slope = \frac{y_2-y_1}{x_2-x_1}$.

The procedure for determining a line given two points is the same five-step process as writing an equation given the slope and a point.

Example 5: *Write the equation for the line containing the points (3, 2) and (–2, 4).*

Solution: You need the slope of the line. Find the line's slope by using the formula. Choose one ordered pair to represent (x_1, y_1) and the other ordered pair to represent (x_2, y_2).

$$slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{-2-3} = -\frac{2}{5}$$

Now use the five-step process to find the equation for this line.

Step 1: Begin by writing the formula for slope-intercept form.

$$y = mx + b$$

Step 2: Substitute the given slope for m.

$$y = \frac{-2}{5}x + b$$

Step 3: Use one of the ordered pairs you are given (–2, 4) and substitute these values for the variables x and y in the equation.

$$4 = \left(\frac{-2}{5}\right)(-2) + b$$

Step 4: Solve for b (the $y-$intercept of the graph).

$$4 = \frac{4}{5} + b$$
$$4 - \frac{4}{5} = \frac{4}{5} - \frac{4}{5} + b$$
$$\frac{16}{5} = b$$

Step 5: Rewrite $y = mx + b$, substituting the slope for m and the $y-$intercept for b.

$$y = \frac{-2}{5}x + \frac{16}{5}$$

Example 6: Write the equation for a line containing the points (–4, 1) and (–2, 3).

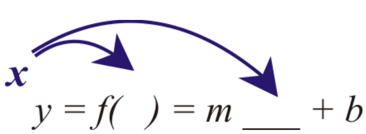
Solution:

1. Start with the slope–intercept form of the line $y = mx + b$.
2. Find the slope of the line: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{-2-(-4)} = \frac{2}{2} = 1$.
3. Substitute the value of slope for $m : y = (1)x + b$.
4. Substitute the coordinate (–2, 3) into the equation for the variables x and $y : 3 = -2 + b \Rightarrow b = 5$.
5. Rewrite the equation, substituting the slope for m and the $y-$intercept for b: $y = x + 5$.

Writing a Function in Slope-Intercept Form

Remember that a linear function has the form $f(x) = mx + b$. Here $f(x)$ represents the y values of the equation or the graph. So $y = f(x)$ and they are often used interchangeably. Using the functional notation in an equation often provides you with more information.

For instance, the expression $f(x) = mx + b$ shows clearly that x is the independent variable because you **substitute** values of x into the function and perform a series of operations on the value of x in order to calculate the values of the dependent variable, y.



In this case when you substitute x into the function, the function tells you to multiply it by m and then add b to the result. This process generates all the values of y you need.

Example 7: Consider the function $f(x) = 3x - 4$. *Find* $f(2), f(0),$ *and* $f(-1)$.

Solution: Each number in parentheses is a value of x that you need to substitute into the equation of the function.

$$f(2) = 3(2) - 4 = 6 - 4 = 2 \qquad f(0) = 3(0) - 4 = 0 - 4 = -4 \qquad f(-1) = 3(-1) - 4 = -3 - 4 = -7$$

$$f(2) = 2; f(0) = -4; \textit{ and } f(-1) = -7$$

Function notation tells you much more than the value of the independent variable. It also indicates a point on the graph. For example, in the above example, $f(-1) = -7$. This means the ordered pair (–1, –7) is a solution to $f(x) = 3x - 4$ and appears on the graphed line. You can use this information to write an equation for a function.

Example 8: *Write an equation for a line with* $m = 3.5$ *and* $f(-2) = 1$.

Solution: You know the slope and you know a point on the graph (–2, 1). Using the methods presented in this lesson, write the equation for the line.

Begin with slope-intercept form.

$$
\begin{aligned}
& & y &= mx + b \\
&\text{Substitute the value for the slope.} & y &= 3.5x + b \\
&\text{Use the ordered pair to solve for } b. & 1 &= 3.5(-2) + b \\
& & b &= 8 \\
&\text{Rewrite the equation.} & y &= 3.5x + 8 \\
&\text{or} & f(x) &= 3.5x + 8
\end{aligned}
$$

.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

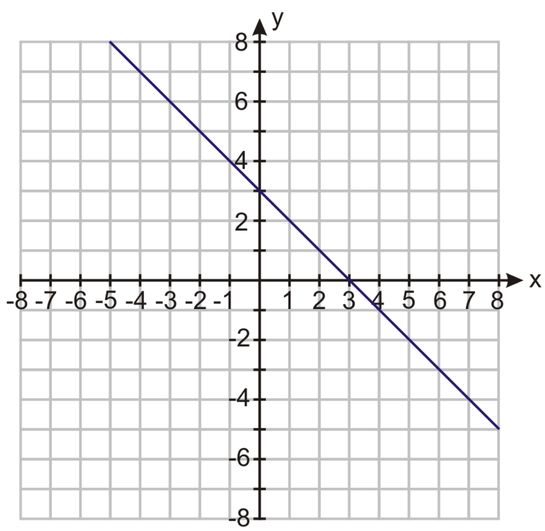
CK-12 Basic Algebra: Linear Equations in Slope-Intercept Form (14:58)

Figure 1.5: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/757

1. What is the formula for slope-intercept form? What do the variables m and b represent?
2. What are the five steps needed to determine the equation of a line given the slope and a point on the graph (not the y–intercept)?
3. What is the first step in finding the equation of a line given two points?

In 4 – 20, find the equation of the line in slope–intercept form.

4. The line has slope of 7 and y–intercept of –2.
5. The line has slope of –5 and y–intercept of 6.
6. The line has slope $= -2$ and a y–intercept $= 7$.
7. The line has slope $= \frac{2}{3}$ and a y–intercept $= \frac{4}{5}$.
8. The line has slope of $-\frac{1}{4}$ and contains point (4, –1).
9. The line has slope of $\frac{2}{3}$ and contains point $\left(\frac{1}{2}, 1\right)$.
10. The line has slope of –1 and contains point $\left(\frac{4}{5}, 0\right)$.
11. The line contains points (2, 6) and (5, 0).
12. The line contains points (5, –2) and (8, 4).
13. The line contains points (3, 5) and (–3, 0).
14. The slope of the line is $-\frac{2}{3}$ and the line contains point (2, –2).
15. The slope of the line is –3 and the line contains point (3, –5).
16. The line contains points (10, 15) and (12, 20).
17.



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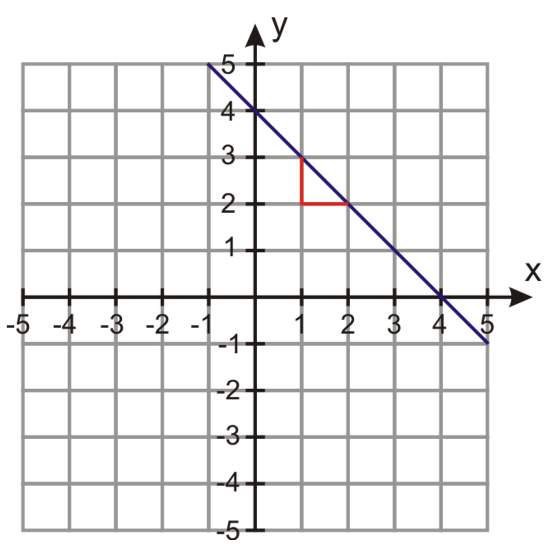

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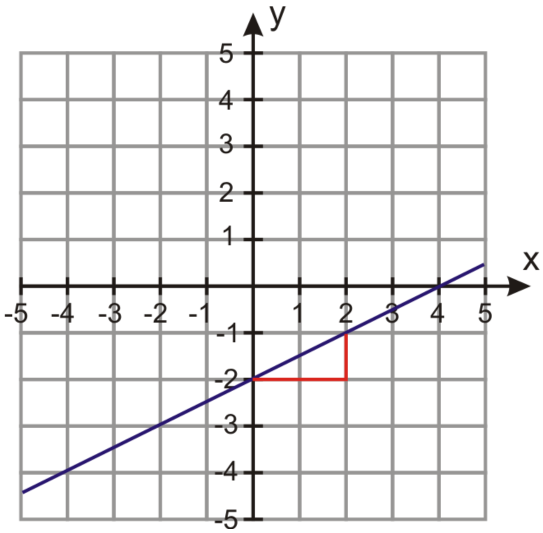

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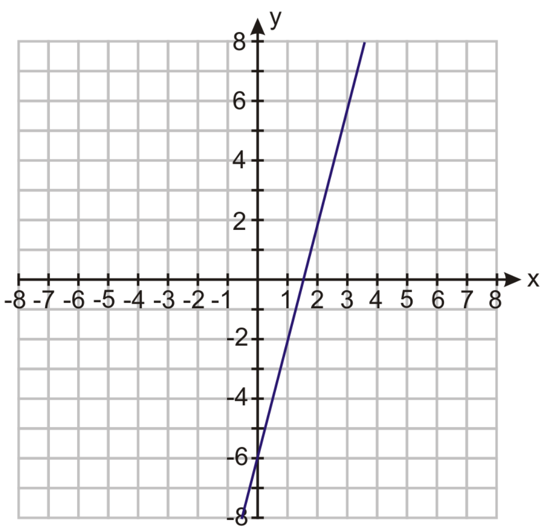


In 21 – 28, find the equation of the linear function in slope–intercept form.

21. $m = 5, f(0) = -3$
22. $m = -2$ and $f(0) = 5$
23. $m = -7, f(2) = -1$
24. $m = \frac{1}{3}, f(-1) = \frac{2}{3}$
25. $m = 4.2, f(-3) = 7.1$
26. $f\left(\frac{1}{4}\right) = \frac{3}{4}, f(0) = \frac{5}{4}$
27. $f(1.5) = -3, f(-1) = 2$
28. $f(-1) = 1$ and $f(1) = -1$

Linear Equations in Standard Form

As the past few lessons of this chapter have shown, there are several ways to write a linear equation. This lesson introduces another method: **standard form**. You have already seen examples of standard form equations in a previous lesson. For example, here are some equations written in standard form.

$$0.75(h) + 1.25(b) = 30$$
$$7x - 3y = 21$$
$$2x + 3y = -6$$

The **standard form** of a linear equation has the form $Ax + By = C$, *where* A, B, *and* C *are integers and* A *and* B *are* ***not*** *both zero.*

Equations written in standard form do not have fractional **coefficients** and the variables are written on the same side of the equation.

You should be able to rewrite any of the formulas into an alternate form.

$$Slope-intercept\ form \leftrightarrow Standard\ form$$
$$Slope-intercept\ form \leftrightarrow Point-slope\ form$$
$$Point-slope\ form \leftrightarrow Standard\ form$$

Example 1: *Rewrite* $\frac{3}{4}(h) + \frac{5}{4}(b) = 30$ *in standard form.*

Solution: According to the definition of standard form, the coefficients must be integers. So we need to **clear the fractions** of the denominator using multiplication.

$$\frac{3}{4}h + \frac{5}{4}b = 30 \rightarrow 4\left(\frac{3}{4}h + \frac{5}{4}b\right) = 4(30)$$
$$3h + 5b = 120$$

This equation is now in standard form, $A = 3, B = 5,$ *and* $C = 120$.

Example 2: Rewrite $y - 5 = 3(x - 2)$ in standard form.

Solution: Use the Distributive Property to simplify the right side of the equation

$$y - 5 = 3x - 6$$

Rewrite this equation so the variables x and y are on the same side of the equation.

$$y - 5 + 6 = 3x - 6 + 6$$
$$y - y + 1 = 3x - y$$
$$1 = 3x - y, \qquad \text{where A=3, B=-1, and C=1.}$$

Example 3: Rewrite $5x - 7 = y$ in standard form.

Solution: Rewrite this equation so the variables x and y are on the same side of the equation.

$$5x - 7 + 7 = y + 7$$
$$5x - y = y - y + 7$$
$$5x - y = 7, \qquad \text{where A=5, B=-1, and C=7.}$$

Finding Slope and Intercept of a Standard Form Equation

Slope-intercept form and point-slope form of a linear equation both contain the slope of the equation explicitly, but the standard form does not. Since the slope is such an important feature of a line, it is useful to figure out how you would find the slope if you were given the equation of the line in standard form.

Begin with standard form: $Ax + By = C$.

If you rewrite this equation in slope-intercept form, it becomes:

$$\begin{aligned} Ax - Ax + By &= C - Ax \\ \frac{By}{B} &= \frac{-Ax + C}{B} \\ y &= \frac{-A}{B}x + \frac{C}{B} \end{aligned}$$

When you compare this form to slope-intercept form, $y = mx + b$, you can see that the slope of a standard form equation is $\frac{-A}{B}$ and the $y-$intercept is $\frac{C}{B}$.

The **standard form** of a linear equation $Ax + By = C$ has the following:

$slope = \frac{-A}{B}$ and $y - intercept = \frac{C}{B}$.

Example 4: Find the slope and $y-$intercept of $2x - 3y = -8$.

Solution: Using the definition of standard form, $A = 2, B = -3,\ and\ C = -8$.

$$\begin{aligned} slope &= \frac{-A}{B} = \frac{-2}{-3} \rightarrow \frac{2}{3} \\ y - intercept &= \frac{C}{B} = \frac{-8}{-3} \rightarrow \frac{8}{3} \end{aligned}$$

The slope is $\frac{2}{3}$ and the $y-$intercept is $\frac{8}{3}$.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Linear Equations in Standard Form (10:08)

Figure 1.6: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/759

1. What is the standard form of a linear equation? What do A, B, and C represent?
2. What is the meaning of ";clear the fractions";? How would you go about doing so?
3. Consider the equation $Ax + By = C$. What are the slope and $y-$intercept of this equation?

Rewrite the following equations in standard form.

4. $y = 3x - 8$
5. $y = -x - 6$
6. $y = \frac{5}{3}x - 4$
7. $0.30x + 0.70y = 15$
8. $5 = \frac{1}{6}x - y$
9. $y - 7 = -5(x - 12)$
10. $2y = 6x + 9$
11. $y = \frac{9}{4}x + \frac{1}{4}$
12. $y + \frac{3}{5} = \frac{2}{3}(x - 2)$
13. $3y + 5 = 4(x - 9)$

Find the slope and $y-$intercept of the following lines.

14. $5x - 2y = 15$
15. $3x + 6y = 25$
16. $x - 8y = 12$
17. $3x - 7y = 20$
18. $9x - 9y = 4$
19. $6x + y = 3$
20. $x - y = 9$
21. $8x + 3y = 15$
22. $4x + 9y = 1$

In 23 – 27, write each equation in standard form by first writing it in point-slope form.

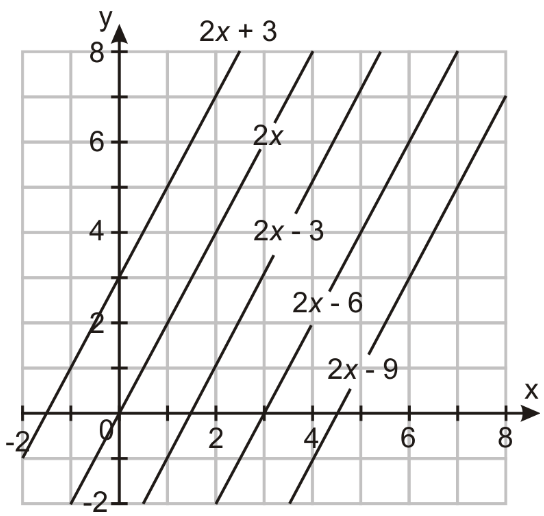
23. $Slope = -1$ *through point* (–3, 5)
24. $Slope = -\frac{1}{4}$ *through point* (4, 0)
25. Line through (5, –2) and (–5, 4)
26. Line through (–3, –2) and (5, 1)
27. Line through (1, –1) and (5, 2)

Equations of Parallel and Perpendicular Lines

In a previous lesson, you learned how to identify parallel lines.

Parallel lines have the same slope.

Each of the graphs below have the same slope. According to the definition, all these lines are parallel.



Example 1: Are $y = \frac{1}{3}x - 4$ and $-3x + 9y = 18$ parallel?

Solution: The slope of the first line is $\frac{1}{3}$. Any line parallel to this must also have a slope of $\frac{1}{3}$.

Find the slope of the second equation: $A = -3, B = 9$

$$slope = \frac{-A}{B} = \frac{3}{9} \rightarrow \frac{1}{3}$$

These two lines have the same slope so they are parallel.

Slopes of Perpendicular Lines

Lines can be parallel, **coincident** (overlap each other), or intersecting (crossing). Lines that intersect at 90° angles have a special name: **perpendicular lines**. The slopes of perpendicular lines have a special property.

Perpendicular lines form a right angle. The product of their slopes is –1.

$$m_1 \cdot m_2 = -1$$

Example 2: Verify that the following lines are perpendicular.

Line a: passes through points (–2, –7) and (1, 5)

Line b: passes through points (4, 1) and (–8, 4)

Solution: Find the slopes of each line.

$$Line\ a : \frac{5 - (-7)}{1 - (-2)} = \frac{12}{3} = \frac{4}{1} \qquad Line\ b : \frac{4 - 1}{-8 - 4} = \frac{3}{-12} = \frac{-1}{4}$$

To verify that the lines are perpendicular, the product of their slopes must equal –1.

$$\frac{4}{1} \times \frac{-1}{4} = -1$$

Because the product of their slopes is -1, lines a and b are perpendicular.

Example 3: Determine whether the two lines are parallel, perpendicular, or neither:

Line 1: $2x = y - 10$; Line 2: $y = -2x + 5$

Solution: Begin by finding the slopes of lines 1 and 2.

$$2x + 10 = y - 10 + 10$$
$$2x + 10 = y$$

The slope of the first line is 2.

$$y = -2x + 5$$

The slope of the second line is –2.

These slopes are not identical, so these lines are not parallel.

To check if the lines are perpendicular, find the product of the slopes. $2 \times -2 = -4$. The product of the slopes is not –1, so the lines are not perpendicular.

Lines 1 and 2 are neither parallel nor perpendicular.

Writing Equations of Parallel Lines

Example 4: *Find the equation parallel to the line $y = 6x - 9$ passing through (–1, 4).*

Solution: Parallel lines have the same slope, so the slope will be 6. You have a point and the slope, so you can use point-slope form.

$$y - y_1 = m(x - x_1)$$
$$y - 4 = 6(x + 1)$$

You could rewrite it in slope-intercept form:

$$y = 6x + 6 + 4$$
$$y = 6x + 10$$

Writing Equations of Perpendicular Lines

Writing equations of perpendicular lines is slightly more difficult than writing parallel line equations. The reason is because you must find the slope of the perpendicular line before you can proceed with writing an equation.

Example: *Find the equation perpendicular to the line $y = -3x + 5$ that passes through point (2, 6).*

Solution: Begin by finding the slopes of the perpendicular line. Using the perpendicular line definition, $m_1 \cdot m_2 = -1$. The slope of the original line is –3. Substitute that for m_1.

$$-3 \cdot m_2 = -1$$

Solve for m_2, the slope of the perpendicular line.

$$\frac{-3m_2}{-3} = \frac{-1}{-3}$$
$$m_2 = \frac{1}{3}$$

The slope of the line perpendicular to $y = -3x + 5$ is $\frac{1}{3}$.

You now have the slope and a point. Use point-slope form to write its equation.

$$y - 6 = \frac{1}{3}(x - 2)$$

You can rewrite this in slope-intercept form: $y = \frac{1}{3}x - \frac{2}{3} + 6$.

$$y = \frac{1}{3}x + \frac{16}{3}$$

Example 4: *Find the equation of the line perpendicular to the line* $y = 5$ *and passing through (5, 4).*

Solution: The line $y = 5$ is a horizontal line with slope of zero. The only thing that makes a 90° angle with a horizontal line is a vertical line. Vertical lines have undefined slopes.

Since the vertical line must go through (5, 4), the equation is $x = 5$.

Multimedia Link: For more help with writing lines, visit AlgebraLab.

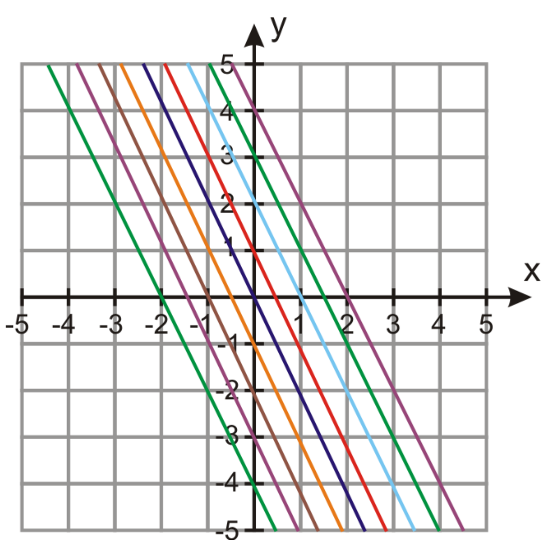
Families of Lines

A straight line has two very important properties, its **slope** and its $y-$**intercept**. The slope tells us how steeply the line rises or falls, and the $y-$intercept tells us where the line intersects the $y-$axis. In this section, we will look at two families of lines.

A **family of lines** is a set of lines that have something in common with each other. Straight lines can belong to two types of families: where the slope is the same and where the $y-$intercept is the same.

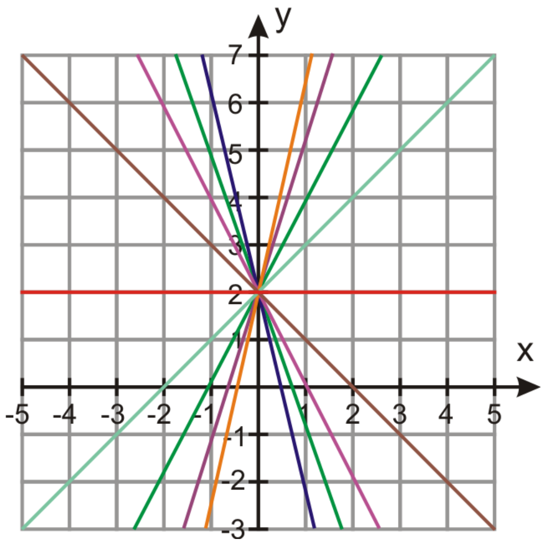
Family 1: The slope is the same

Remember that lines with the same slope are parallel. Each line on the Cartesian plane below has an identical slope with different $y-$intercepts. All the lines look the same but they are shifted up and down the $y-$axis. As b gets larger the line rises on the $y-$axis and as b gets smaller the line goes lower on the $y-$axis. This behavior is often called a **vertical shift**.



Family 2: The intercept is the same

The graph below shows several lines with the same $y-$intercept but varying slopes.



Practice Set

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CK-12 Basic Algebra: Equations of Parallel and Perpendicular Lines (9:13)

Figure 1.7: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/760

1. Define *parallel lines*.
2. Define *perpendicular lines*.
3. What is true about the slopes of perpendicular lines?
4. What is a *family of lines?*

Determine the slope of a line a) parallel and b) perpendicular to each line given.

5. $y = -5x + 7$
6. $2x + 8y = 9$

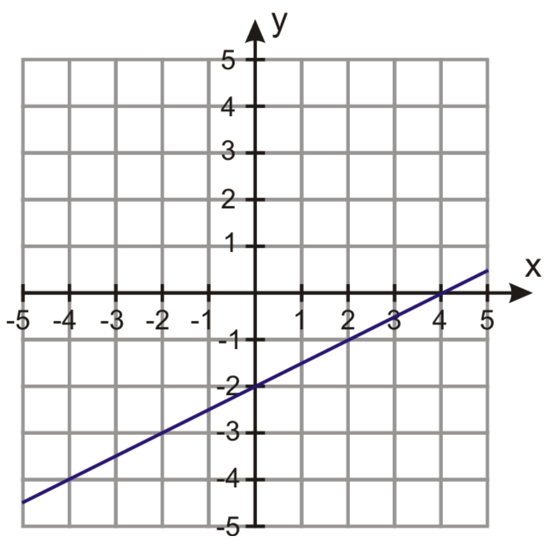
7. $x = 8$
8. $y = -4.75$
9. $y - 2 = \frac{1}{5}(x + 3)$

In 9 – 16, determine whether the lines are parallel, perpendicular, or neither.

10. *Line a* : passing through points (–1, 4) and (2, 6); *Line b* : passing through points (2, –3) and (8, 1).
11. *Line a* : passing through points (4, –3) and (–8, 0); *Line b* : passing through points (–1, –1) and (–2, 6).
12. *Line a* : passing through points (–3, 14) and (1, –2); *Line b* : passing through points (0, –3) and (–2, 5).
13. *Line a* : passing through points (3, 3) and (–6, –3); *Line b* : passing through points (2, –8) and (–6, 4).
14. Line 1: $4y + x = 8$; Line 2: $12y + 3x = 1$
15. Line 1: $5y + 3x + 1$; Line 2: $6y + 10x = -3$
16. Line 1: $2y - 3x + 5 = 0$; Line 2: $y + 6x = -3$
17. Find the equation of the line parallel to $5x - 2y = 2$ that passes through point (3, –2).
18. Find the equation of the line perpendicular to $y = -\frac{2}{5}x - 3$ that passes through point (2, 8).
19. Find the equation of the line parallel to $7y + 2x - 10 = 0$ that passes through the point (2, 2).
20. Find the equation of the line perpendicular to $y + 5 = 3(x - 2)$ that passes through the point (6, 2).
21. Find the equation of the line through (2, –4) perpendicular to $y = \frac{2}{7}x + 3$.
22. Find the equation of the line through (2, 3) parallel to $y = \frac{3}{2}x + 5$.

In 23 – 26, write the equation of the family of lines satisfying the given condition.

23. All lines pass through point (0, 4).
24. All lines are perpendicular to $4x + 3y - 1 = 0$.
25. All lines are parallel to $y - 3 = 4x + 2$.
26. All lines pass through point (0, –1).
27. Write an equation for a line parallel to the equation graphed below.
28. Write an equation for a line perpendicular to the equation graphed below and passing through the ordered pair (0, –1).



Quick Quiz

1. Write an equation for a line with slope of $\frac{4}{3}$ and y–intercept (0, 8).

2. Write an equation for a line containing (6, 1) and (7, –3).

3. A plumber charges $75 for a 2.5-hour job and $168.75 for a 5-hour job.

Assuming the situation is linear, write an equation to represent the plumber's charge and use it to predict the cost of a 1-hour job.

4. Rewrite in standard form: $y = \frac{6}{5}x + 11$.

5. Sasha took tickets for the softball game. Student tickets were $3.00 and adult tickets were $3.75. She collected a total of $337.50 and sold 75 student tickets. How many adult tickets were sold?

1.3 Linear Functions and Applications

A Problem-Solving Plan

Much of mathematics apply to real-world situations. To think critically and to problem solve are mathematical abilities. Although these capabilities may be the most challenging, they are also the most rewarding.

To be successful in applying mathematics in real-life situations, you must have a ";toolbox"; of strategies to assist you. The last few lessons of many chapters in this FlexBook are devoted to filling this toolbox so you to become a better problem solver and tackle mathematics in the real world.

Step #1: Read and Understand the Given Problem

Every problem you encounter gives you clues needed to solve it successfully. Here is a checklist you can use to help you understand the problem.

√ Read the problem carefully. Make sure you read all the sentences. Many mistakes have been made by failing to fully read the situation.

√ Underline or highlight key words. These include mathematical operations such as *sum, difference, product,* and mathematical verbs such as *equal, more than, less than, is.* Key words also include the nouns the situation is describing such as *time, distance, people,* etc.

Visit the Wylie Intermediate Website (http://wylie.region14.net/webs/shamilton/math_clue_words.htm) for more clue words.

√ Ask yourself if you have seen a problem like this before. Even though the nouns and verbs may be different, the general situation may be similar to something else you've seen.

√ What are you being asked to do? What is the question you are supposed to answer?

√ What facts are you given? These typically include numbers or other pieces of information.

Once you have discovered what the problem is about, the next step is to declare what variables will represent the nouns in the problem. Remember to use letters that make sense!

Step #2: Make a Plan to Solve the Problem

The next step in the problem-solving plan is to **make a plan** or **develop a strategy.** How can the information you know assist you in figuring out the unknown quantities?

Here are some common strategies that you will learn.

- Drawing a diagram
- Making a table
- Looking for a pattern
- Using guess and check
- Working backwards
- Using a formula
- Reading and making graphs
- Writing equations
- Using linear models
- Using dimensional analysis
- Using the right type of function for the situation

In most problems, you will use a combination of strategies. For example, drawing a diagram and looking for patterns are good strategies for most problems. Also, making a table and drawing a graph are often used together. The ";writing an equation"; strategy is the one you will work with the most frequently in your study of algebra.

Step #3: Solve the Problem and Check the Results

Once you develop a plan, you can use it to **solve the problem.**

The last step in solving any problem should always be to **check and interpret** the answer. Here are some questions to help you to do that.

- Does the answer make sense?
- If you substitute the solution into the original problem, does it make the sentence true?
- Can you use another method to arrive at the same answer?

Step #4: Compare Alternative Approaches

Sometimes a certain problem is best solved by using a specific method. Most of the time, however, it can be solved by using several different strategies. When you are familiar with all of the problem-solving strategies, it is up to you to choose the methods that you are most comfortable with and that make sense to you. In this book, we will often use more than one method to solve a problem. This way we can demonstrate the strengths and weaknesses of different strategies when applied to different types of problems.

Regardless of the strategy you are using, you should always implement the problem-solving plan when you are solving word problems. Here is a summary of the problem-solving plan.

Step 1: Understand the problem.

Step 2: Devise a plan – Translate. Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart, or construct a table as a start to begin your problem-solving plan.

Step 3: Carry out the plan – Solve.

Step 4: Check and Interpret: Check to see if you have used all your information. Then look to see if the answer makes sense.

Solve Real-World Problems Using a Plan

Example 1: Jeff is 10 years old. His younger brother, Ben, is 4 years old. How old will Jeff be when he is twice as old as Ben?

Solution: Begin by understanding the problem. Highlight the key words.

Jeff is **10** years old. His younger brother, **Ben**, is **4** years old. **How old** will Jeff be **when he is twice as old as Ben**?

The question we need to answer is. ";What is Jeff's age when he is twice as old as Ben?";

You could guess and check, use a formula, make a table, or look for a pattern.

The key is ";twice as old."; This clue means two times, or double Ben's age. Begin by doubling possible ages. Let's look for a pattern.

$4 \times 2 = 8$. Jeff is already older than 8.

$5 \times 2 = 10$. This doesn't make sense because Jeff is already 10.

$6 \times 2 = 12$. In two years, Jeff will be 12 and Ben will be 6. Jeff will be twice as old.

Jeff will be 12 years old.

Example 2: Matthew is planning to harvest his corn crop this fall. The field has 660 rows of corn with 300 ears per row. Matthew estimates his crew will have the crop harvested in 20 hours. How many ears of corn will his crew harvest per hour?

Solution: Begin by highlighting the key information.

Matthew is planning to harvest his corn crop this fall. The field has **660 rows** of corn with **300 ears per row**. Matthew estimates his crew will have the **crop harvested in 20 hours. How many ears of corn** will his crew **harvest per hour**?

You could draw a picture (it may take a while), write an equation, look for a pattern, or make a table. Let's try to use reasoning.

We need to figure out how many ears of corn are in the field. $660(300) = 198,000$. This is how many ears are in the field. It will take 20 hours to harvest the entire field, so we need to divide 198,000 by 20 to get the number of ears picked per hour.

$$\frac{198,000}{20} = 9,900$$

The crew can harvest 9,900 ears per hour.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra: Word Problem-Solving Plan 1 (10:12)

Figure 1.8: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/719

1. What are the four steps to solving a problem?
2. Name three strategies you can use to help make a plan. Which one(s) are you most familiar with already?
3. Which types of strategies work well together? Why?
4. Suppose Matthew's crew takes 36 hours to harvest the field. How many ears per hour will they harvest?
5. Why is it difficult to solve Ben and Jeff's age problem by drawing a diagram?
6. How do you check a solution to a problem? What is the purpose of checking the solution?
7. There were 12 people on a jury, with four more women than men. How many women were there?
8. A rope 14 feet long is cut into two pieces. One piece is 2.25 feet longer than the other. What are the lengths of the two pieces?
9. A sweatshirt costs \$35. Find the total cost if the sales tax is 7.75%.
10. This year you got a 5% raise. If your new salary is \$45,000, what was your salary before the raise?
11. It costs \$250 to carpet a room that is $14\ ft \times 18\ ft$. How much does it cost to carpet a room that is $9\ ft \times 10\ ft$?
12. A department store has a 15% discount for employees. Suppose an employee has a coupon worth \$10 off any item and she wants to buy a \$65 purse. What is the final cost of the purse if the employee discount is applied before the coupon is subtracted?
13. To host a dance at a hotel, you must pay \$250 plus \$20 per guest. How much money would you have to pay for 25 guests?
14. It costs \$12 to get into the San Diego County Fair and \$1.50 per ride. If Rena spent \$24 in total, how many rides did she go on?
15. An ice cream shop sells a small cone for \$2.92, a medium cone for \$3.50, and a large cone for \$4.25. Last Saturday, the shop sold 22 small cones, 26 medium cones, and 15 large cones. How much money did the store earn?
16. The sum of angles in a triangle is 180 degrees. If the second angle is twice the size of the first angle and the third angle is three times the size of the first angle, what are the measures of the angles in the triangle?

Problem-Solving Strategies: Make a Table; Look for a Pattern

This lesson focuses on two of the strategies introduced in the previous chapter: making a table and looking for a pattern. These are the most common strategies you have used before algebra. Let's review the four-step problem-solving plan from Lesson 1.7.

Step 1: Understand the problem.

Step 2: Devise a plan – Translate. Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart, or construct a table as a start to begin your problem-solving plan.

Step 3: Carry out the plan – Solve.

Step 4: Check and Interpret: Check to see if you used all your information. Then look to see if the answer makes sense.

Using a Table to Solve a Problem

When a problem has data that needs to be organized, a table is a highly effective problem-solving strategy. A table is also helpful when the problem asks you to record a large amount of information. Patterns and numerical relationships are easier to see when data are organized in a table.

Example 1: *Josie takes up jogging. In the first week she jogs for 10 minutes per day, in the second week she jogs for 12 minutes per day. Each week, she wants to increase her jogging time by 2 minutes per day. If she jogs six days per week each week, what will be her total jogging time in the sixth week?*

Solution: Organize the information in a table

Table 1.5:

Week 1	**Week 2**	**Week 3**	**Week 4**
10 minutes **60 min/week**	**12 minutes** **72 min/week**	**14 minutes** **84 min/week**	**16 minutes** **96 min/week**

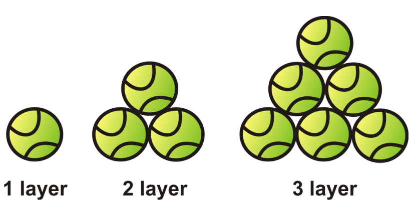
We can see the pattern that the number of minutes is increasing by 12 each week. Continuing this pattern, Josie will run 120 minutes in the sixth week.

Don't forget to check the solution! The pattern starts at 60 and adds 12 each week after the first week. The equation to represent this situation is $t = 60 + 12(w - 1)$. By substituting 6 for the variable of w, the equation becomes $t = 60 + 12(6 - 1) = 60 + 60 = 120$

Solve a Problem by Looking for a Pattern

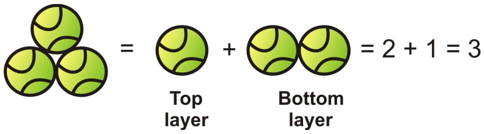
Some situations have a readily apparent pattern, which means that the pattern is easy to see. In this case, you may not need to organize the information into a table. Instead, you can use the pattern to arrive at your solution.

Example 2: *You arrange tennis balls in triangular shapes as shown. How many balls will there be in a triangle that has 8 layers?*

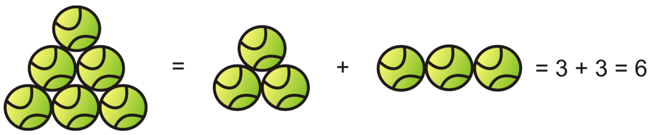


One layer: It is simple to see that a triangle with one layer has only one ball.

Two layers: For a triangle with two layers we add the balls from the top layer to the balls of the bottom layer. It is useful to make a sketch of the different layers in the triangle.



Three layers: we add the balls from the top triangle to the balls from the bottom layer.



We can fill the first three rows of the table.

1	2	3	4
1	3	6	$6 + 4 = 10$

To find the number of tennis balls in 8 layers, continue the pattern.

5	6	7	8
$10 + 5 = 15$	$15 + 6 = 21$	$21 + 7 = 28$	$28 + 8 = 36$

There will be 36 tennis balls in the 8 layers.

Check: Each layer of the triangle has one more ball than the previous one. In a triangle with 8 layers, each layer has the smae number of balls as its position. When we add these we get:

$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$ balls

The answer checks out.

Comparing Alternative Approaches to Solving Problems

In this section, we will compare the methods of ";Making a Table"; and ";Looking for a Pattern"; by using each method in turn to solve a problem.

Example 3: *Andrew cashes a \$180 check and wants the money in \$10 and \$20 bills. The bank teller gives him 12 bills. How many of each kind of bill does he receive?*

Solution: Method 1: Making a Table

Tens	0	2	4	6	8	10	12	14	16	18
Twenties	9	8	7	6	5	4	3	2	1	0

The combination that has a sum of 12 is six \$10 bills and six \$20 bills.

Method 2: Using a Pattern

The pattern is that for every pair of \$10 bills, the number of \$20 bills reduces by one. Begin with the most number of \$20 bills. For every \$20 bill lost, add two \$10 bills.

$$6(\$10) + 6(\$20) = \$180$$

Check: Six \$10 bills and six \$20 bills $= 6(\$10) + 6(\$20) = \$60 + \$120 = \$180$.

Using These Strategies to Solve Problems

Example 4: *Students are going to march in a homecoming parade. There will be one kindergartener, two first-graders, three second-graders, and so on through 12^{th} grade. How many students will be walking in the homecoming parade?*

Could you make a table? Absolutely. Could you look for a pattern? Absolutely.

Solution 1: Make a table:

1	2	3	4	5	6	7	8	9	10	11	12	
1	2	3	4	5	6	7	8	9	10	11	12	13

The solution is the sum of all the numbers, 91. There will be 91 students walking in the homecoming parade.

Solution 2: Look for a pattern.

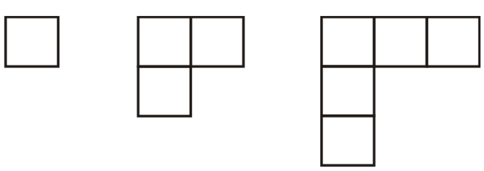
The pattern is: The number of students is one more than their grade level. Therefore, the solution is the sum of numbers from 1 (kindergarten) through 13 (12^{th} grade). The solution is 91.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra: Word Problem-Solving Strategies (12:51)

Figure 1.9: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/722

1. Go back and find the solution to the problem in Example 1.
2. Britt has $2.25 in nickels and dimes. If she has 40 coins in total how many of each coin does she have?
3. A pattern of squares is placed together as shown. How many squares are in the 12^{th} diagram?



4. Oswald is trying to cut down on drinking coffee. His goal is to cut down to 6 cups per week. If he starts with 24 cups the first week, cuts down to 21 cups the second week, and drops to 18 cups the third week, how many weeks will it take him to reach his goal?
5. Taylor checked out a book from the library and it is now 5 days late. The late fee is 10 cents per day. How much is the fine?
6. How many hours will a car traveling at 75 miles per hour take to catch up to a car traveling at 55 miles per hour if the slower car starts two hours before the faster car?

7. Grace starts biking at 12 miles per hour. One hour later, Dan starts biking at 15 miles per hour, following the same route. How long would it take him to catch up with Grace?
8. Lemuel wants to enclose a rectangular plot of land with a fence. He has 24 feet of fencing. What is the largest possible area that he could enclose with the fence?

Problem-Solving Strategies: Use a Linear Model

This chapter has focused on writing equations and determining lines of best fit. When we fit a line to data using interpolation, extrapolation, or linear regression, it is called **linear modeling.**

A **model** is an equation that best describes the data graphed in the scatter plot.

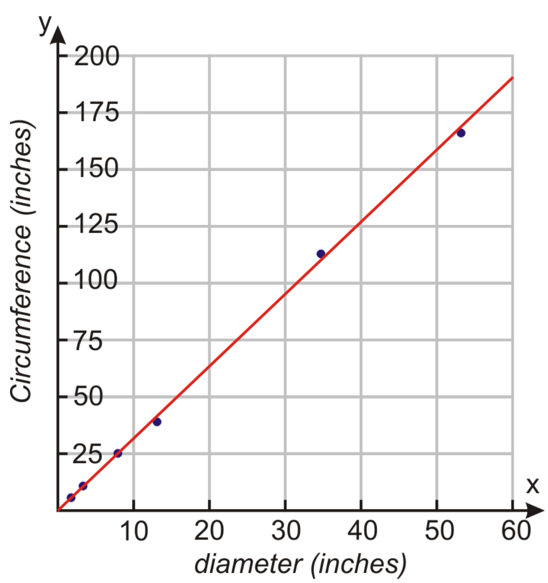
Example 1: *Dana heard something very interesting at school. Her teacher told her that if you divide the circumference of a circle by its diameter you always get the same number. She tested this statement by measuring the circumference and diameter of several circular objects. The following table shows her results.*

From this data, estimate the circumference of a circle whose diameter is 12 inches. What about 25 inches? 60 inches?

Solution: Begin by creating a scatter plot and drawing the line of best fit.

Table 1.6: **Diameter and Circumference of Various Objects**

Object	Diameter (inches)	Circumference (inches)
Table	53	170
Soda can	2.25	7.1
Cocoa tin	4.2	12.6
Plate	8	25.5
Straw	0.25	1.2
Propane tank	13.3	39.6
Hula hoop	34.25	115



Find the equation of the line of best fit using points (0.25, 1.2) and (8, 25.5).

$$\text{Slope} \qquad m = \frac{25.5 - 12}{8 - 0.25} = \frac{24.3}{7.75} = 3.14$$

$$= 3.14x + b$$

$$1.2 = 3.14(0.25) + b \Rightarrow b = 0.42$$

$$\text{Equation} \qquad y = 3.14x + 0.42$$

$$\text{Diameter} = 12 \text{ inches} \Rightarrow y = 3.14(12) + 0.42 = \underline{38.1 \text{ inches}}$$

$$\text{Diameter} = 25 \text{ inches} \Rightarrow y = 3.14(25) + 0.42 = \underline{78.92 \text{ inches}}$$

$$\text{Diameter} = 60 \text{ inches} \Rightarrow y = 3.14(60) + 0.42 = \underline{188.82 \text{ inches}}$$

In this problem, the slope = 3.14. This number should be very familiar to you—it is the number *pi* rounded to the hundredths place. Theoretically, the circumference of a circle divided by its diameter is always the same and it equals 3.14 or π.

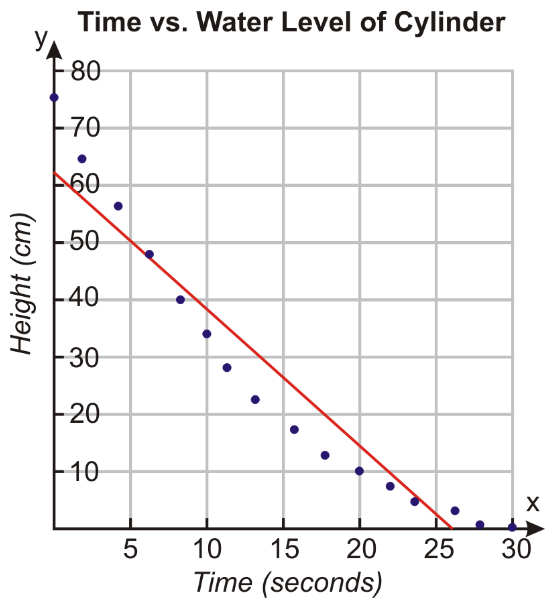
Example 2: *A cylinder is filled with water to a height of 73 centimeters. The water is drained through a hole in the bottom of the cylinder and measurements are taken at two-second intervals. The table below shows the height of the water level in the cylinder at different times.*

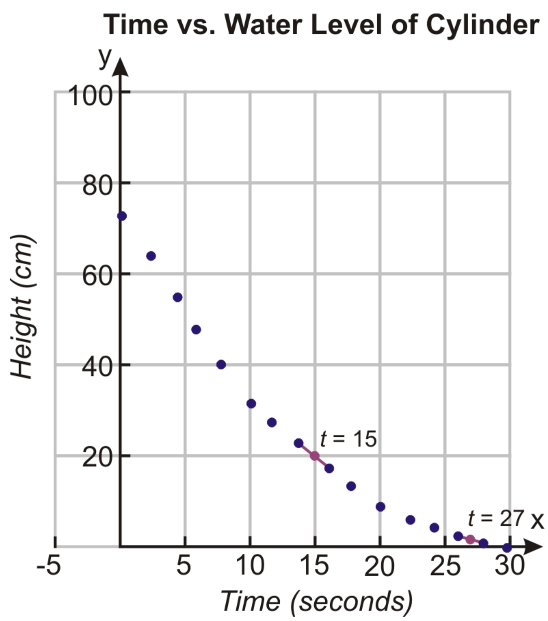
Find the water level at 15 seconds.

Solution: Begin by graphing the scatter plot. As you can see below, a straight line does not fit the majority of this data. Therefore, there is no line of best fit. Instead, use interpolation.

Table 1.7: **Water Level in Cylinder at Various Times**

Time (seconds)	**Water level (cm)**
0.0	73
2.0	63.9
4.0	55.5
6.0	47.2
8.0	40.0
10.0	33.4
12.0	27.4
14.0	21.9
16.0	17.1
18.0	12.9
20.0	9.4
22.0	6.3
24.0	3.9
26.0	2.0
28.0	0.7
30.0	0.1





To find the value at 15 seconds, connect points (14, 21.9) and (16, 17.1) and find the equation of the straight line.

$$m = \frac{17.1 - 21.9}{16 - 14} = \frac{-4.8}{2} = -2.4$$

$$y = -2.4x + b \Rightarrow 21.9 = -2.4(14) + b \Rightarrow b = 55.5$$

Equation $y = -2.4x + 55.5$

Substitute $x = 15$ and obtain $y = -2.4(15) + 55.5 = 19.5\ cm$.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra: Using a Linear Model (12:14)

Figure 1.10: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/765

1. What is a mathematical model?
2. What is *linear modeling*? What are the options to determine a linear model?
3. Using the **Water Level** data, use interpolation to determine the height of the water at 17 seconds.

Use the Life Expectancy table below to answer the questions.

4. Make a scatter plot of the data.
5. Use a line of best fit to estimate the life expectancy of a person born in 1955.
6. Use linear interpolation to estimate the life expectancy of a person born in 1955.
7. Use a line of best fit to estimate the life expectancy of a person born in 1976.
8. Use linear interpolation to estimate the life expectancy of a person born in 1976.
9. Use a line of best fit to estimate the life expectancy of a person born in 2012.
10. Use linear extrapolation to estimate the life expectancy of a person born in 2012.
11. Which method gives better estimates for this data set? Why?

Table 1.8:

Birth Year	**Life expectancy in years**
1930	59.7
1940	62.9
1950	68.2
1960	69.7
1970	70.8
1980	73.7
1990	75.4
2000	77

The table below lists the high temperature for the first day of each month in 2006 in San Diego, California (Weather Underground). Use this table to answer the questions.

12. Draw a scatter plot of the data.
13. Use a line of best fit to estimate the temperature in the middle of the 4^{th} month (month 4.5).
14. Use linear interpolation to estimate the temperature in the middle of the 4^{th} month (month 4.5).
15. Use a line of best fit to estimate the temperature for month 13 (January 2007).
16. Use linear extrapolation to estimate the temperature for month 13 (January 2007).
17. Which method gives better estimates for this data set? Why?

Table 1.9:

Month number	**Temperature** (F)
1	63

Table 1.9: (continued)

Month number	**Temperature** (F)
8	78
9	81
10	75
11	68
12	69

Solve Real-World Problems Using Linear Models

Let's apply the methods we just learned to a few application problems that can be modeled using a linear relationship.

Example 9: *Nadia has \$200 in her savings account. She gets a job that pays \$7.50 per hour and she deposits all her earnings in her savings account. Write the equation describing this problem in slope–intercept form. How many hours would Nadia need to work to have \$500 in her account?*

Solution: Begin by defining the variables:

$y =$ amount of money in Nadia's savings account

$x =$ number of hours

The problem gives the $y-$intercept and the slope of the equation.

We are told that Nadia has \$200 in her savings account, so $b = 200$.

We are told that Nadia has a job that pays \$7.50 per hour, so $m = 7.50$.

By substituting these values in slope–intercept form $y = mx + b$, we obtain $y = 7.5x + 200$.

To answer the question, substitute \$500 for the value of y and solve.

$$500 = 7.5x + 200 \Rightarrow 7.5x = 300 \Rightarrow x = 40$$

Nadia must work 40 hours if she is to have \$500 in her account.

Example 10: *A stalk of bamboo of the family Phyllostachys nigra grows at steady rate of 12 inches per day and achieves its full height of 720 inches in 60 days. Write the equation describing this problem in slope–intercept form. How tall is the bamboo 12 days after it started growing?*

Solution: Define the variables.

$y =$ the height of the bamboo plant in inches

$x =$ number of days

The problem gives the slope of the equation and a point on the line.

The bamboo grows at a rate of 12 inches per day, so $m = 12$.

We are told that the plant grows to 720 inches in 60 days, so we have the point (60, 720).

Start with the slope-intercept form of the line.	$y = mx + b$
Substitute 12 for the slope.	$y = 12x + b$
Substitute the point $(60, 720)$.	$720 = 12(60) + b \Rightarrow b = 0$
Substitute the value of b back into the equation.	$y = 12x$

To answer the question, substitute the value $x = 12$ to obtain $y = 12(12) = 144$ inches

Applying Standard Form to Real-World Situations

Example 5: *Nimitha buys fruit at her local farmer's market. This Saturday, oranges cost $2 per pound and cherries cost $3 per pound. She has $12 to spend on fruit. Write an equation in standard form that describes this situation. If she buys 4 pounds of oranges, how many pounds of cherries can she buy?*

Solution: Define the variables: $x =$ pounds of oranges and $y =$ pounds of cherries.

The equation that describes this situation is: $2x + 3y = 12$

If she buys 4 pounds of oranges, we substitute $x = 4$ in the equation and solve for y.

$2(4) + 3y = 12 \Rightarrow 3y = 12 - 8 \Rightarrow 3y = 4 \Rightarrow y = \frac{4}{3}$. Nimitha can buy $1\frac{1}{3}$ pounds of cherries.

Example 6: *Jethro skateboards part of the way to school and walks for the rest of the way. He can skateboard at 7 miles per hour and he can walk at 3 miles per hour. The distance to school is 6 miles. Write an equation in standard form that describes this situation. If Jethro skateboards for $\frac{1}{2}$ an hour, how long does he need to walk to get to school?*

Solution: Define the variables: $x =$ hours Jethro skateboards and $y =$ hours Jethro walks.

The equation that describes this situation is $7x + 3y = 6$.

If Jethro skateboards $\frac{1}{2}$ hour, we substitute $x = 0.5$ in the equation and solve for y.

$7(0.5) + 3y = 6 \Rightarrow 3y = 6 - 3.5 \Rightarrow 3y = 2.5 \Rightarrow y = \frac{5}{6}$. Jethro must walk $\frac{5}{6}$ of an hour.

Solving Situations Involving Point-Slope Form

Let's solve some word problems where point-slope form is needed.

Example 5: *Marciel rented a moving truck for the day. Marciel remembers only that the rental truck company charges $40 per day and some amount of cents per mile. Marciel drives 46 miles and the final amount of the bill (before tax) is $63. What is the amount per mile the truck rental company charges?*

Write an equation in point-slope form that describes this situation. How much would it cost to rent this truck if Marciel drove 220 miles?

Solution: Define the variables: x = distance in miles; y = cost of the rental truck in dollars. There are two ordered pairs: (0, 40) and (46, 63).

Step 1: Begin by finding the slope: $\frac{63-40}{46-0} = \frac{23}{46} = \frac{1}{2}$.

Step 2: Substitute the slope for m and one of the coordinates for (x_1, y_1).

$$y - 40 = \frac{1}{2}(x - 0)$$

To find out how much will it cost to rent the truck for 220 miles, substitute 220 for the variable x.

$$y - 40 = \frac{1}{2}(220 - 0)$$
$$y - 40 = 0.5(220) \Rightarrow y = \$150$$

Fitting a Line to Data

The real-world situations you have been studying so far form linear equations. However, most data in life is messy and does not fit a line in slope-intercept form with 100% accuracy. Because of this tendency, people spend their entire career attempting to fit lines to data. The equations that are created to fit the data are used to make predictions, as you will see in the next lesson.

This lesson focuses on graphing **scatter plots** and using the scatter plot to find a linear equation that will best fit the data.

A **scatter plot** is a plot of all the ordered pairs in the table. This means that a scatter plot is a relation, and not necessarily a function. Also, the scatter plot is **discrete**, as it is a set of distinct points. Even when we expect the relationship we are analyzing to be linear, we should not expect that all the points would fit perfectly on a straight line. Rather, the points will be ";scattered"; about a straight line. There are many reasons why the data does not fall perfectly on a line. Such reasons include **measurement errors** and **outliers**.

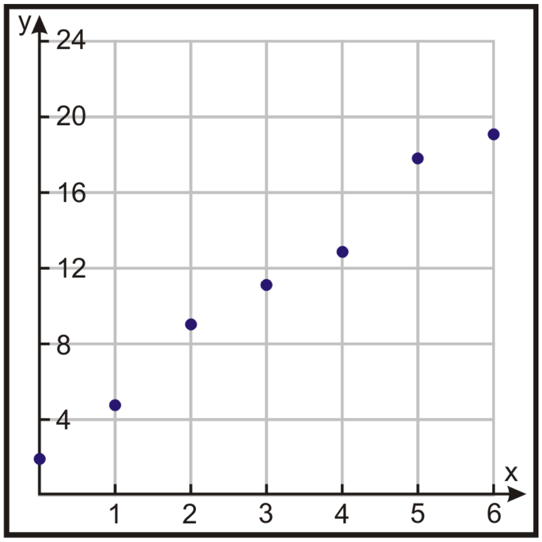
Measurement error is the amount you are off by reading a ruler or graph.

An **outlier** is a data point that does not fit with the general pattern of the data. It tends to be ";outside"; the majority of the scatter plot.

Example: *Make a scatter plot of the following ordered pairs.*

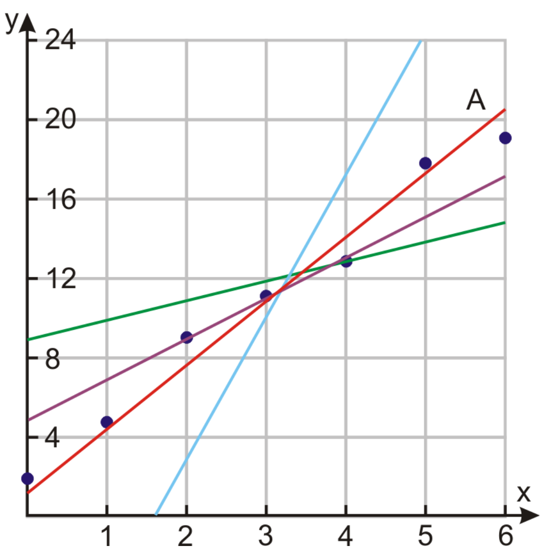
(0, 2), (1, 4.5), (2, 9), (3, 11), (4, 13), (5, 18), (6, 19.5)

Solution: Graph each ordered pair on one Cartesian plane.



Fitting a Line to Data

Notice that the points graphed on the plane above look like they might be part of a straight line, although they would not fit perfectly. If the points were perfectly lined up, it would be quite easy to draw a line through all of them and find the equation of that line. However, if the points are ";scattered,"; we try to find a line that best fits the data. The graph below shows several potential **lines of best fit**.



You see that we can draw many lines through the points in our data set. These lines have equations that are very different from each other. We want to use the line that is closest to **all** the points on the graph. The best candidate in our graph is the red line A. Line A is the **line of best fit** for this scatter plot.

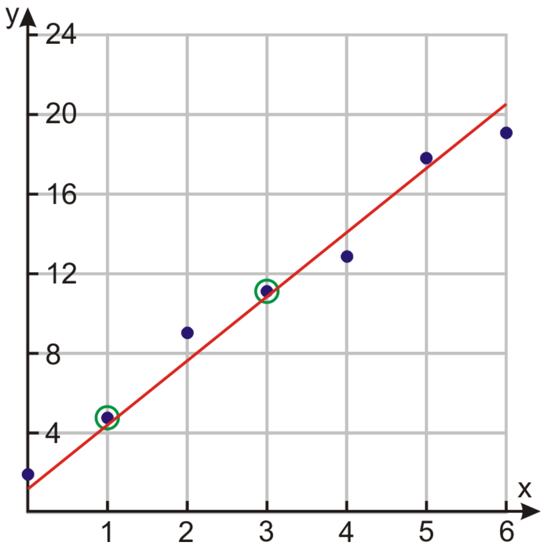
Writing Equations for Lines of Best Fit

Once you have decided upon your line of best fit, you need to write its equation by finding two points on it and using either:

- Point-slope form;
- Standard form; or
- Slope-intercept form.

The form you use will depend upon the situation and the ease of finding the $y-$intercept.

Using the red line from the example above, locate two points on the line.



Find the slope: $m = \frac{11-4.5}{3-1} = \frac{6.5}{2} = 3.25$.

Then $y = 3.25x + b$.

Substitute (3, 11) into the equation. $11 = 3.25(3) + b \Rightarrow b = 1.25$

The equation for the line that fits the data best is $y = 3.25x + 1.25$.

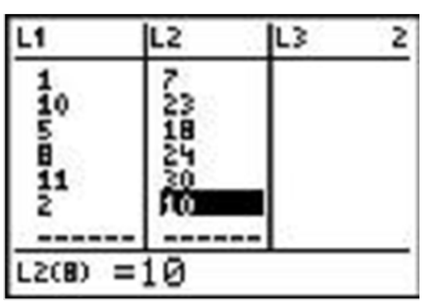
Finding Equations for Lines of Best Fit Using a Calculator

Graphing calculators can make writing equations of best fit easier and more accurate. Two people working with the same data might get two different equations because they would be drawing different lines. To get the most accurate equation for the line, we can use a graphing calculator. The calculator uses a mathematical algorithm to find the line that minimizes error between the data points and the line of best fit.

Example: *Use a graphing calculator to find the equation of the line of best fit for the following data:* (3, 12), (8, 20), (1, 7), (10, 23), (5, 18), (8, 24), (11, 30), (2, 10).

Solution:

Step 1: *Input the data in your calculator.* Press [**STAT**] and choose the [**EDIT**] option.



L1	L2	L3 2
1	7	
10	23	
5	18	
8	24	
11	30	
2	10	
------	------	

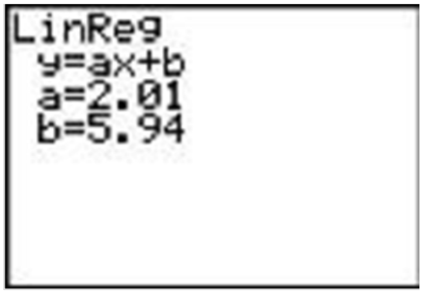
L2(8) =10

Input the data into the table by entering the x values in the first column and the y values in the second column.

Step 2: *Find the equation of the line of best fit.*

Press **[STAT]** again and use the right arrow to select **[CALC]** at the top of the screen.

Chose option number 4: *LinReg*$(ax+b)$ and press **[ENTER]**. The calculator will display *LinReg*$(ax+b)$.



Press **[ENTER]** and you will be given the a and b values.

Here a represents the slope and b represents the $y-$intercept of the equation. The linear regression line is $y = 2.01x + 5.94$.

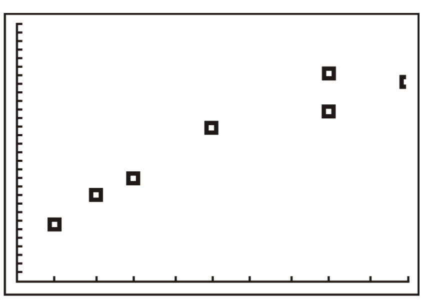
Step 3: *Draw the scatter plot.*

To draw the scatter plot press **[STATPLOT]** **[2nd]** **[Y=]**.

Choose Plot 1 and press **[ENTER]**.

Press the **On** option and choose the Type as scatter plot (the one highlighted in black).

Make sure that the X list and Y list names match the names of the columns of the table in Step 1.



Choose the box or plus as the mark since the simple dot may make it difficult to see the points.

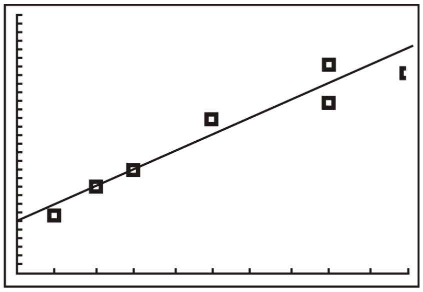
Press [**GRAPH**] and adjust the window size so you can see all the points in the scatter plot.

Step 4: *Draw the line of best fit through the scatter plot.*

Press [**Y=**].

Enter the equation of the line of best fit that you just found: $Y_1 = 2.01X + 5.94$.

Press [**GRAPH**].



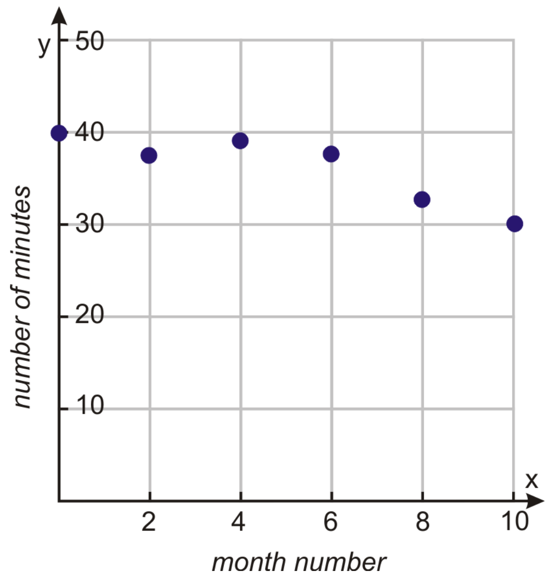
Using Lines of Best Fit to Solve Situations

Example: *Gal is training for a 5K race (a total of 5000 meters, or about 3.1 miles). The following table shows her times for each month of her training program. Assume here that her times will decrease in a straight line with time. Find an equation of a line of fit. Predict her running time if her race is in August.*

Table 1.10:

Month	**Month number**	**Average time (minutes)**
January	0	40
February	1	38
March	2	39
April	3	38
May	4	33
June	5	30

Solution: Begin by making a scatter plot of Gal's running times. The independent variable, x, is the month number and the dependent variable, y, is the running time in minutes. Plot all the points in the table on the coordinate plane.



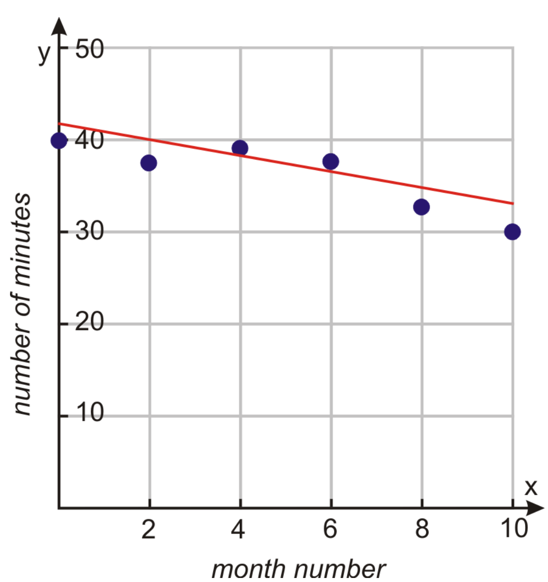
Draw a line of fit.

Choose two points on the line (0, 41) and (4, 34).

Find the equation of the line.

$$\begin{aligned}
m &= \frac{34-41}{4-0} = -\frac{7}{4} = -1\frac{3}{4} \\
y &= -\frac{7}{4}x + b \\
41 &= -\frac{7}{4}(0) + b \Rightarrow b = 41 \\
y &= -\frac{7}{4}x + 41
\end{aligned}$$

In a real-world problem, the slope and y–intercept have a physical significance.



$$\text{Slope} = \frac{number\ of\ minutes}{month}$$

Since the slope is negative, the number of minutes Gal spends running a 5K race decreases as the months pass. The slope tells us that Gal's running time decreases 1.75 minutes per month.

The $y-$intercept tells us that when Gal started training, she ran a distance of 5K in 41 minutes, which is just an estimate, since the actual time was 40 minutes.

The problem asks us to predict Gal's running time in August. Since June is assigned to month number five, then August will be month number seven. Substitute $x = 7$ into the line of best fit equation.

$$y = -\frac{7}{4}(7) + 41 = -\frac{49}{4} + 41 = -\frac{49}{4} + \frac{164}{4} = \frac{115}{4} = 28\frac{3}{4}$$

The equation predicts that Gal will be running the 5K race in 28.75 minutes.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Fitting a Line to Data (7:48)

Figure 1.11: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/762

1. To buy a car, Andrew puts in a down payment of \$1500 and pays \$350 per month in installments. Write an equation describing this problem in slope-intercept form. How much money has Andrew paid at the end of one year?
2. Anne transplants a rose seedling in her garden. She wants to track the growth of the rose, so she measures its height every week. In the third week, she finds that the rose is 10 inches tall and in the eleventh week she finds that the rose is 14 inches tall. Assuming the rose grows linearly with time, write an equation describing this problem in slope-intercept form. What was the height of the rose when Anne planted it?
3. Ravi hangs from a giant exercise spring whose length is 5 m. When his child Nimi hangs from the spring, its length is 2 m. Ravi weighs 160 lbs. and Nimi weighs 40 lbs. Write the equation for this problem in slope-intercept form. What should we expect the length of the spring to be when his wife Amardeep, who weighs 140 lbs., hangs from it?
4. Petra is testing a bungee cord. She ties one end of the bungee cord to the top of a bridge and to the other end she ties different weights. She then measures how far the bungee stretches. She finds that for a weight of 100 lbs., the bungee stretches to 265 feet and for a weight of 120 lbs., the bungee stretches to 275 feet. Physics tells us that in a certain range of values, including the ones given here, the amount of stretch is a linear function of the weight. Write the equation describing this problem in slope–intercept form. What should we expect the stretched length of the cord to be for a weight of 150 lbs?

5. The farmer's market sells tomatoes and corn. Tomatoes are selling for \$1.29 per pound and corn is selling for \$3.25 per pound. If you buy 6 pounds of tomatoes, how many pounds of corn can you buy if your total spending cash is \$11.61?
6. The local church is hosting a Friday night fish fry for Lent. They sell a fried fish dinner for \$7.50 and a baked fish dinner for \$8.25. The church sold 130 fried fish dinners and took in \$2,336.25. How many baked fish dinners were sold?
7. Andrew has two part-time jobs. One pays \$6 per hour and the other pays \$10 per hour. He wants to make \$366 per week. Write an equation in standard form that describes this situation. If he is only allowed to work 15 hours per week at the \$10 per hour job, how many hours does he need to work per week at his \$6 per hour job in order to achieve his goal?
8. Anne invests money in two accounts. One account returns 5% annual interest and the other returns 7% annual interest. In order not to incur a tax penalty, she can make no more than \$400 in interest per year. Write an equation in standard form that describes this problem. If she invests \$5000 in the 5% interest account, how much money does she need to invest in the other account?
9. What is a *scatter plot*? How is this different from other graphs you have created?
10. Define *line of best fit*.
11. What is an *outlier*? How can an outlier be spotted on a graph?
12. What are the two methods of finding a line of best fit?
13. Explain the steps needed to find a line of best fit ";by hand."; What are some problems with using this method?

For each data set, draw the scatter plot and find the equation of the line of best fit by hand.

6. (57, 45) (65, 61) (34, 30) (87, 78) (42, 41) (35, 36) (59, 35) (61, 57) (25, 23) (35, 34)
7. (32, 43) (54, 61) (89, 94) (25, 34) (43, 56) (58, 67) (38, 46) (47, 56) (39, 48)
8. (12, 18) (5, 24) (15, 16) (11, 19) (9, 12) (7, 13) (6, 17) (12, 14)
9. (3, 12) (8, 20) (1, 7) (10, 23) (5, 18) (8, 24) (2, 10)

In 10 – 12, for each data set, use a graphing calculator to find the equation of the line of best fit.

10. (57, 45) (65, 61) (34, 30) (87, 78) (42, 41) (35, 36) (59, 35) (61, 57) (25, 23) (35, 34)
11. (32, 43) (54, 61) (89, 94) (25, 34) (43, 56) (58, 67) (38, 46) (47, 56) (95, 105) (39, 48)
12. (12, 18) (3, 26) (5, 24) (15, 16) (11, 19) (0, 27) (9, 12) (7, 13) (6, 17) (12, 14)
13. Shiva is trying to beat the samosa eating record. The current record is 53.5 samosas in 12 minutes. The following table shows how many samosas he eats during his daily practice for the first week of his training. Will he be ready for the contest if it occurs two weeks from the day he started training? What are the meanings of the slope and the y–intercept in this problem?

Table 1.11:

Day	**No. of Samosas**
1	30
2	34
3	36
4	36
5	40
6	43
7	45

14. Nitisha is trying to find the elasticity coefficient of a Superball. She drops the ball from different heights and measures the maximum height of the resulting bounce. The table below shows her data. Draw a scatter plot and find the equation. What is the initial height if the bounce height is 65 cm? What are the meanings of the slope and the $y-$intercept in this problem?

Table 1.12:

Initial height ***(cm)***	**Bounce height** ***(cm)***
30	22
35	26
40	29
45	34
50	38
55	40
60	45
65	50
70	52

15. Baris is testing the burning time of ";BriteGlo"; candles. The following table shows how long it takes to burn candles of different weights. Let's assume it's a linear relation. We can then use a line to fit the data. If a candle burns for 95 hours, what must be its weight in ounces?

Table 1.13: **Candle Burning Time Based on Candle Weight**

Candle weight (oz)	**Time (hours)**
2	15
3	20
4	35
5	36
10	80
16	100
22	120
26	180

16. The table below shows the median California family income from 1995 to 2002 as reported by the U.S. Census Bureau. Draw a scatter plot and find the equation. What would you expect the median annual income of a Californian family to be in year 2010? What are the meanings of the slope and the $y-$intercept in this problem?

Table 1.14:

Year	**Income**
1995	53,807
1996	55,217
1997	55,209
1998	55,415

Table 1.14: (continued)

Year	Income
1999	63,100
2000	63,206
2001	63,761
2002	65,766

Predicting with Linear Models

Numerical information appears in all areas of life. You can find it in newspapers, in magazines, in journals, on the television, or on the Internet. In the last lesson, you saw how to find the equation of a line of best fit. Using a line of best fit is a good method if the relationship between the dependent and independent variables is linear. Not all data fits a straight line, though. This lesson will show other methods to help estimate data values. These methods are useful in both linear and non-linear relationships.

Linear Interpolation

Linear interpolation is useful when looking for a value between given data points. It can be considered as ";filling in the gaps"; of a table of data.

The strategy for linear interpolation is to use a straight line to connect the known data points on either side of the unknown point. Linear interpolation is often not accurate for non-linear data. If the points in the data set change by a large amount, linear interpolation may not give a good estimate.

Linear Extrapolation

Linear extrapolation can help us estimate values that are either higher or lower than the values in the data set. Think of this as ";the long-term estimate"; of the data.

The strategy for linear extrapolation is to use a subset of the data instead of the entire data set. This is especially true for non-linear data you will encounter in later chapters. For this type of data, it is sometimes useful to extrapolate using the last two or three data points in order to estimate a value higher than the data range.

Collecting and Organizing Data

Data can be collected through various means, including surveys or experiments.

A **survey** is a data collection method used to gather information about individuals' opinions, beliefs, or habits.

The information collected by the U.S. Census Bureau or the Center for Disease Control are examples of data gathered using surveys. The U.S. Census Bureau collects information about many aspects of the U.S. population.

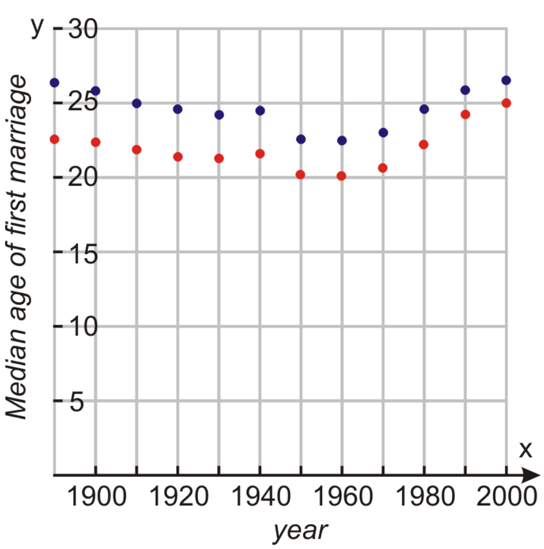
An **experiment** is a controlled test or investigation.

Let's say we are interested in how the median age for first marriages has changed during the 20^{th} century. The U.S. Census provides the following information about the median age at first marriage for males and females. Below is the table of data and its corresponding scatter plot.

Table 1.15:

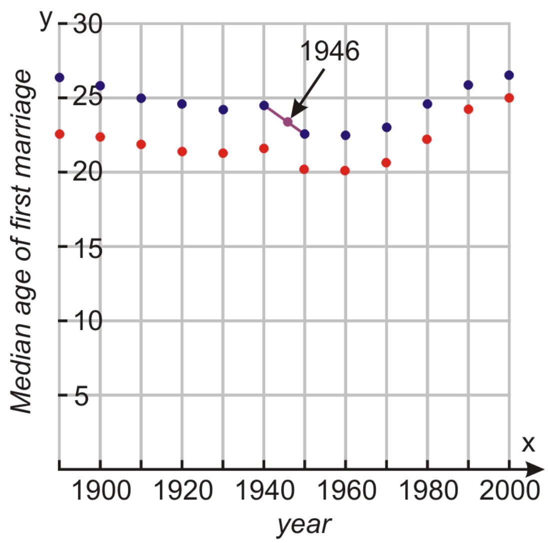
Year	Median Age of Males	Median Age of Females
1890	26.1	22.0
1900	25.9	21.9
1910	25.1	21.6
1920	24.6	21.2
1930	24.3	21.3
1940	24.3	21.5
1950	22.8	20.3
1960	22.8	20.3
1970	23.2	20.8
1980	24.7	22.0
1990	26.1	23.9
2000	26.8	25.1

Median Age of Males and Females at First Marriage by Year



Example: *Estimate the median age for the first marriage of a male in the year 1946.*

Solution: We will first use the method of interpolation because there is a ";gap"; needing to be filled. 1946 is between 1940 and 1950, so these are the data points we will use.



By connecting the two points, an equation can be found.

$$\text{Slope} \qquad m = \frac{22.8 - 24.3}{1950 - 1940} = \frac{-1.5}{10} = -0.15$$

$$y = -0.15x + b$$

$$24.3 = -0.15(1940) + b$$

$$b = 315.3$$

$$\text{Equation} \qquad y = -0.15x + 315.3$$

To estimate the median age of marriage of males in year 1946, substitute $x = 1946$ in the equation.

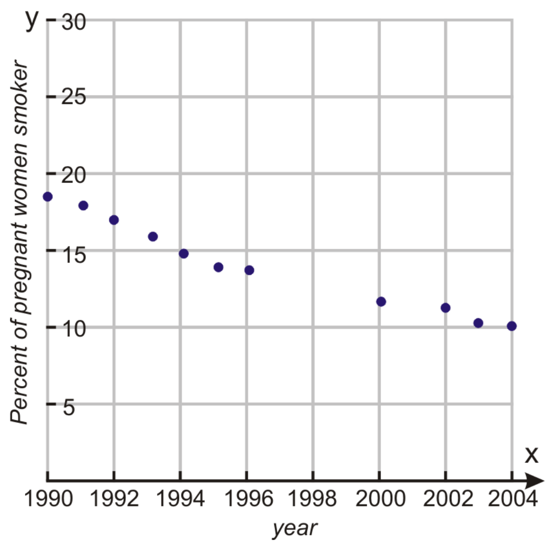
$$y = -0.15(1946) + 315.3 = 23.4 \textit{ years old}$$

Example: *The Center for Disease Control (CDC) has the following information regarding the percentage of pregnant women smokers organized by year. Estimate the percentage of pregnant women that were smoking in the year 1998.*

Table 1.16: **Percent of Pregnant Women Smokers by Year**

Year	Percent
1990	18.4
1991	17.7
1992	16.9
1993	15.8
1994	14.6
1995	13.9
1996	13.6
2000	12.2
2002	11.4
2003	10.4
2004	10.2

Percent of Pregnant Women Smokers by Year



Solution: We want to use the information close to 1998 to **interpolate** the data. We do this by connecting the points on either side of 1998 with a straight line and find the equation of that line.

$$\begin{aligned}
\text{Slope} \qquad & m = \frac{12.2 - 13.6}{2000 - 1996} = \frac{-1.4}{4} = -0.35 \\
& y = -0.35x + b \\
& 12.2 = -0.35(2000) + b \\
& b = 712.2 \\
\text{Equation} \qquad & y = -0.35x + 712.2
\end{aligned}$$

To estimate the percentage of pregnant women who smoked in year 1998, substitute $x = 1998$ into the equation.

$$y = -0.35(1998) + 712.2 = 12.9\%$$

Predicting Using an Equation

When linear interpolation and linear extrapolation do not produce accurate predictions, using the line of best fit (**linear regression**) may be the best choice. The ";by hand"; and calculator methods of determining the line of best fit were presented in the last lesson.

Example: The winning times for the women's 100-meter race are given in the following table. Estimate the winning time in the year 2010. Is this a good estimate?

Table 1.17:

Winner	**Ctry.**	**Year**	**Seconds**	**Winner**	**Ctry.**	**Year**	**Seconds**
Mary Lines	UK	1922	12.8	Vera Krepkina	Sov.	1958	11.3
Leni Schmidt	Germ.	1925	12.4	Wyomia Tyus	USA	1964	11.2
Gertrurd Glasitsch	Germ.	1927	12.1	Barbara Ferrell	USA	1968	11.1
Tollien Schuurman	Neth.	1930	12.0	Ellen Strophal	E. Germ.	1972	11.0

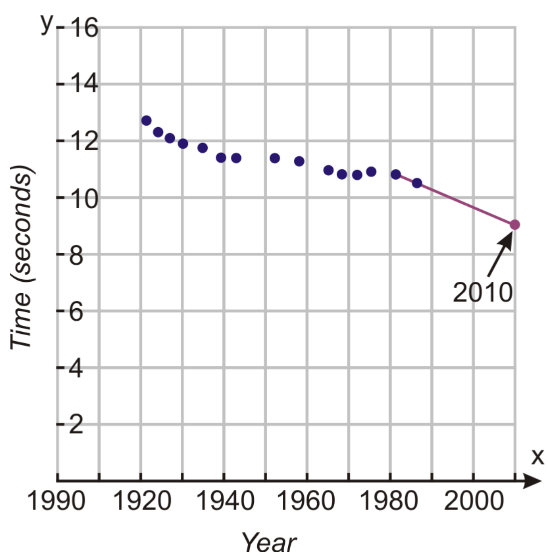
Table 1.17: (continued)

Winner	Ctry.	Year	Seconds	Winner	Ctry.	Year	Seconds
Helen Stephens	USA	1935	11.8	Inge Helten	W. Germ.	1975	11.0
Lulu Mae Hymes	USA	1939	11.5	Marlies Gohr	E. Germ.	1982	10.9
Fanny Blankers-Koen	Neth.	1943	11.5	Florence Griffith Joyner	USA	1988	10.5
Marjorie Jackson	Austr.	1952	11.4				

Solution: Start by making a scatter plot of the data. Connect the last two points on the graph and find the equation of the line.

Winning Times for the Women's 100-meter Race by Year

[3]Source: http://en.wikipedia.org/wiki/World_Record_progression_100_m_women.



$$\text{Slope} \qquad m = \frac{10.5 - 10.9}{1988 - 1982} = \frac{-0.4}{6} = -0.067$$

$$y = -0.067x + b$$

$$10.5 = -0.067(1988) + b$$

$$b = 143.7$$

$$\text{Equation} \qquad y = -0.067x + 143.7$$

The winning time in year 2010 is estimated to be: $y = -0.067(2010) + 143.7 = \underline{9.03 \text{ seconds}}$.

How accurate is this estimate? It is likely that it's not very accurate because 2010 is a long time from 1988. This example demonstrates the weakness of linear extrapolation. Estimates given by linear extrapolation are never as good as using the equation from the line of best fit method. In this particular example, the last data point clearly does not fit in with the general trend of the data so the slope of the extrapolation line is much steeper than it should be.

As a historical note, the last data point corresponds to the winning time for Florence Griffith Joyner in 1988. After her race, she was accused of using performance-enhancing drugs but this fact was never proven.

In addition, there is a question about the accuracy of the timing because some officials said that the tail wind was not accounted for in this race even though all the other races of the day were impacted by a strong wind.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra: Predicting with Linear Models (11:46)

Figure 1.12: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/764

1. What does it mean to *interpolate the data*? In which cases would this method be useful?
2. How is interpolation different from extrapolation? In which cases would extrapolation be more beneficial?
3. What was the problem with using the interpolation method to come up with an equation for the women's Olympic winning times?
4. Use the **Winning Times** data and determine an equation for the line of best fit.
5. Use the **Median Age at First Marriage** data to estimate the age at marriage for females in 1946. Fit a line, by hand, to the data before 1970.
6. Use the **Median Age at First Marriage** data to estimate the age at marriage for females in 1984. Fit a line, by hand, to the data from 1970 on in order to estimate this accurately.
7. Use the **Median Age at First Marriage** data to estimate the age at marriage for males in 1995. Use linear interpolation between the 1990 and 2000 data points.
8. Use the data from **Pregnant Women and Smoking** to estimate the percent of pregnant smokers in 1997. Use linear interpolation between the 1996 and 2000 data points.
9. Use the data from **Pregnant Women and Smoking** to estimate the percent of pregnant smokers in 2006. Use linear extrapolation with the final two data points.
10. Use the **Winning Times** data to estimate the winning time for the female 100-meter race in 1920. Use linear extrapolation because the first two or three data points have a different slope than the rest of the data.
11. The table below shows the highest temperature vs. the hours of daylight for the 15^{th} day of each month in the year 2006 in San Diego, California. Using linear interpolation, estimate the high temperature for a day with 13.2 hours of daylight.

Table 1.18:

Hours of daylight	**High temperature** (F)
10.25	60
11.0	62

Table 1.18: (continued)

Hours of daylight	**High temperature** (F)
12	62
13	66
13.8	68
14.3	73
14	86
13.4	75
12.4	71
11.4	66
10.5	73
10	61

12. Use the table above to estimate the high temperature for a day with 9 hours of daylight using linear extrapolation. Is the prediction accurate? Find the answer using line of best fit.

Chapter Review

Define the following words:

1. Domain
2. Range
3. Solution
4. Evaluate
5. Substitute
6. Operation
7. Variable
8. Algebraic expression
9. Equation
10. Algebraic inequality
11. Function
12. Independent variable

Find an equation of the line in slope-intercept form using the given information.

1. (3, 4) with $slope = \frac{2}{3}$
2. $slope = -5$, $y - intercept = 9$
3. $slope = -1$ containing (6, 0)
4. containing (3.5, 1) and (9, 6)
5. $slope = 3$, $y{-}intercept = -1$
6. $slope = \frac{-1}{3}$ containing (–3, –4)
7. containing (0, 0) and (9, –8)
8. $slope = \frac{5}{3}$, $y{-}intercept = 6$
9. containing (5, 2) and (–6, –3)
10. $slope = 3$ and $f(6) = 1$
11. $f(2) = -5$ and $f(-6) = 3$
12. $slope = \frac{3}{8}$ and $f(1) = 1$

Find an equation of the line in point-slope form using the given information.

13. $slope = m$ containing (x_1, y_1)
14. $slope = \frac{1}{2}$ containing (-7, 5)
15. $slope = 2$ containing (7, 0)

Graph the following equations.

16. $y + 3 = -(x - 2)$
17. $y - 7 = \frac{-2}{3}(x + 5)$
18. $y + 1.5 = \frac{3}{2}(x + 4)$

Find the equation of the line represented by the function below in point-slope form.

19. $f(1) = -3$ and $f(6) = 0$
20. $f(9) = 2$ and $f(9) = -5$
21. $f(2) = 0$ and $slope = \frac{8}{3}$

Write the standard form of the equation of each line.

23. $y - 3 = \frac{-1}{4}(x + 4)$
24. $y = \frac{2}{7}(x - 21)$
25. $-3x - 25 = 5y$

Write the standard form of the line for each equation using the given information.

25. containing (0, –4) and (–1, 5)
26. $slope = \frac{4}{3}$ containing (3, 2)
27. $slope = 5$ containing (5, 0)
28. Find the slope and $y-$intercept of $7x + 5y = 16$.
29. Find the slope and $y-$intercept of $7x - 7y = -14$.
30. Are $\frac{1}{2}x + \frac{1}{2}y = 5$ and $2x + 2y = 3$ parallel, perpendicular, or neither?
31. Are $x = 4$ and $y = -2$ parallel, perpendicular, or neither?
32. Are $2x + 8y = 26$ and $x + 4y = 13$ parallel, perpendicular, or neither?
33. Write an equation for the line perpendicular to $y = 3x + 4$ containing (–5, 1).
34. Write an equation for the line parallel to $y = x + 5$ containing (–4, –4).
35. Write an equation for the line perpendicular to $9x + 5y = 25$ containing (–4, 4).
36. Write an equation for the line parallel to $y = 5$ containing (–7, 16).
37. Write an equation for the line parallel to $x = 0$ containing (4, 6).
38. Write an equation for the line perpendicular to $y = -2$ containing (10, 10).
39. An Internet café charges \$6.00 to use 65 minutes of their Wifi. It charges \$8.25 to use 100 minutes. Suppose the relationship is linear.
    (a) Write an equation to model this data in point-slope form.
    (b) What is the price to acquire the IP address?
    (c) How much does the café charge per minute?
40. A tomato plant grows $\frac{1}{2}$ inch per week. The plant was 5 inches tall when planted.
    (a) Write an equation in slope-intercept form to represent this situation.
    (b) How many weeks will it take the plant to reach 18 inches tall?
41. Joshua bought a television and paid 6% sales tax. He then bought an albino snake and paid 4.5% sales tax. His combined purchases totaled \$679.25.

(a) Write an equation to represent Joshua's purchases.
(b) Graph all the possible solutions to this situation.
(c) Give three examples that would be solutions to this equation.

42. Comfy Horse Restaurant began with a 5-gallon bucket of dishwashing detergent. Each day $\frac{1}{4}$ gallon is used.

(a) Write an equation to represent this situation in slope-intercept form.
(b) How long will it take to empty the bucket?

43. The data below shows the divorce rate per 1,000 people in the state of Wyoming for various years (source: Nation Masters).

(a) Graph the data in a scatter plot.
(b) Fit a line to the data by hand.
(c) Find the line of best fit by hand.
(d) Using your model, what do you predict the divorce rate is in the state of Wyoming in the year 2011?
(e) Repeat this process using your graphing calculator. How close was your line to the one the calculator provided?

Year	2000	2001	2002	2003	2004	2005	2006	2007
Rate (per 1,000 people)	5.8	5.8	5.4	5.4	5.3	5.4	5.3	5.0

44. The table below shows the percentage of voter turnout at presidential elections for various years (source The American Presidency Project).

Year	1828	1844	1884	1908	1932	1956	1972	1988	2004
% of Voter Turnout	57.6	78.9	77.5	65.4	56.9	60.6	55.21	50.15	55.27

(a) Draw a scatter plot of this data.

(b) Use the linear regression feature on your calculator to determine a line of best fit and draw it on your graph.

(c) Use the line of best fit to predict the voter turnout for the 2008 election.

(d) What are some outliers to this data? What could be a cause for these outliers?

45. The data below shows the bacteria population in a Petri dish after h hours.

hours	0	1	2	3	4	5	6
Bacteria present	100	200	400	800	1600	3200	6400

(a) Use the method of interpolation to find the number of bacteria present after 4.25 hours.

(b) Use the method of extrapolation to find the number of bacteria present after 10 hours.

(c) Could this data be best modeled with a linear equation? Explain your answer.

Chapter Test

1. Write $y = \frac{-3}{2}x + 4$ in standard form.
2. Write an equation in slope-intercept form for a line perpendicular to $y = \frac{1}{3}x + 6$ containing (1, 2).
3. Write an equation in point-slope form for a line containing (5, 3) and (–6, 0.5).

4. What is the speed of a car travelling 80 *miles/hour* in *feet/second*?
5. How many kilometers are in a marathon (26.2 miles)?
6. Lucas bought a 5-gallon container of paint. He plans to use $\frac{2}{3}$ gallon per room.
   (a) Write an equation to represent this situation.
   (b) How many rooms can Lucas paint before the container is empty?

7. Are these two lines parallel, perpendicular, or neither? Explain your answer by showing your work: $y = 3x - 1$ and $-x + 3y = 6$.
8. The table below gives the gross public debt of the U.S. Treasury for the years 2004–2007.

Year	2004	2005	2006	2007
Debt (in billions \$)	$7,596.1$	$8,170.4$	$8,680.2$	$9,229.2$

   (a) Make a scatter plot of the data.
   (b) Use the method of extrapolation to determine the gross public debt for 2009.
   (c) Find a linear regression line using a graphing calculator.
   (d) Use the equation found in (c) to determine the gross public debt for 2009.
   (e) Which answer seems more accurate, the linear model or the extrapolation?

9. What is the process used to interpolate data?
10. Use the table below to answer the following questions.

Hours (h)	0	1	2	3	4
Percentage of mineral remaining	100	50	25	12.5	6.25

   (a) Draw a scatter plot to represent the data.
   (b) Would a linear regression line be the best way to represent the data?
   (c) Use the method of interpolation to find the percentage of mineral remaining when $h = 2.75$.

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9615.

Chapter 2

Operations on Polynomials and Factoring

2.1 Operations on Polynomials and Factoring

This chapter will present a new type of function: the **polynomial.** Chances are, polynomials will be new to you. However, polynomials are used in many careers and real life situations - to model the population of a city over a century, to predict the price of gasoline, and to predict the volume of a solid. This chapter will also present basic **factoring** - breaking a polynomial into its linear factors. This will help you solve many quadratic equations found in Chapter 10.

2.2 Addition and Subtraction of Polynomials

So far we have discussed linear functions and exponential functions. This lesson introduces polynomial functions.

Definition: A **polynomial** is an expression made with constants, variables, and *positive integer* exponents of the variables.

An example of a polynomial is: $4x^3 + 2x^2 - 3x + 1$. There are four terms: $4x^3$, $2x^2$, $3x$, and 1. The numbers appearing in each term in front of the variable are called the **coefficients.** 4, 2, and 3 are coefficients because those numbers are in front of a variable. The number appearing all by itself without a variable is called a **constant.** 1 is the constant because it is by itself.

Example 1: *Identify the following expressions as polynomials or non-polynomials.*

(a) $5x^2 - 2x$

(b) $3x^2 - 2x^{-2}$

(c) $x\sqrt{x}-1$

(d) $\frac{5}{x^3+1}$

(e) $4x^{\frac{1}{3}}$

(f) $4xy^2-2x^2y-3+y^3-3x^3$

Solution:

(a) $5x^2-2x$ This ***is*** a polynomial.

(b) $3x^2-2x^{-2}$ This is ***not*** a polynomial because it has a negative exponent.

(c) $x\sqrt{x}-1$ This is ***not*** a polynomial because is has a square root.

(d) $\frac{5}{x^3+1}$ This is ***not*** a polynomial because the power of x appears in the denominator.

(e) $4x^{\frac{1}{3}}$ This is ***not*** a polynomial because it has a fractional exponent.

(f) $4xy^2-2x^y-3+y^3-3x^3$ This ***is*** a polynomial.

Classifying Polynomials by Degree

The **degree** of a polynomial is the largest exponent of a single term.

- $4x^3$ has a degree of 3 and is called a **cubic term** or 3^{rd} **order term**.
- $2x^2$ has a degree of 2 and is called a **quadratic term** or 2^{nd} **order term**.
- $-3x$ has a degree of 1 and is called a **linear term** or 1^{st} **order term.**
- 1 has a degree of 0 because there is no variable.

Polynomials can have more than one variable. Here is another example of a polynomial: $t^4-6s^3t^2-12st+4s^4-5$. This is a polynomial because all exponents on the variables are positive integers. This polynomial has five terms. **Note:** *The degree of a term is the sum of the powers on each variable in the term.*

t^4 has a degree of 4, so it's a 4^{th} order term.

$-6s^3t^2$ has a degree of 5, so it's a 5^{th} order term.

-12^{st} has a degree of 2, so it's a 2^{nd} order term.

$4s^4$ has a degree of 4, so it's a 4^{th} order term.

–5 is a constant, so its degree is 0.

Since the highest degree of a term in this polynomial is 5, this is a polynomial of degree 5 or a 5^{th} order polynomial.

Example 2: *Identify the coefficient on each term, the degree of each term, and the degree of the polynomial.*

$$x^4-3x^3y^2+8x-12$$

Solution: The coefficients of each term in order are 1, –3, 8 and the constant is –12.

The degrees of each term are 4, 5, 1, and 0. Therefore, the degree of the polynomial is 5.

A **monomial** is a one-termed polynomial. It can be a constant, a variable, or a combination of constants and variables. Examples of monomials are: $b^2;\ 6;\ -2ab^2;\ \frac{1}{4}x^2$

Rewriting Polynomials in Standard Form

Often, we arrange the terms in a polynomial in **standard from** in which the term with the highest degree is first and is followed by the other terms in order of decreasing power. The first term of a polynomial in this form is called the *leading term*, and the coefficient in this term is called the *leading coefficient*.

Example 3: *Rearrange the terms in the following polynomials so that they are in standard form. Indicate the leading term and leading coefficient of each polynomial.*

(a) $7 - 3x^3 + 4x$

(b) $ab - a^3 + 2b$

Solution:

(a) $7 - 3x^3 + 4x$ is rearranged as $-3x^3 + 4x + 7$. The leading term is $-3x^3$ and the leading coefficient is –3.

(b) $ab - a^3 + 2b$ is rearranged as $-a^3 + ab + 2b$. The leading term is $-a^3$ and the leading coefficient is –1.

Simplifying Polynomials

A polynomial is simplified if it has no terms that are alike. **Like terms** are terms in the polynomial that have the same variable(s) with the same exponents, but they can have different coefficients.

$2x^2y$ and $5x^2y$ are like terms.

$6x^2y$ and $6xy^2$ are not like terms.

If we have a polynomial that has like terms, we simplify by combining them.

$$x^2 + \underline{6xy} - \underline{4xy} + y^2$$

$$\nearrow \quad \nwarrow$$

Like terms

This polynomial is simplified by combining the like terms $6xy - 4xy = 2xy$. We write the simplified polynomial as $x^2 + 2xy + y^2$.

Example 4: *Simplify by collecting and combining like terms.*

$$a^3b^3 - 5ab^4 + 2a^3b - a^3b^3 + 3ab^4 - a^2b$$

Solution: Use the Commutative Property of Addition to reorganize like terms then simplify.

$$\begin{aligned}
&= (a^3b^3 - a^3b^3) + (-5ab^4 + 3ab^4) + 2a^3b - a^2b \\
&= 0 - 2ab^4 + 2a^3b - a^2b \\
&= -2ab^4 + 2a^3b - a^2b
\end{aligned}$$

Polynomial Functions

So far, we've reviewed Polynomial Equations and looked at writing them in simplified and descending order. In the next couple of sections, we are going to look at performing operations on Polynomials and it helps if we look at each Polynomial using Function Notation.

If you remember from chapter 1:

$$\begin{array}{c} input \\ \downarrow \\ \underbrace{f(x)}_{\substack{function \\ box}} = y \leftarrow output \end{array}$$

Example 2: Rewrite the following equations in function notation.

a. $y = 7x - 3$

b. $d = 65t$

c. $F = 1.8C + 32$

Solution:

a. According to the definition of a function, $y = f(x)$, so $f(x) = 7x - 3$.

b. This time the dependent variable is d. Function notation replaces the dependent variable, so $d = f(t) = 65t$.

c. $F = f(C) = 1.8C + 32$

Adding and Subtracting Polynomial Functions

Now let's look a adding and subtracting polynomials when using function notation.

Example 5: *Given:* $f(x) = 3x^2 - 4x + 7$ *and* $g(x) = 2x^3 - 4x^2 - 6x + 5$. Add and Simplify (f+g)(x)

Solution: Add (f+g)(x)

$3x^2 - 4x + 7$ and $2x^3 - 4x^2 - 6x + 5$.

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) \\ &= (3x^2 - 4x + 7) + (2x^3 - 4x^2 - 6x + 5) \\ &= 2x^3 + (3x^2 - 4x^2) + (-4x - 6x) + (7 + 5) \\ &= 2x^3 - x^2 - 10x + 12\end{aligned}$$

Multimedia Link: For more explanation of polynomials, visit http://www.purplemath.com/modules/polydefs.htm - Purplemath's website.

Example 6: *Given* $f(x) = 5b^2 - 2a^2$ *and* $g(x) = 4a^2 - 8ab - 9b^2$ Subtract and Simplify (f-g)(x)

Solution: Subtract (f-g)(x)

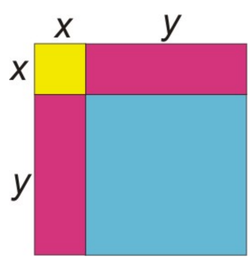
$$\begin{aligned}(f-g)(x) &= f(x) - g(x) \\ &= (4a^2 - 8ab - 9b^2) - (5b^2 - 2a^2) \\ &= [(4a^2 - (-2a^2)] + (-9b^2 - 5b^2) - 8ab \\ &= 6a^2 - 14b^2 - 8ab\end{aligned}$$

Solving Real-World Problems Using Addition or Subtraction of Polynomials

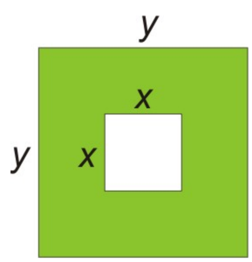
Polynomials are useful for finding the areas of geometric objects. In the following examples, you will see this usefulness in action.

Example 7: *Write a polynomial that represents the area of each figure shown.*

(a)



(b)



Solution: The blue square has area: $y \cdot y = y^2$.

The yellow square has area: $x \cdot x = x^2$.

The pink rectangles each have area: $x \cdot y = xy$.

$$\begin{aligned}\text{Test area} &= y^2 + x^2 + xy + xy \\ &= y^2 + x^2 + 2xy\end{aligned}$$

To find the area of the green region we find the area of the big square and subtract the area of the little square.

The big square has area $y \cdot y = y^2$.

The little square has area $x \cdot x = x^2$.

Area of the green region $= y^2 - x^2$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Addition and Subtraction of Polynomials (15:59)

Define the following key terms.

1. Polynomial
2. Monomial

Figure 2.1: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/790

3. Degree
4. Leading coefficient

For each of the following expressions, decide whether it is a polynomial. Explain your answer.

5. $x^2 + 3x^{\frac{1}{2}}$
6. $\frac{1}{3}x^2y - 9y^2$
7. $3x^{-3}$
8. $\frac{2}{3}t^2 - \frac{1}{t^2}$

Express each polynomial in standard form. Give the degree of each polynomial.

9. $3 - 2x$
10. $8x^4 - x + 5x^2 + 11x^4 - 10$
11. $8 - 4x + 3x^3$
12. $-16 + 5f^8 - 7f^3$
13. $-5 + 2x - 5x^2 + 8x^3$
14. $x^2 - 9x^4 + 12$

Add and simplify.

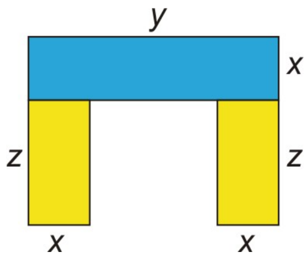
15. $f(x) = (x+8)$ *and* $g(x) = (-3x-5)$ $\;(f+g)(x) =$
16. $f(r) = (8r^4 - 6r^2 - 3r + 9)$ *and* $g(r) = (3r^3 + 5r^2 + 12r - 9)$ $\;(f+g)(r) =$
17. $f(x) = (-2x^2 + 4x - 12)$ *and* $g(x) = (7x + x^2)$ $\;(f+g)(x) =$
18. $v(a,b) = (2a^2b - 2a + 9)$ *and* $w(a,b) = (5a^2b - 4b + 5)$ $\;(v+w)(a,b) =$
19. $r(a,b) = (6.9a^2 - 2.3b^2 + 2ab)$ *and* $t(a,b) = (3.1a - 2.5b^2 + b)$ $\;(r+t)(a,b) =$

Subtract and simplify.

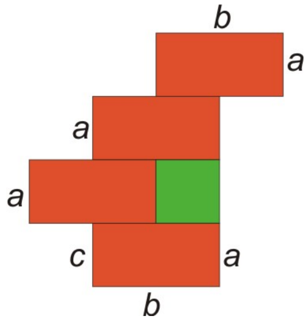
20. $f(t) = (-t + 15t^2)$ *and* $g(t) = (5t^2 + 2t - 9)$ $\;(f-g)(t) =$
21. $r(y) = (-y^2 + 4y - 5)$ *and* $s(y) = (5y^2 + 2y + 7)$ $\;(r-s)(y) =$
22. $f(h) = (-h^7 + 2h^5 + 13h^3 + 4h^2 - h - 1)$
    $g(h) = (-3h^5 + 20h^3 - 3h^2 + 8h - 4)$
    $(f-g)(h) =$
23. $t(m) = (-5m^2 - m)$ *and* $r(m) = (3m^2 + 4m - 5)$ $\;(t-r)(m) =$
24. $f(a,b) = (2a^2b - 3ab^2 + 5a^2b^2)$
    $g(a,b) = (2a^2b^2 + 4a^2b - 5b^2)$
    $(f-g)(a,b) =$

Find the area of the following figures.

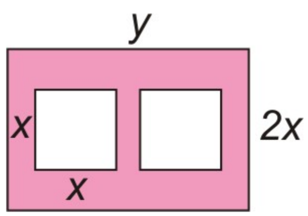
25.



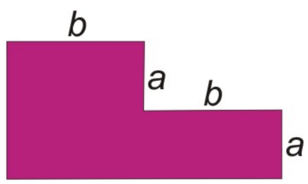
26.



27.



28.



2.3 Multiplication of Polynomials

When multiplying polynomials together, we must remember the exponent rules we learned in the last chapter, such as the Product Rule. This rule says that if we multiply expressions that have the same base, we just add the exponents and keep the base unchanged. If the expressions we are multiplying have coefficients and more than one variable, we multiply the coefficients just as we would any number. We also apply the product rule on each variable separately.

Example: $f(x) = (2x^2y^3) \text{ and } g(x) = (3x^2y)$

$$\begin{aligned}(f*g)(x) &= f(x)*g(x)\\ &= (2x^2y^3)\times(3x^2y)\\ &= (2\cdot 3)\times(x^2\cdot x^2)\times(y^3\cdot y)\\ &= 6x^4y^4\end{aligned}$$

Multiplying a Polynomial by a Monomial

This is the simplest of polynomial multiplications. Problems are like that of the one above.

Example 1: *Multiply the following monomials.*

(a) $f(x) = (2x^2)$ *and* $g(x) = (5x^3)$

(c) $f(x,y) = (3xy^5)$ *and* $g(x,y) = (-6x^4y^2)$

(d) $f(a,b,c) = (-12a^2b^3c^4)$ *and* $g(a,b,c) = (-3a^2b^2)$

Solution:

(a)

$$\begin{aligned}(f*g)(x) &= f(x)\cdot g(x)\\ &= (2x^2)\cdot(5x^3)\\ &= (2\cdot 5)\cdot(x^2\cdot x^3)\\ &= 10x^{2+3}\\ &= 10x^5\end{aligned}$$

(c)

$$\begin{aligned}(f*g)(x) &= f(x)*g(x)\\ &= (3xy^5)*(-6x^4y^2)\\ &= -18x^{1+4}y^{5+2}\\ &= -18x^5y^7\end{aligned}$$

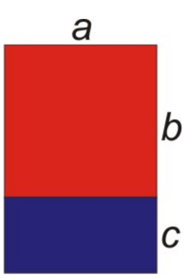
(d)

$$\begin{aligned}(f*g)(a,b,c) &= f(a,b,c)*g(a,b,c)\\ &= (-12a^2b^3c^4)(-3a^2b^2)\\ &= 36a^{2+2}b^{3+2}c^4\\ &= 36a^4b^5c^4\end{aligned}$$

To multiply monomials, we use the **Distributive Property.**

Distributive Property: For any expressions a, b, and c, $a(b+c) = ab + ac$.

This property can be used for numbers as well as variables. This property is best illustrated by an area problem. We can find the area of the big rectangle in two ways.



One way is to use the formula for the area of a rectangle.

$$\begin{aligned}Area\ of\ the\ big\ rectangle &= Length \times Width\\ Length &= a,\ Width = b+c\\ Area &= a\times(b+c)\end{aligned}$$

The area of the big rectangle can also be found by adding the areas of the two smaller rectangles.

$$\begin{aligned} Area\ of\ red\ rectangle &= ab \\ Area\ of\ blue\ rectangle &= ac \\ Area\ of\ big\ rectangle &= ab + ac \end{aligned}$$

This means that $a(b + c) = ab + ac$.

In general, if we have a number or variable in front of a parenthesis, this means that each term in the parenthesis is multiplied by the expression in front of the parenthesis.

$a(b + c + d + e + f + \ldots) = ab + ac + ad + ae + af + \ldots$ The ";...”; means ";and so on.”;

Example 2: $f(x, y) = 2x^3y$ *and* $g(x, y) = (-3x^4y^2 + 2x^3y - 10x^2 + 7x + 9)$. *Multiply f(x,y) * g(x,y)*

Solution:

$$\begin{aligned} (f * g)(x, y) &= f(x, y) * g(x, y) \\ &= 2x^3y(-3x^4y^2 + 2x^3y - 10x^2 + 7x + 9) \\ &= (2x^3y)(-3x^4y^2) + (2x^3y)(2x^3y) + (2x^3y)(-10x^2) + (2x^3y)(7x) + (2x^3y)(9) \\ &= -6x^7y^3 + 4x^6y^2 - 20x^5y + 14x^4y + 18x^3y \end{aligned}$$

Multiplying a Polynomial by a Binomial

A binomial is a polynomial with two terms. The Distributive Property also applies for multiplying binomials. Let's think of the first parentheses as one term. The Distributive Property says that the term in front of the parentheses multiplies with each term inside the parentheses separately. Then, we add the results of the products.

$(a + b)(c + d) = (a + b) \cdot c + (a + b) \cdot d$ Let's rewrite this answer as $c \cdot (a + b) + d \cdot (a + b)$

We see that we can apply the Distributive Property on each of the parentheses in turn.

$$c \cdot (a + b) + d \cdot (a + b) = c \cdot a + c \cdot b + d \cdot a + d \cdot b \text{ (or } ca + cb + da + db)$$

What you should notice is that when multiplying any two polynomials, **every term in one polynomial is multiplied by every term in the other polynomial**.

Example: *Multiply and simplify* $(2x + 1)(x + 3)$.

Solution: We must multiply each term in the first polynomial with each term in the second polynomial. First, multiply the first term in the first parentheses by all the terms in the second parentheses.

$(2x + 1)(x + 3) = (2x)(x) + (2x)(3) + \ldots$

Now we multiply the second term in the first parentheses by all terms in the second parentheses and add them to the previous terms.

$(2x + 1)(x + 3) = (2x)(x) + (2x)(3) + (1)(x) + (1)(3)$

Now we can simplify.

$$\begin{aligned} (2x)(x) + (2x)(3) + (1)(x) + (1)(3) &= 2x^2 + 6x + x + 3 \\ &= 2x^2 + 7x + 3 \end{aligned}$$

Multimedia Link: For further help, visit http://www.purplemath.com/modules/polydefs.htm – Purplemath's website – or watch this CK-12 Basic Algebra: Adding and Subtracting Polynomials YouTube video.

Figure 2.2: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/791

Example 3: *Multiply and simplify* $f(x) = (4x - 5)$ *and* $g(x) = (x - 20)$.

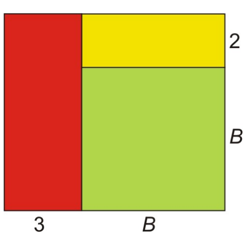
Solution:

$$\begin{aligned}(f * g)(x) &= f(x) * g(x)\\ &= (4x-5)(x-20)\\ &= (4x)(x) + (4x)(-20) + (-5)(x) + (-5)(-20)\\ &= 4x^2 - 80x - 5x + 100\\ &= 4x^2 - 85x + 100\end{aligned}$$

Solving Real-World Problems Using Multiplication of Polynomials

We can use multiplication to find the area and volume of geometric shapes. Look at these examples.

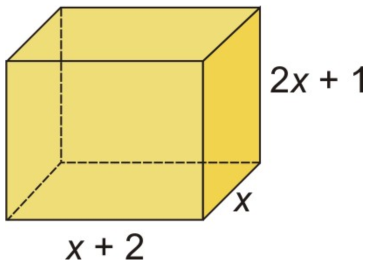
Example 4: *Find the area of the following figure.*



Solution: We use the formula for the area of a rectangle: Area = length · width. For the big rectangle:

$$\begin{aligned}\text{Length} = B + 3,\ \text{Width} &= B + 2\\ \text{Area} &= (B+3)(B+2)\\ &= B^2 + 2B + 3B + 6\\ &= B^2 + 5B + 6\end{aligned}$$

Example 5: *Find the volume of the following figure.*



Solution:

$$\begin{aligned} The\ volume\ of\ this\ shape &= (area\ of\ the\ base) \cdot (height). \\ \text{Area of the base} &= x(x+2) \\ &= x^2 + 2x \end{aligned}$$

$$\begin{aligned} Volume &= (area\ of\ base) \times height \\ Volume &= (x^2+2x)(2x+1) \end{aligned}$$

You are asked to finish this example in the practice questions.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Multiplication of Polynomials (9:49)

Figure 2.3: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/792

Multiply the following monomials.

1. $f(x) = (2x)\ and\ g(x) = (-7x)$
   $(f * g)(x) =$
2. $f(a) = 4\ and\ g(a) = (-6a)$
   $(f * g)(a) =$
3. $f(a,b) = (-5a^2b)\ and\ g(a,b) = (-12a^3b^3)$
   $(f * g)(a,b) =$
4. $f(x,y) = (-5x)\ and\ g(x,y) = (5y)$
   $(f * g)(x,y) =$

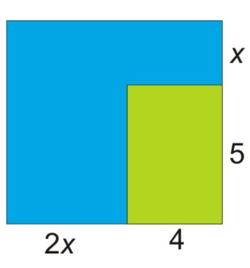
5. $f(x,y) = y$ *and* $g(x,y) = (xy^4)$
   $(f * g)(x,y) =$
6. $f(x,y,z) = (3xy^2z^2)$ *and* $g(x,y,z) = (15x^2yz^3)$
   $(f * g)(x,y,z) =$

Multiply and simplify.

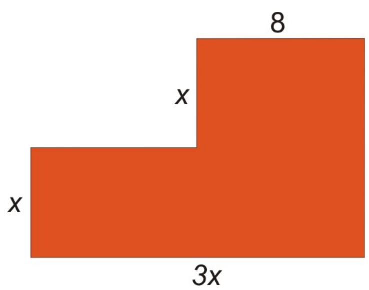
7. $x^8(xy^3 + 3x)$
8. $f(x) = 2x$ *and* $g(x) = (4x - 5)$
   $(f * g)(x) =$
9. $6ab(-10a^2b^3 + c^5)$
10. $f(x) = 9x^3$ *and* $g(x) = (3x^2 - 2x + 7)$
    $(f * g)(x) =$
11. $f(a,b) = -3a^2b$ *and* $g(a,b) = (9a^2 - 4b^2)$
    $(f * g)(a,b) =$
12. $(x-2)(x+3)$
13. $f(a) = (a+2)$ *and* $g(a) = (a-3)$
    $(f * g)(a) =$
14. $(-4xy)(2x^4yz^3 - y^4z^9)$
15. $f(x) = (x-3)$ *and* $g(x) = (x+2)$
    $(f * g)(x) =$
16. $(a^2+2)(3a^2-4)$
17. $f(x) = (7x-2)$ *and* $g(x) = (9x-5)$
    $(f * g)(x) =$
18. $(2x-1)(2x^2 - x + 3)$
19. $f(x) = (3x+2)$ *and* $g(x) = (9x^2 - 6x + 4)$
    $(f * g)(x) =$
20. $(a^2 + 2a - 3)(a^2 - 3a + 4)$
21. $(3m+1)(m-4)(m+5)$
22. Finish the volume example from Example 5 of the lesson. $Volume = (x^2 + 2x)(2x + 1)$

Find the areas of the following figures.

23.

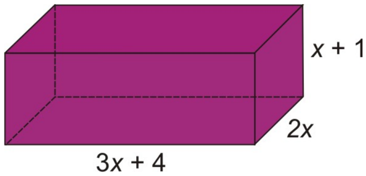


24.

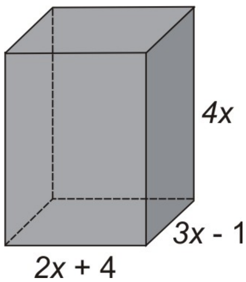


Find the volumes of the following figures.

25.



26.



2.4 Special Products of Polynomials

When we multiply two linear (degree of 1) binomials, we create a quadratic (degree of 2) polynomial with four terms. The middle terms are like terms so we can combine them and simplify to get a quadratic or 2^{nd} degree **trinomial** (polynomial with three terms). In this lesson, we will talk about some special products of binomials.

Finding the Square of a Binomial

A special binomial product is the **square of a binomial**. Consider the following multiplication: $(x+4)(x+4)$. We are multiplying the same expression by itself, which means that we are squaring the expression. This means that:

$$(x+4)(x+4) = (x+4)^2$$
$$(x+4)(x+4) = x^2 + 4x + 4x + 16 = x^2 + 8x + 16$$

This follows the general pattern of the following rule.

Square of a Binomial: $(a+b)^2 = a^2 + 2ab + b^2$, *and* $(a-b)^2 = a^2 - 2ab + b^2$

Stay aware of the common mistake $(a+b)^2 = a^2 + b^2$. To see why $(a+b)^2 \neq a^2 + b^2$, try substituting numbers for a and b into the equation (for example, $a = 4$ and $b = 3$), and you will see that it is *not* a true statement. The middle term, $2ab$, is needed to make the equation work.

Example 1: $f(x) = (x+10)$. Simplify $(f(x))^2$ by multiplying

Solution: Use the square of a binomial formula, substituting $a = x$ and $b = 10$

$$f(x) = (x+10)$$

$$\begin{aligned}(f(x))^2 &= f(x) * f(x)\\ &= (x+10)(x+10)\\ &= (x)^2 + 2(x)(10) + (10)^2\\ &= x^2 + 20x + 100\end{aligned}$$

Finding the Product of Binomials Using Sum and Difference Patterns

Another special binomial product is the product of a sum and a difference of terms. For example, let's multiply the following binomials.

$$f(x) = (x+4) \text{ and } g(x) = (x-4)$$
$$\begin{aligned}(f * g)(x) &= f(x) * g(x) \\ &= (x+4)(x-4) \\ &= x^2 - 4x + 4x - 16 \\ &= x^2 - 16\end{aligned}$$

Notice that the middle terms are opposites of each other, so they cancel out when we collect like terms. This always happens when we multiply a sum and difference of the same terms.

$$\begin{aligned}(a+b)(a-b) &= a^2 - ab + ab - b^2 \\ &= a^2 - b^2\end{aligned}$$

When multiplying a sum and difference of the same two terms, the middle terms cancel out. We get the square of the first term minus the square of the second term. You should remember this formula.

Sum and Difference Formula: $(a+b)(a-b) = a^2 - b^2$

Example 2: *Multiply the following binomias and simplify.*

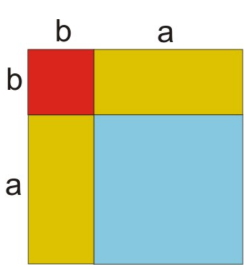
$$(5x+9)(5x-9)$$

Solution: Use the above formula, substituting $a = 5x$ and $b = 9$. Multiply.

$$(5x+9)(5x-9) = (5x)^2 - (9)^2 = 25x^2 - 81$$

Solving Real-World Problems Using Special Products of Polynomials

Let's now see how special products of polynomials apply to geometry problems and to mental arithmetic. Look at the following example.



Example: *Find the area of the square.*

Solution: *The area of the square = side × side*

$$\begin{aligned}\text{Area} &= (a+b)(a+b) \\ &= a^2 + 2ab + b^2\end{aligned}$$

Notice that this gives a visual explanation of the square of binomials product.

$Area\ of\ big\ square : (a+b)^2 = Area\ of\ blue\ square = a^2 + 2\ (area\ of\ yellow) = 2ab + area\ of\ red\ square = b^2$

The next example shows how to use the special products in doing fast mental calculations.

Example 3: *Find the products of the following numbers without using a calculator.*

(a) 43×57

(b) 45^2

Solution: The key to these mental ";tricks"; is to rewrite each number as a sum or difference of numbers you know how to square easily.

(a) Rewrite $43 = (50 - 7)$ and $57 = (50 + 7)$.

Then $43 \times 57 = (50 - 7)(50 + 7) = (50)^2 - (7)^2 = 2500 - 49 = 2,451$.

(b) $45^2 = (40 + 5)^2 = (40)^2 + 2(40)(5) + (5)^2 = 1600 + 400 + 25 = 2,025$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Special Products of Binomials (10:36)

Figure 2.4: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/793

Use the special product for squaring binomials to multiply these expressions.

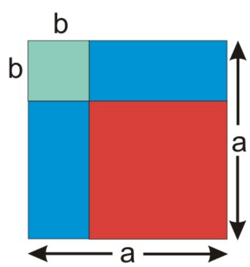
1. $(x + 9)^2$
2. $f(x) = (x - 1)$
   $(f(x))^2 =$
3. $(2y + 6)^2$
4. $f(x) = (3x - 7)$
   $(f(x))^2 =$
5. $(7c + 8)^2$
6. $f(a) = (9a^2 + 6)$
   $(f(a))^2 =$
7. $(b^2 - 1)^2$
8. $v(m) = (m^3 + 4)$
   $(v(m))^2 =$
9. $\left(\frac{1}{4}t + 2\right)^2$

10. $g(k) = (6k-3)$
    $(g(k))^2 =$
11. $(a^3-7)^2$
12. $f(x,y) = (4x^2+y^2)$
    $(f(x,y))^2 =$
13. $(8x-3)^2$

Use the special product of a sum and difference to multiply these expressions.

14. $(2x-1)(2x+1)$
15. $(2x-3)(2x+3)$
16. $(4+6x)(4-6x)$
17. $(6+2r)(6-2r)$
18. $(-2t+7)(2t+7)$
19. $(8z-8)(8z+8)$
20. $(3x^2+2)(3x^2-2)$
21. $(x-12)(x+12)$
22. $(5a-2b)(5a+2b)$
23. $(ab-1)(ab+1)$

Find the area of the orange square in the following figure. It is the lower right shaded box.

24. 

Multiply the following numbers using the special products.

25. 45×55
26. 97×83
27. 19^2
28. 56^2
29. 876×824
30. 1002×998
31. 36×44

2.5 Composition of Functions

Functions are often described in terms of ";input"; and ";output."; For example, consider the function $f(x) = 2x+3$. When we input an x value, we output a y value, or a function value. We find the output by taking the input x, multiplying by 2, and adding 3. We can do this for any value of x. Now consider a second function $g(x) = 5x$. For this function too, we can take an x value, input the x into $g(x)$, and obtain an output. For example, if $x = 4$, $g(4) = 5(4) = 20$. What happens if we then take the output of 20 and use it as the input of f?

Then we have $f(20) = 2(20) + 3 = 43$. The table below shows several examples of this same process:

Table 2.1:

x	Output from g	Output from f
2	10	23
3	15	33
4	20	43
5	25	53

Examining the values in the table, we can see a pattern: all of the final output values from f are 3 more than 10 times the initial input. We have created a new function in which $g(x) = 5x$ is the input:

$h(x) = f(5x) = 2(5x) + 3 = 10x + 3$

When we input one function into another, we call this the **composition** of the two functions. Formally, we write the composed function as $f(g(x)) = 10x + 3$ or $(f \text{ o } g)x = 10x + 3$

Example 3: Find $f(g(x))$ and $g(f(x))$:

Table 2.2:

a. $f(x) = 3x + 1$ and $g(x) = x^2$	b. $f(x) = 2x + 4$ and $g(x) = (1/2)x$-2

Solution:

a. $f(x) = 3x + 1$ and $g(x) = x^2$

In both cases, the resulting function is quadratic.
$g(f(x)) = g(3x + 1) = (3x + 1)^2 = 9x^2 + 6x + 1$
$f(g(x)) = f(x^2) = 3(x^2) + 1 = 3x^2 + 1$

b. $f(x) = 2x + 4$ and $g(x) = (1/2)\ x$ - 2

$f(g(x)) = 2((1/2)x$ -2) + 4 = $(2/2)x$ - 4 + 4 = $(2/2)x = x$

In this case, the composites were equal to each other, and they both equal x, the original input into the function. This means that there is a special relationship between these two functions. We will examine this relationship in Chapter 3. It is important to note, however, that $f(g(x)$ is not necessarily equal to $g(f(x))$.
$g(f(x)) = g(2x+4) = (1/2)(2x+4)$-2 = x+2=2=x.

When we compose functions, we are combining two (or more) functions by inputting the output of one function into another. We can also *decompose* a function. Consider the function $f(x)=(2x+1)^2$. We can decompose this function into an ";inside"; and an ";outside"; function. For example, we can construct $f(x) = (2x+1)^2$ with a linear function and a quadratic function. If $g(x)=x^2$ and $h(x)=(2x+1)$, then $f(x) = g(h(x))$. The linear function $h(x) = (2x+1)$ is the inside function, and the quadratic function $g(x) = x^2$ is the outside function.

Example 4: Decompose the function $f(x)=(3x\text{-}1)^2$-5 into a quadratic function $g(x)$ and a linear function $h(x)$.

Solution: Let $h(x)=3x$-1 and $g(x)=x^2$-5. Then $f(x)=g(h(x))$ because $g(h(x))=g(3x\text{-}1)=(3x\text{-}1)^2$-5.

The decomposition of a function is not necessarily unique. For example, there are many ways that we could express a linear function as the composition of other linear functions. (You will be asked to find decompositions of linear functions in the review questions.)

Lesson Summary

In this lesson we have performed operations on functions and analyzed the resulting graphs. When we add or subtract functions, the resulting function may or may not be a member of the same family as the original functions. When we multiply linear and polynomial functions, the result will be a polynomial function. When we divide linear and polynomial functions, the result will be a polynomial or a rational function. If we perform operations on non-polynomial functions such as the absolute value function, the resulting function is likely to be somewhat complicated. Using a graphing calculator or graphing software can help you see the features of these graphs.

In this lesson we have also composed and decomposed functions. When we compose functions, we treat one function as the input of another. When we decompose functions, we identify functions that could be composed to produce a more complicated function. We can also use composition of functions to model situations in which there are multiple ";inputs."; This kind of model, as well as other types of models, will be the focus of the next lesson.

Points to Consider

1. What is the difference between performing operations on functions and composing functions?
2. In what situation will a composed function be a member of the same family as one of the components?

Review Questions

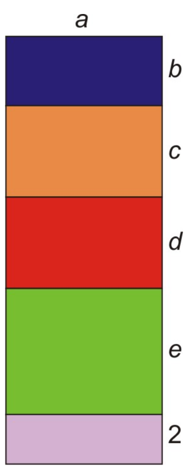
For questions 1 – 4, let $f(x)$=4x-5 and $g(x)$=(1/2)x^2 + x

1. $f(x)$+$g(x)$=?
2. $g(x)$-$f(x)$=?
3. $f(x) \cdot g(x)$=?
4. Write an equation $h(x)$= $f(x)/g(x)$ and state the domain of $h(x)$.
5. Let $f(x) = x^3$ and $g(x)$=x+8. a. Find $f(g(x))$
   b. Find $g(f(x))$
   c. Are the composed functions equal?
6. How are the graphs of the cubic functions in problem 5 related?
7. Let $f(x)$=2x-5 and $g(x)$=5x-2 . Find $f(g(f(x)))$.
8. Write the function $t(x)$=(3x+1)^4 as the composition of two functions.
9. Let $f(x) = |x|$ and $g(x)$=3-(x^2). Use a graphing utility to sketch a graph of f+g and g/f. What are the domains and ranges of f+g and g/f?
10. Let $f(x)$ = (x+a)/(bx+c) and $g(x)$=dx.a. Find f($g(x)$).
    b. What is the domain of f($g(x)$)?

2.6 Polynomial Equations in Factored Form

We have been multiplying polynomials by using the Distributive Property, where all the terms in one polynomial must be multiplied by all terms in the other polynomial. In this lesson, you will start learning how to do this process using a different method called **factoring**.

Factoring: Take the factors that are common to all the terms in a polynomial. Then multiply the common factors by a parenthetical expression containing all the terms that are left over when you divide out the common factors.



Let's look at the areas of the rectangles again: $Area = length \times width$. The total area of the figure on the right can be found in two ways.

Method 1: Find the areas of all the small rectangles and add them.

Blue rectangle $= ab$

Orange rectangle $= ac$

Red rectangle $= ad$

Green rectangle $= ae$

Purple rectangle $= 2a$

Total area $= ab + ac + ad + ae + 2a$

Method 2: Find the area of the big rectangle all at once.

$$\begin{aligned} \text{Length} &= a \\ \text{Width} &= b + c + d + e + 2 \\ \text{Area} &= a(b + c + d + e = 2) \end{aligned}$$

The answers are the same no matter which method you use:

$$ab + ac + ad + ae + 2a = a(b + c + d + e + 2)$$

Using the Zero Product Property

Polynomials can be written in **expanded form** or in **factored form**. Expanded form means that you have sums and differences of different terms:

$$6x^4 + 7x^3 - 26x^2 - 17x + 30$$

Notice that the degree of the polynomial is four.

The **factored form** of a polynomial means it is written as a product of its factors.

The factors are also polynomials, usually of lower degree. Here is the same polynomial in factored form.

$$(x-1)(x+2)(2x-3)(3x+5)$$

Suppose we want to know where the polynomial $6x^4+7x^3-26x^2-17x+30$ equals zero. It is quite difficult to solve this using the methods we already know. However, we can use the Zero Product Property to help.

Zero Product Property: The only way a product is zero is if one or both of the terms are zero.

By setting the factored form of the polynomial equal to zero and using this property, we can easily solve the original polynomial.

$$(x-1)(x+2)(2x-3)(3x+5)=0$$

According to the property, for the original polynomial to equal zero, we have to set each term equal to zero and solve.

$$\begin{aligned}(x-1)=0 &\rightarrow x=1\\ (x+2)=0 &\rightarrow x=-2\\ (2x-3)=0 &\rightarrow x=\frac{3}{2}\\ (3x+5)=0 &\rightarrow x=-\frac{5}{3}\end{aligned}$$

The solutions to $6x^4+7x^3-26x^2-17x+30=0$ are $x=-2,-\frac{5}{3},1,\frac{3}{2}$.

Multimedia Link: For further explanation of the Zero Product Property, watch this CK-12 Basic Algebra: Zero Product Property - YouTube video.

Figure 2.5: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/794

Example 2: *Solve* $(x-9)(3x+4)=0$.

Solution: Separate the factors using the Zero Product Property: $(x-9)(3x+4)=0$.

$$\begin{aligned}x-9&=0 \qquad\qquad \text{or} \qquad\qquad & 3x+4&=0\\ x&=9 & 3x&=-4\\ & & x&=\frac{-4}{3}\end{aligned}$$

Finding the Greatest Common Monomial Factor

Once we get a polynomial in factored form, it is easier to solve the polynomial equation. But first, we need to learn how to factor. Factoring can take several steps because we want to factor completely so we cannot factor any more.

A **common factor** can be a number, a variable, or a combination of numbers and variables that appear in **every term** of the polynomial.

When a common factor is factored from a polynomial, you divide each term by the common factor. What is left over remains in parentheses.

Example 3: *Factor:*

1. $15x - 25$
2. $3a + 9b + 6$

Solution:

1. We see that the factor of 5 divides evenly from all terms.

$$15x - 25 = 5(3x - 5)$$

2. We see that the factor of 3 divides evenly from all terms.

$$3a + 9b + 6 = 3(a + 3b + 2)$$

Now we will use examples where different powers can be factored and there is more than one common factor.

Example 4: *Find the greatest common factor.*

(a) $a^3 - 3a^2 + 4a$

(b) $5x^3y - 15x^2y^2 + 25xy^3$

Solution:

(a) Notice that the factor a appears in all terms of $a^3 - 3a^2 + 4a$ but each term has a different power of a. The common factor is the lowest power that appears in the expression. In this case the factor is a.

Let's rewrite $a^3 - 3a^2 + 4a = a(a^2) + a(-3a) + a(4)$

Factor a to get $a(a^2 - 3a + 4)$

(b) The common factors are $5xy$.

When we factor $5xy$, we obtain $5xy(x^2 - 3xy + 5y^2)$.

Solving Simple Polynomial Equations by Factoring

We already saw how we can use the Zero Product Property to solve polynomials in factored form. Here you will learn how to solve polynomials in expanded form. These are the steps for this process.

Step 1: Rewrite the equation in standard form such that: Polynomial expression = 0.

Step 2: Factor the polynomial completely.

Step 3: Use the zero-product rule to set **each factor equal to zero.**

Step 4: Solve each equation from step 3.

Step 5: Check your answers by substituting your solutions into the original equation.

Example 5: *Solve the following polynomial equation.*

$$x^2 - 2x = 0$$

Solution: $x^2 - 2x = 0$

Rewrite: This is not necessary since the equation is in the correct form.

Factor: The common factor is x, so this factors as: $x(x-2) = 0$.

Set each factor equal to zero.

$$x = 0 \qquad \text{or} \qquad x - 2 = 0$$

Solve:

$$x = 0 \qquad \text{or} \qquad x = 2$$

Check: Substitute each solution back into the original equation.

$$\begin{aligned} x &= 0 \qquad & (0)^2 - 2(0) &= 0 \\ x &= 2 \qquad & (2)^2 - 2(2) &= 0 \end{aligned}$$

Answer $x = 0, \ x = 2$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Polynomial Equations in Factored Form (9:29)

Figure 2.6: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/797

1. What is the Zero Product Property? How does this simplify solving complex polynomials?

Factor the common factor from the following polynomials.

2. $36a^2 + 9a^3 - 6a^7$
3. $yx^3y^2 + 12x + 16y$
4. $3x^3 - 21x$
5. $5x^6 + 15x^4$
6. $4x^3 + 10x^2 - 2x$
7. $-10x^6 + 12x^5 - 4x^4$

8. $12xy + 24xy^2 + 36xy^3$
9. $5a^3 - 7a$
10. $45y^{12} + 30y^{10}$
11. $16xy^2z + 4x^3y$

Why can't the Zero Product Property be used to solve the following polynomials?

12. $(x-2)(x) = 2$
13. $(x+6) + (3x-1) = 0$
14. $(x^{-3})(x+7) = 0$
15. $(x+9) - (6x-1) = 4$
16. $(x^4)(x^2-1) = 0$

Solve the following polynomial equations.

17. $x(x+12) = 0$
18. $(2x+3)(5x-4) = 0$
19. $(2x+1)(2x-1) = 0$
20. $24x^2 - 4x = 0$
21. $60m = 45m^2$
22. $(x-5)(2x+7)(3x-4) = 0$
23. $2x(x+9)(7x-20) = 0$
24. $18y - 3y^2 = 0$
25. $9x^2 = 27x$
26. $4a^2 + a = 0$
27. $b^2 - \frac{5}{3b} = 0$

2.7 Factoring Quadratic Expressions

In this lesson, we will learn how to factor quadratic polynomials for different values of a, b, and c. In the last lesson, we factored common monomials, so you already know how to factor quadratic polynomials where $c = 0$.

Factoring Quadratic Expressions in Standard From

Quadratic polynomials are polynomials of degree 2. The standard form of a quadratic polynomial is $ax^2 + bx + c$, *where* a, b, *and* c *are real numbers.*

Example 1: *Factor* $x^2 + 5x + 6$.

Solution: We are looking for an answer that is a product of two binomials in parentheses: $(x + \_\_\_\_\_\_)(x + \_\_\_\_\_\_)$.

To fill in the blanks, we want two numbers m and n that multiply to 6 and add to 5. A good strategy is to list the possible ways we can multiply two numbers to give us 6 and then see which of these pairs of numbers add to 5. The number six can be written as the product of.

$$\begin{aligned} 6 &= 1 \times 6 \quad \textit{and} \quad 1 + 6 = 7 \\ 6 &= 2 \times 3 \quad \textit{and} \quad 2 + 3 = 5 \end{aligned}$$

So the answer is $(x+2)(x+3)$.

We can check to see if this is correct by multiplying $(x+2)(x+3)$.

x is multiplied by x and $3 = x^2 + 3x$.

2 is multiplied by x and $3 = 2x + 6$.

Combine the like terms: $x^2 + 5x + 6$.

Example 2: *Factor* $x^2 - 6x + 8$.

Solution: We are looking for an answer that is a product of the two parentheses $(x+\_\_\_\_\_\_)(x+\_\_\_\_\_\_)$.

The number 8 can be written as the product of the following numbers.

$8 = 1 \cdot 8$ and $1 + 8 = 9$ Notice that these are two different choices.

$$\begin{array}{lcl} 8 = (-1)(-8) & \textit{and} & -1 + (-8) = -9 \\ 8 = 2 \times 4 & \textit{and} & 2 + 4 = 6 \end{array}$$

And

$8 = (-2) \cdot (-4)$ and $-2 + (-4) = -6 \leftarrow$ This is the correct choice.

The answer is $(x-2)(x-4)$.

Example 3: *Factor* $x^2 + 2x - 15$.

Solution: We are looking for an answer that is a product of two parentheses $(x \pm \_\_\_\_\_\_)(x \pm \_\_\_\_\_\_)$.

In this case, we must take the negative sign into account. The number –15 can be written as the product of the following numbers.

$-15 = -1 \cdot 15$ and $-1 + 15 = 14$ Notice that these are two different choices.

And also,

$-15 = 1 \cdot (-15)$ and $1 + (-15) = -14$ Notice that these are two different choices.

$-15 = (-3) \times 5$ *and* $(-3) + 5 = 2$ *This is the correct choice.*

$-15 = 3 \times (-5)$ *and* $3 + (-5) = -2$

The answer is $(x-3)(x+5)$.

Example 4: *Factor* $-x^2 + x + 6$.

Solution: First factor the common factor of –1 from each term in the trinomial. Factoring –1 changes the signs of each term in the expression.

$$-x^2 + x + 6 = -(x^2 - x - 6)$$

We are looking for an answer that is a product of two parentheses $(x \pm \_\_\_\_\_\_)(x \pm \_\_\_\_\_\_)$.

Now our job is to factor $x^2 - x - 6$.

The number –6 can be written as the product of the following numbers.

$$\begin{array}{lcll} -6 = (-1) \times 6 & \textit{and} & (-1) + 6 = 5 & \\ -6 = 1 \times (-6) & \textit{and} & 1 + (-6) = -5 & \\ -6 = (-2) \times 3 & \textit{and} & (-2) + 3 = 1 & \\ -6 = 2 \times (-3) & \textit{and} & 2 + (-3) = -1 & \textit{This is the correct choice.} \end{array}$$

The answer is $-(x-3)(x+2)$.

To Summarize:

A quadratic of the form $x^2 + bx + c$ factors as a product of two parenthesis $(x+m)(x+n)$.

- If b and c are positive then both m and n are positive.
  - Example $x^2 + 8x + 12$ factors as $(x+6)(x+2)$.
- If b is negative and c is positive then both m and n are negative.
  - Example $x^2 - 6x + 8$ factors as $(x-2)(x-4)$.
- If c is negative then either m is positive and n is negative or vice-versa.
  - Example $x^2 + 2x - 15$ factors as $(x+5)(x-3)$.
  - Example $x^2 + 34x - 35$ factors as $(x+35)(x-1)$.
- If $a = -1$, factor a common factor of -1 from each term in the trinomial and then factor as usual. The answer will have the form $-(x+m)(x+n)$.
  - Example $-x^2 + x + 6$ factors as $-(x-3)(x+2)$.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Factoring Quadratic Equations (16:30)

Figure 2.7: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/798

Factor the following quadratic polynomials.

1. $x^2 + 10x + 9$
2. $x^2 + 15x + 50$
3. $x^2 + 10x + 21$
4. $x^2 + 16x + 48$
5. $x^2 - 11x + 24$
6. $x^2 - 13x + 42$
7. $x^2 - 14x + 33$
8. $x^2 - 9x + 20$
9. $x^2 + 5x - 14$
10. $x^2 + 6x - 27$

11. $x^2 + 7x - 78$
12. $x^2 + 4x - 32$
13. $x^2 - 12x - 45$
14. $x^2 - 5x - 50$
15. $x^2 - 3x - 40$
16. $x^2 - x - 56$
17. $-x^2 - 2x - 1$
18. $-x^2 - 5x + 24$
19. $-x^2 + 18x - 72$
20. $-x^2 + 25x - 150$
21. $x^2 + 21x + 108$
22. $-x^2 + 11x - 30$
23. $x^2 + 12x - 64$
24. $x^2 - 17x - 60$

2.8 Factoring Special Products

When we learned how to multiply binomials, we talked about two special products: the Sum and Difference Formula and the Square of a Binomial Formula. In this lesson, we will learn how to recognize and factor these special products.

Factoring the Difference of Two Squares

We use the Sum and Difference Formula to factor a difference of two squares. A difference of two squares can be a quadratic polynomial in this form: $a^2 - b^2$. Both terms in the polynomial are perfect squares. In a case like this, the polynomial factors into the sum and difference of the square root of each term.

$$a^2 - b^2 = (a + b)(a - b)$$

In these problems, the key is figuring out what the a and b terms are. Let's do some examples of this type.

Example 1: *Factor the difference of squares.*

(a) $x^2 - 9$

(b) $x^2y^2 - 1$

Solution:

(a) Rewrite as $x^2 - 9$ as $x^2 - 3^2$. Now it is obvious that it is a difference of squares.

We substitute the values of a and b for the Sum and Difference Formula:

$$(x + 3)(x - 3)$$

The answer is $x^2 - 9 = (x + 3)(x - 3)$.

(b) Rewrite as $x^2y^2 - 1$ as $(xy)^2 - 1^2$. This factors as $(xy + 1)(xy - 1)$.

Factoring Perfect Square Trinomials

A **perfect square trinomial** has the form

$$a^2 + 2ab + b^2 \qquad \text{or} \qquad a^2 - 2ab + b^2$$

The **factored form** of a perfect square trinomial has the form

$$(a+b)^2 \ if \ a^2 + 2(ab) + b^2$$

And

$$(a-b)^2 \ if \ a^2 - 2(ab) + b^2$$

In these problems, the key is figuring out what the a and b terms are. Let's do some examples of this type.

Example: $x^2 + 8x + 16$

Solution: Check that the first term and the last term are perfect squares.

$$x^2 + 8x + 16 \qquad \text{as} \qquad x^2 + 8x + 4^2.$$

Check that the middle term is twice the product of the square roots of the first and the last terms. This is true also since we can rewrite them.

$$x^2 + 8x + 16 \qquad \text{as} \qquad x^2 + 2 \cdot 4 \cdot x + 4^2$$

This means we can factor $x^2 + 8x + 16$ as $(x+4)^2$.

Example 2: *Factor* $x^2 - 4x + 4$.

Solution: Rewrite $x^2 - 4x + 4$ as $x^2 + 2 \cdot (-2) \cdot x + (-2)^2$.

We notice that this is a perfect square trinomial and we can factor it as: $(x-2)^2$.

Solving Polynomial Equations Involving Special Products

We have learned how to factor quadratic polynomials that are helpful in solving polynomial equations like $ax^2 + bx + c = 0$. Remember that to solve polynomials in expanded form, we use the following steps:

Step 1: Rewrite the equation in standard form such that: Polynomial expression = 0.

Step 2: Factor the polynomial completely.

Step 3: Use the Zero Product Property to set **each factor equal to zero.**

Step 4: Solve each equation from step 3.

Step 5: Check your answers by substituting your solutions into the original equation.

Example 3: *Solve the following polynomial equations.*

$$x^2 + 7x + 6 = 0$$

Solution: No need to rewrite because it is already in the correct form.

Factor: We write 6 as a product of the following numbers:

$$\begin{array}{ccc} 6 = 6 \times 1 & \text{and} & 6 + 1 = 7 \\ x^2 + 7x + 6 = 0 & \text{factors as} & (x+1)(x+6) = 0 \end{array}$$

Set each factor equal to zero:

$$x+1=0 \qquad \text{or} \qquad x+6=0$$

Solve:

$$x=-1 \qquad \text{or} \qquad x=-6$$

Check: Substitute each solution back into the original equation.

$$(-1)^2+7(-1)+6=1+(-7)+6=0$$
$$(-6)^2+7(-6)+6=36+(-42)+6=0$$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Factoring Special Products (10:08)

Figure 2.8: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/799

Factor the following perfect square trinomials.

1. $x^2+8x+16$
2. $x^2-18x+81$
3. $-x^2+24x-144$
4. $x^2+14x+49$
5. $4x^2-4x+1$
6. $25x^2+60x+36$
7. $4x^2-12xy+9y^2$
8. x^4+22x^2+121

Factor the following difference of squares.

9. x^2-4
10. x^2-36
11. $-x^2+100$

12. $x^2 - 400$
13. $9x^2 - 4$
14. $25x^2 - 49$
15. $-36x^2 + 25$
16. $16x^2 - 81y^2$

Solve the following quadratic equations using factoring.

17. $x^2 - 11x + 30 = 0$
18. $x^2 + 4x = 21$
19. $x^2 + 49 = 14x$
20. $x^2 - 64 = 0$
21. $x^2 - 24x + 144 = 0$
22. $4x^2 - 25 = 0$
23. $x^2 + 26x = -169$
24. $-x^2 - 16x - 60 = 0$

Solving Real-World Problems Using Polynomial Equations

Now that we know most of the factoring strategies for quadratic polynomials, we can see how these methods apply to solving real-world problems.

Example 6: *The product of two positive numbers is 60. Find the two numbers if one of the numbers is 4 more than the other.*

Solution: $x =$ one of the numbers and $x + 4$ equals the other number. The product of these two numbers equals 60. We can write the equation.

$$x(x + 4) = 60$$

Write the polynomial in standard form.

$$x^2 + 4x = 60$$
$$x^2 + 4x - 60 = 0$$

Factor: $-60 = 6 \times (-10)$ *and* $6 + (-10) = -4$

$-60 = -6 \times 10$ *and* $-6 + 10 = 4$ This is the correct choice.

The expression factors as $(x + 10)(x - 6) = 0$.

Solve:

$$x + 10 = 0 \qquad \text{or} \qquad x - 6 = 0$$
$$x = -10 \qquad \qquad x = 6$$

Since we are looking for positive numbers, the answer must be positive.

$x = 6$ for one number, and $x + 4 = 10$ for the other number.

Check: $6 \cdot 10 = 60$ so the answer checks.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Factor by Grouping and Factoring Completely (13:57)

Figure 2.9: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/800

Factor completely.

1. $2x^2 + 16x + 30$
2. $12c^2 - 75$
3. $-x^3 + 17x^2 - 70x$
4. $6x^2 - 600$
5. $-5t^2 - 20t - 20$
6. $6x^2 + 18x - 24$
7. $-n^2 + 10n - 21$
8. $2a^2 - 14a - 16$
9. $2x^2 - 512$
10. $12x^3 + 12x^2 + 3x$

Factor by grouping.

11. $6x^2 - 9x + 10x - 15$
12. $5x^2 - 35x + x - 7$
13. $9x^2 - 9x - x + 1$
14. $4x^2 + 32x - 5x - 40$
15. $12x^3 - 14x^2 + 42x - 49$
16. $4x^2 + 25x - 21$
17. $24b^3 + 32b^2 - 3b - 4$
18. $2m^3 + 3m^2 + 4m + 6$
19. $6x^2 + 7x + 1$
20. $4x^2 + 8x - 5$
21. $5a^3 - 5a^2 + 7a - 7$
22. $3x^2 + 16x + 21$
23. $4xy + 32x + 20y + 160$
24. $10ab + 40a + 6b + 24$
25. $9mn + 12m + 3n + 4$

26. $4jk - 8j^2 + 5k - 10j$
27. $24ab + 64a - 21b - 56$

Solve the following application problems.

28. One leg of a right triangle is seven feet longer than the other leg. The hypotenuse is 13 feet. Find the dimensions of the right triangle.
29. A rectangle has sides of $x + 2$ and $x - 1$. What value of x gives an area of 108?
30. The product of two positive numbers is 120. Find the two numbers if one numbers is seven more than the other.
31. Framing Warehouse offers a picture-framing service. The cost for framing a picture is made up of two parts. The cost of glass is $1 per square foot. The cost of the frame is $2 per linear foot. If the frame is a square, what size picture can you get framed for $20.00?

2.9 Chapter Review

Define the following words:

1. Polynomial
2. Monomial
3. Trinomial
4. Binomial
5. Coefficient
6. Independent events
7. Factors
8. Factoring
9. Greatest common factor
10. Constant

Identify the coefficients, constants, and the polynomial degrees in each of the following polynomials.

13. $x^5 - 3x^3 + 4x^2 - 5x + 7$
14. $x^4 - 3x^3y^2 + 8x - 12$

Rewrite the following in standard form.

15. $-4b + 4 + b^2$
16. $3x^2 + 5x^4 - 2x + 9$

Add or subtract the following polynomials and simplify.

17. Add $x^2 - 2xy + y^2$ and $2y^2 - 4x^2$ and $10xy + y^3$.
18. Subtract $x^3 - 3x^2 + 8x + 12$ from $4x^2 + 5x - 9$.
19. Add $2x^3 + 3x^2y + 2y$ and $x^3 - 2x^2y + 3y$

Multiply and simplify the following polynomials.

20. $(-3y^4)(2y^2)$
21. $-7a^2bc^3(5a^2 - 3b^2 - 9c^2)$
22. $-7y(4y^2 - 2y + 1)$
23. $(3x^2 + 2x - 5)(2x - 3)$
24. $(x^2 - 9)(4x^4 + 5x^2 - 2)$
25. $(2x^3 + 7)(2x^3 - 7)$

Square the binomials and simplify.

26. $(x^2 + 4)^2$
27. $(5x - 2y)^2$
28. $(13x^2 + 2y)$

Solve the following polynomial equations.

29. $4x(x + 6)(4x - 9) = 0$
30. $x(5x - 4) = 0$

Factor out the greatest common factor of each expression

31. $-12n + 28n + 4$
32. $45x^{10} + 45x^7 + 18x^4$
33. $-16y^5 - 8y^5x^2 + 40y^6x^3$
34. $15u^4 - 10u^6 - 10u^3v$
35. $-6a^9 + 20a^4b + 10a^3$
36. $12x + 27y^2 - 27x^6$

Factor the difference of squares.

37. $x^2 - 100$
38. $x^2 - 1$
39. $16x^2 - 25$
40. $4x^2 - 81$

Complete the following problems.

46. One leg of a right triangle is 3 feet longer than the other leg. The hypotenuse is 15 feet. Find the dimensions of the right triangle.
47. A rectangle has sides of $x + 5$ and $x - 3$. What value of x gives an area of 48?

2.10 Chapter Test

Simplify the following expressions.

1. $(4x^2 + 5x + 1) - (2x^2 - x - 3)$
2. $(2x + 5) - (x^2 + 3x - 4)$
3. $(b + 4c) + (6b + 2c + 3d)$

4. $(5x^2 + 3x + 3) + (3x^2 - 6x + 4)$
5. $(3x + 4)(x - 5)$
6. $(9x^2 + 2)(x - 3)$
7. $(4x + 3)(8x^2 + 2x + 7)$

Factor the following expressions.

8. $27x^2 - 18x + 3$
9. $9n^2 - 100$
10. $648x^2 - 32$
11. $81p^2 - 90p + 25$
12. $6x^2 - 35x + 49$

Solve the following problems.

13. A rectangle has sides of $x + 7$ and $x - 5$. What value of x gives an area of 63?
14. The product of two positive numbers is 50. Find the two numbers if one of the numbers is 6 more than the other.

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9619.

Chapter 3

Exponents and Exponential Functions

3.1 Exponents and Exponential Functions

Exponential functions occur in daily situations; money in a bank account, population growth, the decay of carbon-14 in living organisms, and even a bouncing ball. Exponential equations involve **exponents**, or the concept of repeated multiplication. This chapter focuses on combining expressions using the properties of exponents. The latter part of this chapter focuses on creating exponential equations and using the models to predict.

3.2 Exponential Properties Involving Products

In this lesson, you will be learning what an exponent is and about the properties and rules of exponents. You will also learn how to use exponents in problem solving.

Definition: An **exponent** is a power of a number that shows how many times that number is multiplied by itself.

An example would be 2^3. You would multiply 2 by itself 3 times, $2 \times 2 \times 2$. The number 2 is the **base** and the number 3 is the **exponent.** The value 2^3 is called the **power.**

Example 1: *Write in exponential form:* $\alpha \times \alpha \times \alpha \times \alpha$.

Solution: You must count the number of times the base, α, is being multiplied by itself. It's being multiplied four times so the solution is α^4.

Note: There are specific rules you must remember when taking powers of negative numbers.

$$(negative\ number) \times (positive\ number) = negative\ number$$
$$(negative\ number) \times (negative\ number) = positive\ number$$

For **even powers** of negative numbers, the answer will always be positive. Pairs can be made with each number and the negatives will be cancelled out.

$$(-2)^4 = (-2)(-2)(-2)(-2) = (-2)(-2) \cdot (-2)(-2) = +16$$

For **odd powers** of negative numbers, the answer is always negative. Pairs can be made but there will still be one negative number unpaired, making the answer negative.

$$(-2)^5 = (-2)(-2)(-2)(-2)(-2) = (-2)(-2) \cdot (-2)(-2) \cdot (-2) = -32$$

When we multiply the same numbers, each with different powers, it is easier to combine them before solving. This is why we use the **Product of Powers Property.**

Product of Powers Property: For all real numbers $\chi, \chi^n \cdot \chi^m = \chi^{n+m}$.

Example 2: *Multiply* $\chi^4 \cdot \chi^5$.

Solution: $\chi^4 \cdot \chi^5 = \chi^{4+5} = \chi^9$

Note that when you use the product rule you DO NOT MULTIPLY BASES.

Example: $2^2 \cdot 2^3 \neq 4^5$

Another note is that this rule APPLIES ONLY TO TERMS THAT HAVE THE SAME BASE.

Example: $2^2 \cdot 3^3 \neq 6^5$

$$(x^4)^3 = x^4 \cdot x^4 \cdot x^4 \qquad \text{3 factors of } x \text{ to the power 4.}$$
$$\underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^{12}}$$

This situation is summarized below.

Power of a Product Property: For all real numbers χ,

$$(\chi^n)^m = \chi^{n \cdot m}$$

The Power of a Product Property is similar to the Distributive Property. Everything inside the parentheses must be taken to the power outside. For example, $(x^2y)^4 = (x^2)^4 \cdot (y)^4 = x^8y^4$. Watch how it works the long way.

$$\underbrace{(x \cdot x \cdot y)}_{x^2y} \cdot \underbrace{(x \cdot x \cdot y)}_{x^2y} \cdot \underbrace{(x \cdot x \cdot y)}_{x^2y} \cdot \underbrace{(x \cdot x \cdot y)}_{x^2y} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y)}_{x^8y4}$$

The Power of a Product Property does not work if you have a sum or difference inside the parenthesis. For example, $(\chi + \gamma)^2 \neq \chi^2 + \gamma^2$. Because it is an addition equation, it should look like $(\chi + \gamma)(\chi + \gamma)$.

Example 3: *Simplify* $(\chi^2)^3$.

Solution: $(\chi^2)^3 = \chi^{2 \cdot 3} = \chi^6$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Exponent Properties Involving Products (14:00)

Figure 3.1: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/782

1. Consider a^5. 1. What is the base?
   2. What is the exponent?
   3. What is the power?
   4. How can this power be written using repeated multiplication?

Determine whether the answer will be positive or negative. You do **not** have to provide the answer.

2. $-(3^4)$
3. -8^2
4. $10 \times (-4)^3$
5. What is the difference between -5^2 and $(-5)^2$?

Write in exponential notation.

6. $2 \cdot 2$
7. $(-3)(-3)(-3)$
8. $y \cdot y \cdot y \cdot y \cdot y$
9. $(3a)(3a)(3a)(3a)$
10. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
11. $3x \cdot 3x \cdot 3x$
12. $(-2a)(-2a)(-2a)(-2a)$
13. $6 \cdot 6 \cdot 6 \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y$

Find each number.

14. 1^{10}
15. 0^3
16. 7^3
17. -6^2

18. 5^4
19. $3^4 \cdot 3^7$
20. $2^6 \cdot 2$
21. $(4^2)^3$
22. $(-2)^6$
23. $(0.1)^5$
24. $(-0.6)^3$

Multiply and simplify.

25. $6^3 \cdot 6^6$
26. $2^2 \cdot 2^4 \cdot 2^6$
27. $3^2 \cdot 4^3$
28. $x^2 \cdot x^4$
29. $x^2 \cdot x^7$
30. $(y^3)^5$
31. $(-2y^4)(-3y)$
32. $(4a^2)(-3a)(-5a^4)$

Simplify.

33. $(a^3)^4$
34. $(xy)^2$
35. $(3a^2b^3)^4$
36. $(-2xy^4z^2)^5$
37. $(3x^2y^3) \cdot (4xy^2)$
38. $(4xyz) \cdot (x^2y^3) \cdot (2yz^4)$
39. $(2a^3b^3)^2$
40. $(-8x)^3(5x)^2$
41. $(4a^2)(-2a^3)^4$
42. $(12xy)(12xy)^2$
43. $(2xy^2)(-x^2y)^2(3x^2y^2)$

Mixed Review

44. How many ways can you choose a 4-person committee from seven people?
45. Three canoes cross a finish line to earn medals. Is this an example of a permutation or a combination? How many ways are possible?
46. Find the slope between (–9, 11) and (18, 6).
47. Name the number set(s) to which $\sqrt{36}$ belongs.
48. Simplify $\sqrt{74x^2}$.
49. 78 is 10% of what number?
50. Write the equation for the line containing (5, 3) and (3, 1).

3.3 Exponential Properties Involving Quotients

In this lesson, you will learn how to simplify quotients of numbers and variables.

Quotient of Powers Property: For all real numbers χ, $\frac{\chi^n}{\chi^m} = \chi^{n-m}$.

When dividing expressions with the same base, keep the base and subtract the exponent in the denominator (bottom) from the exponent in the numerator (top). When we have problems with different bases, we apply the rule separately for each base. To simplify $\frac{x^7}{x^4}$, repeated multiplication can be used.

$$\frac{x^7}{x^4} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = \frac{x \cdot x \cdot x}{1} = x^3$$

$$\frac{x^5y^3}{x^3y^2} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}} \cdot \frac{\cancel{y} \cdot \cancel{y} \cdot y}{\cancel{y} \cdot \cancel{y}} = \frac{x \cdot x}{1} \cdot \frac{y}{1} = x^2y \text{ OR } \frac{x^5y^3}{x^3y^2} = x^{5-3} \cdot y^{3-2} = x^2y$$

Example 1: *Simplify each of the following expressions using the* ***quotient rule.***

(a) $\frac{x^{10}}{x^5}$

(b) $\frac{x^5y^4}{x^3y^2}$

Solution:

(a) $\frac{x^{10}}{x^5} = x^{10-5} = x^5$

(b) $\frac{x^5y^4}{x^3y^2} = x^{5-3} \cdot y^{4-2} = x^2y^2$

Power of a Quotient Property: $\left(\frac{x^n}{y^m}\right)^p = \frac{x^{n \cdot p}}{y^{m \cdot p}}$

The power inside the parenthesis for the numerator and the denominator multiplies with the power outside the parenthesis. The situation below shows why this property is true.

$$\left(\frac{x^3}{y^2}\right)^4 = \left(\frac{x^3}{y^2}\right) \cdot \left(\frac{x^3}{y^2}\right) \cdot \left(\frac{x^3}{y^2}\right) \cdot \left(\frac{x^3}{y^2}\right) = \frac{(x \cdot x \cdot x) \cdot (x \cdot x \cdot x) \cdot (x \cdot x \cdot x) \cdot (x \cdot x \cdot x)}{(y \cdot y) \cdot (y \cdot y) \cdot (y \cdot y) \cdot (y \cdot y)} = \frac{x^{12}}{y^8}$$

Example 2: *Simplify the following expression.*

$$\left(\frac{x^{10}}{y^5}\right)^3$$

Solution: $\left(\frac{x^{10}}{y^5}\right)^3 = \frac{x^{10 \cdot 3}}{y^{5 \cdot 3}} = \frac{x^{30}}{y^{15}}$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Exponent Properties Involving Quotients (9:22)

Evaluate the following expressions.

1. $\frac{5^6}{5^2}$
2. $\frac{6^7}{6^3}$
3. $\frac{3^{10}}{3^4}$
4. $\left(\frac{2^2}{3^3}\right)^3$

Simplify the following expressions.

Figure 3.2: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/783

5. $\frac{a^3}{a^2}$
6. $\frac{x^9}{x^5}$
7. $\frac{x^{10}}{x^5}$
8. $\frac{a^6}{a}$
9. $\frac{a^5b^4}{a^3b^2}$
10. $\frac{4^5}{4^2}$
11. $\frac{5^3}{5^7}$
12. $\left(\frac{3^4}{5^2}\right)^2$
13. $\left(\frac{a^3b^4}{a^2b}\right)^3$
14. $\frac{x^6y^5}{x^2y^3}$
15. $\frac{6x^2y^3}{2xy^2}$
16. $\left(\frac{2a^3b^3}{8a^7b}\right)^2$
17. $(x^2)^2 \cdot \frac{x^6}{x^4}$
18. $\left(\frac{16a^2}{4b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$
19. $\frac{6a^3}{2a^2}$
20. $\frac{15x^5}{5x}$
21. $\left(\frac{18a^{10}}{15a^4}\right)^4$
22. $\frac{25yx^6}{20y^5x^2}$
23. $\left(\frac{x^6y^2}{x^4y^4}\right)^3$
24. $\left(\frac{6a^2}{4b^4}\right)^2 \cdot \frac{5b}{3a}$
25. $\frac{(3ab)^2(4a^3b^4)^3}{(6a^2b)^4}$
26. $\frac{(2a^2bc^2)(6abc^3)}{4ab^2c}$

Mixed Review

27. Evaluate $x|z| - |z|$ when $x = 8$ and $z = -4$.
28. Graph the solution set to the system $\begin{cases} y < -x - 2 \\ y \geq -6x + 3 \end{cases}$.
29. Evaluate $\binom{8}{4}$.
30. Make up a situation that can be solved by 4!
31. Write the following as an algebraic sentence: *A number cubed is 8.*

3.4 Zero, Negative, and Fractional Exponents

In the previous lessons, we have dealt with powers that are positive whole numbers. In this lesson, you will learn how to solve expressions when the exponent is zero, negative, or a fractional number.

Exponents of Zero: For all real numbers $\chi, \chi \neq 0, \chi^0 = 1$.

Example: $\frac{\chi^4}{\chi^4} = \chi^{4-4} = \chi^0 = 1$. This example is simplified using the Quotient of Powers Property.

Simplifying Expressions with Negative Exponents

The next objective is **negative exponents.** When we use the quotient rule and we subtract a greater number from a smaller number, the answer will become negative. The variable and the power will be moved to the denominator of a fraction. You will learn how to write this in an expression.

Example: $\frac{x^4}{x^6} = x^{4-6} = x^{-2} = \frac{1}{x^2}$. Another way to look at this is $\frac{\chi \cdot \chi \cdot \chi \cdot \chi}{\chi \cdot \chi \cdot \chi \cdot \chi \cdot \chi \cdot \chi}$. The four χs on top will cancel out with four χs on bottom. This will leave two χs remaining on the bottom, which makes your answer look like $\frac{1}{\chi^2}$.

Negative Power Rule for Exponents: $\frac{1}{\chi^n} = \chi^{-n}$ where $\chi \neq 0$

Example: $\chi^{-6}\gamma^{-2} = \frac{1}{\chi^6} \cdot \frac{1}{\gamma^2} = \frac{1}{\chi^6\gamma^2}$. The negative power rule for exponents is applied to both variables separately in this example.

Multimedia Link: For more help with these types of exponents, watch this http://www.phschool.com/atschool/academy123/html/bbapplet/content/wl-book-demo/ph-241s.html - PH School video or visit the http://www.mathsisfun.com/algebra/negative-exponents.html - mathisfun website.

Example 1: *Write the following expressions without fractions.*

(a) $\frac{2}{x^2}$

(b) $\frac{x^2}{y^3}$

Solution:

(a) $\frac{2}{x^2} = 2x^{-2}$

(b) $\frac{x^2}{y^3} = x^2y^{-3}$

Notice in Example 1(a), the number 2 is in the numerator. This number is multiplied with χ^{-2}. It could also look like this, $2 \cdot \frac{1}{\chi^2}$ to be better understood.

Simplifying Expressions with Fractional Exponents

The next objective is to be able to use fractions as exponents in an expression.

Roots as Fractional Exponents: $\sqrt[m]{a^n} = a^{\frac{n}{m}}$

Example: $\sqrt{a} = a^{\frac{1}{2}}$, $\sqrt[3]{a} = a^{\frac{1}{3}}$, $\sqrt[5]{a^2} = (a^2)^{\frac{1}{5}} = a^{\frac{2}{5}} = a^{\frac{2}{5}}$

Example 2: *Simplify the following expressions.*

(a) $\sqrt[3]{\chi}$

(b) $\sqrt[4]{\chi^3}$

Solution:

(a) $\chi^{\frac{1}{3}}$

(b) $\chi^{\frac{3}{4}}$

It is important when evaluating expressions that you remember the Order of Operations. Evaluate what is inside the parentheses, then evaluate the exponents, then perform multiplication/division from left to right, then perform addition/subtraction from left to right.

Example 3: *Evaluate the following expression.*

(a) $3 \cdot 5^2 - 10 \cdot 5 + 1$

Solution: $3 \cdot 5^2 - 10 \cdot 5 + 1 = 3 \cdot 25 - 10 \cdot 5 + 1 = 75 - 50 + 1 = 26$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Zero, Negative, and Fractional Exponents (14:04)

Figure 3.3: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/784

Simplify the following expressions. Be sure the final answer includes only positive exponents.

1. $x^{-1} \cdot y^2$
2. x^{-4}
3. $\frac{x^{-3}}{x^{-7}}$
4. $\frac{1}{x}$
5. $\frac{2}{x^2}$
6. $\frac{x^2}{y^3}$
7. $\frac{3}{xy}$
8. $3x^{-3}$
9. $a^2b^{-3}c^{-1}$
10. $4x^{-1}y^3$
11. $\frac{2x^{-2}}{y^{-3}}$
12. $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}}$
13. $\left(a^{\frac{1}{3}}\right)^2$
14. $\frac{a^{\frac{5}{2}}}{a^{\frac{1}{2}}}$
15. $\left(\frac{x^2}{y^3}\right)^{\frac{1}{3}}$
16. $\frac{x^{-3}y^{-5}}{z^{-7}}$

17. $(x^{\frac{1}{2}}y^{-\frac{2}{3}})(x^2y^{\frac{1}{3}})$
18. $\left(\frac{a}{b}\right)^{-2}$
19. $(3a^{-2}b^2c^3)^3$
20. $x^{-3} \cdot x^3$

Simplify the following expressions without any fractions in the answer.

21. $\frac{a^{-3}(a^5)}{a^{-6}}$
22. $\frac{5x^6y^2}{x^8y}$
23. $\frac{(4ab^6)^3}{(ab)^5}$
24. $\left(\frac{3x}{y^{\frac{1}{3}}}\right)^3$
25. $\frac{4a^2b^3}{2a^5b}$
26. $\left(\frac{x}{3y^2}\right)^3 \cdot \frac{x^2y}{4}$
27. $\left(\frac{ab^{-2}}{b^3}\right)^2$
28. $\frac{x^{-3}y^2}{x^2y^{-2}}$
29. $\frac{3x^2y^{\frac{3}{2}}}{xy^{\frac{1}{2}}}$
30. $\frac{(3x^3)(4x^4)}{(2y)^2}$
31. $\frac{a^{-2}b^{-3}}{c^{-1}}$
32. $\frac{x^{\frac{1}{2}}y^{\frac{5}{2}}}{x^{\frac{3}{2}}y^{\frac{3}{2}}}$

Evaluate the following expressions to a single number.

33. 3^{-2}
34. $(6.2)^0$
35. $8^{-4} \cdot 8^6$
36. $(16^{\frac{1}{2}})^3$
37. 5^0
38. 7^2
39. $\left(\frac{2}{3}\right)^3$
40. 3^{-3}
41. $16^{\frac{1}{2}}$
42. $8^{\frac{-1}{3}}$

In 43 – 45, evaluate the expression for $x = 2, y = -1, z = 3$.

43. $2x^2 - 3y^3 + 4z$
44. $(x^2 - y^2)^2$
45. $\left(\frac{3x^2y^5}{4z}\right)^{-2}$
46. Evaluate $x^2 4x^3 y^4 4y^2$ if $x = 2$ and $y = -1$.
47. Evaluate $a^4(b^2)^3 + 2ab$ if $a = -2$ and $b = 1$.
48. Evaluate $5x^2 - 2y^3 + 3z$ if $x = 3,\ y = 2$, and $z = 4$.
49. Evaluate $\left(\frac{a^2}{b^3}\right)^{-2}$ if $a = 5$ and $b = 3$.

50. Evaluate $3 \cdot 5^5 - 10 \cdot 5 + 1$.
51. Evaluate $\frac{2 \cdot 4^2 - 3 \cdot 5^2}{3^2}$.
52. Evaluate $\left(\frac{3^3}{2^2}\right)^{-2} \cdot \frac{3}{4}$.

Mixed Review

53. A quiz has ten questions: 7 true/false and 3 multiple choice. The multiple choice questions each have four options. How many ways can the test be answered?
54. Simplify $3a^4b^4 \cdot a^{-3}b^{-4}$.
55. Simplify $(x^4y^2 \cdot xy^0)^5$.
56. Simplify $\frac{v^2}{-vu^{-2} \cdot u^{-1}v^4}$.
57. Solve for $n : -6(4n + 3) = n + 32$.

3.5 Scientific Notation

Sometimes in mathematics numbers are huge. They are so huge that we use what is called scientific notation. It is easier to work with such numbers when we shorten their decimal places and multiply them by 10 to a specific power. In this lesson, you will learn how to use scientific notation by hand and on a calculator.

Powers of 10:

$$\begin{aligned} 100,000 &= 10^5 \\ 10,000 &= 10^4 \\ 1,000 &= 10^3 \\ 100 &= 10^2 \\ 10 &= 10^1 \end{aligned}$$

Using Scientific Notation for Large Numbers

Example: If we divide 643,297 by 100,000 we get 6.43297. If we multiply this quotient by 100,000, we get back to our original number. But we have just seen that 100,000 is the same as 10^5, so if we multiply 6.43297 by 10^5, we should also get our original answer. In other words $6.43297 \times 10^5 = 643,297$ Because there are five zeros, the decimal moves over five places.

Solution: Look at the following examples:

$$\begin{aligned} 2.08 \times 10^4 &= 20,800 \\ 2.08 \times 10^3 &= 2,080 \\ 2.08 \times 10^2 &= 208 \\ 2.08 \times 10^1 &= 20.8 \\ 2.08 \times 10^0 &= 2.08 \end{aligned}$$

The power tells how many decimal places to move; positive powers mean the decimal moves to the right. A positive 4 means the decimal moves four positions the right.

Example 1: *Write in scientific notation.*

653,937,000

Solution: $653,937,000 = 6.53937000 \times 100,000,000 = 6.53937 \times 10^8$

Oftentimes we do not keep more than a few decimal places when using scientific notation and we round the number to the nearest whole number, tenth, or hundredth depending on what the directions say. Rounding Example 1 could look like 6.5×10^8.

Using Scientific Notation for Small Numbers

We've seen that scientific notation is useful when dealing with large numbers. It is also good to use when dealing with extremely small numbers.

Look at the following examples:

$$\begin{aligned}
2.08 \times 10^{-1} &= 0.208 \\
2.08 \times 10^{-2} &= 0.0208 \\
2.08 \times 10^{-3} &= 0.00208 \\
2.08 \times 10^{-4} &= 0.000208
\end{aligned}$$

Example 2: *The time taken for a light beam to cross a football pitch is* 0.0000004 *seconds. Write in scientific notation.*

Solution: $0.0000004 = 4 \times 0.0000001 = 4 \times \frac{1}{10,000,000} = 4 \times \frac{1}{10^7} = 4 \times 10^{-7}$

Evaluating Expressions Using Scientific Notation

When evaluating expressions with scientific notation, it is easiest to keep the powers of 10 together and deal with them separately.

Example: $(3.2 \times 10^6) \cdot (8.7 \times 10^{11}) = 3.2 \times 8.7 \cdot 10^6 \times 10^{11} = 27.84 \times 10^{17} = 2.784 \times 10^1 \times 10^{17} = 2.784 \times 10^{18}$

Solution: It is best to keep one number before the decimal point. In order to do that, we had to make 27.84 become 2.784×10^1 so we could evaluate the expression more simply.

Example 3: *Evaluate the following expression.*

(a) $(1.7 \times 10^6) \cdot (2.7 \times 10^{-11})$

(b) $(3.2 \times 10^6) \div (8.7 \times 10^{11})$

Solution:

(a) $(1.7 \times 10^6) \cdot (2.7 \times 10^{-11}) = 1.7 \times 2.7 \cdot 10^6 \times 10^{-11} = 4.59 \times 10^{-5}$

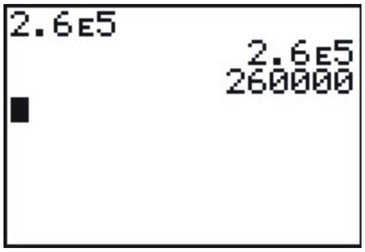
(b) $(3.2 \times 10^6) \div (8.7 \times 10^{11}) = \frac{3.2 \times 10^6}{8.7 \times 10^{11}} = \frac{3.2}{8.7} \times \frac{10^6}{10^{11}} = 0.368 \times 10^{6-11} = 3.68 \times 10^{-1} \times 10^{-5} = 3.68 \times 10^{-6}$

You must remember to keep the powers of ten together, and have 1 number before the decimal.

Scientific Notation Using a Calculator

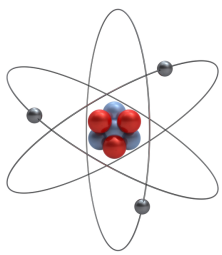
Scientific and graphing calculators make scientific notation easier. To compute scientific notation, use the [**EE**] button. This is [**2nd**] [**,**] on some TI models or [10^x], which is [**2nd**] [**log**].

For example to enter 2.6×10^5 enter 2.6 [**EE**] 5.



When you hit [**ENTER**] the calculator displays $2.6E5$ if it's set in **Scientific** mode OR it displays 260,000 if it's set in **Normal** mode.

Solving Real-World Problems Using Scientific Notation



Example: *The mass of a single lithium atom is approximately one percent of one millionth of one billionth of one billionth of one kilogram. Express this mass in scientific notation.*

Solution: We know that percent means we divide by 100, and so our calculation for the mass (in kg) is $\frac{1}{100} \times \frac{1}{1{,}000{,}000} \times \frac{1}{1{,}000{,}000{,}000} \times \frac{1}{1{,}000{,}000{,}000} = 10^{-2} \times 10^{-6} \times 10^{-9} \times 10^{-9}$

Next, we use the product of powers rule we learned earlier in the chapter.

$$10^{-2} \times 10^{-6} \times 10^{-9} \times 10^{-9} = 10^{((-2)+(-6)+(-9)+(-9))} = 10^{-26} \ kg.$$

The mass of one lithium atom is approximately $1 \times 10^{-26} \ kg$.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Scientific Notation (14:26)

Write the numerical value of the following expressions.

1. 3.102×10^2
2. 7.4×10^4
3. 1.75×10^{-3}
4. 2.9×10^{-5}
5. 9.99×10^{-9}
6. $(3.2 \times 10^6) \cdot (8.7 \times 10^{11})$

Figure 3.4: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/785

7. $(5.2 \times 10^{-4}) \cdot (3.8 \times 10^{-19})$
8. $(1.7 \times 10^{6}) \cdot (2.7 \times 10^{-11})$
9. $(3.2 \times 10^{6}) \div (8.7 \times 10^{11})$
10. $(5.2 \times 10^{-4}) \div (3.8 \times 10^{-19})$
11. $(1.7 \times 10^{6}) \div (2.7 \times 10^{-11})$

Write the following numbers in scientific notation.

12. 120,000
13. 1,765,244
14. 63
15. 9,654
16. 653,937,000
17. 1,000,000,006
18. 12
19. 0.00281
20. 0.000000027
21. 0.003
22. 0.000056
23. 0.00005007
24. 0.00000000000954
25. The moon is approximately a sphere with radius $r = 1.08 \times 10^3$ miles. Use the formula Surface Area $= 4\pi r^2$ to determine the surface area of the moon, in square miles. Express your answer in scientific notation, rounded to two significant figures.
26. The charge on one electron is approximately 1.60×10^{-19} coulombs. One **Faraday** is equal to the total charge on 6.02×10^{23} electrons. What, in coulombs, is the charge on one Faraday?
27. Proxima Centauri, the next closest star to our Sun, is approximately 2.5×10^{13} miles away. If light from Proxima Centauri takes 3.7×10^4 hours to reach us from there, calculate the speed of light in miles per hour. Express your answer in scientific notation, rounded to two significant figures.

Mixed Review

28. 14 milliliters of a 40% sugar solution was mixed with 4 milliliters of pure water. What is the concentration of the mixture?
29. Solve the system $\begin{cases} 6x + 3y + 18 \\ -15 = 11y - 5x \end{cases}$.
30. Graph the function by creating a table: $f(x) = 2x^2$. Use the following values for $x : -5 \leq x \leq 5$.
31. Simplify $\frac{5a^6b^2c^{-6}}{a^{11}b}$. Your answer should have only positive exponents.

32. Each year Americans produce about 230 million tons of trash (Source: http://www.learner.org/interactives/garba
There are 307,006,550 people in the United States. How much trash is produced per person per year?
33. The volume of a 3-dimesional box is given by the formula: $V = l(w)(h)$, where $l = length$, $w = width$, and $h = height$ of the box. The box holds 312 cubic inches and has a length of 12 inches and a width of 8 inches. How tall is the box?

Quick Quiz

1. Simplify: $\frac{(2x^{-4}y^3)^{-3} \cdot x^{-3}y^{-2}}{-2x^0y^2}$.
2. The formula $A = 1,500(1.0025)^t$ gives the total amount of money in a bank account with a balance of \$1,500.00, earning 0.25% interest, compounded annually. How much money would be in the account five years in the past?
3. *True or false?* $\left(\frac{5}{4}\right)^{-3} = -\frac{125}{64}$

3.6 Exponential Growth Functions

In previous lessons, we have seen the variable as the base. In exponential functions, the exponent is the variable and the base is a constant.

General Form of an Exponential Function: $y = a(b)^x$, where $a = initial\ value$ and

$$b = growth\ factor$$

In exponential growth situations, the growth factor must be greater than one.

$$b > 1$$

Example: *A colony of bacteria has a population of 3,000 at noon on Sunday. During the next week, the colony's population doubles every day. What is the population of the bacteria colony at noon on Saturday?*

Solution: Make a table of values and calculate the population each day.

Table 3.1:

Day	0 (Sun)	1 (Mon)	2 (Tues)	3 (Wed)	4 (Thurs)	5 (Fri)	6 (Sat)
Population (thousands)	3	6	12	24	48	96	192

To get the population of bacteria for the next day we multiply the current day's population by 2 because it doubles every day. If we define x as the number of days since Sunday at noon, then we can write the following: $P = 3 \cdot 2^x$. This is a formula that we can use to calculate the population on any day. For instance, the population on Saturday at noon will be $P = 3 \cdot 2^6 = 3 \cdot 64 = 192$ thousand bacteria. We use $x = 6$, since Saturday at noon is six days after Sunday at noon.

Graphing Exponential Functions

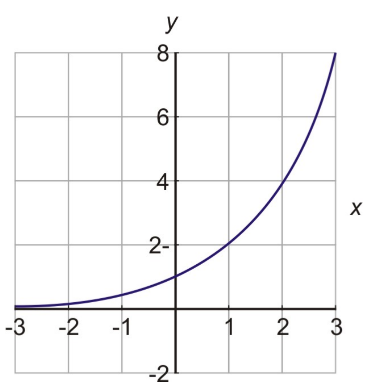
Example: Graph the equation using a table of values $y = 2^x$.

Solution: Make a table of values that includes both negative and positive values of x. Substitute these values for x to get the value for the y variable.

Table 3.2:

x	y
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

Plot the points on the coordinate axes to get the graph below. Exponential functions always have this basic shape: They start very small and then once they start growing, they grow faster and faster, and soon they become huge.



Comparing Graphs of Exponential Functions

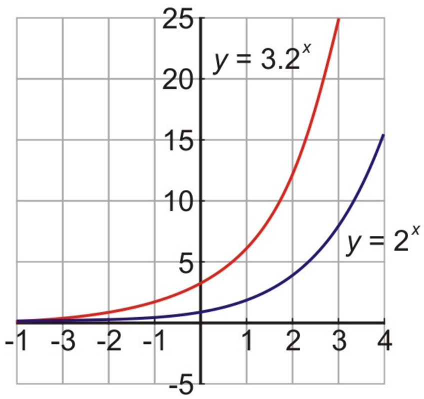
The shape of the exponential graph changes if the constants change. The curve can become steeper or shallower.

Earlier in the lesson, we produced a graph for $y = 2^x$. Let's compare that graph with the graph of $y = 3 \cdot 2^x$.

Table 3.3:

x	y
-2	$3 \cdot 2^{-2} = 3 \cdot \frac{1}{2^2} = \frac{3}{4}$
-1	$3 \cdot 2^{-1} = 3 \cdot \frac{1}{2^1} = \frac{3}{2}$
0	$3 \cdot 2^0 = 3$
1	$3 \cdot 2^1 = 6$
2	$3 \cdot 2^2 = 3 \cdot 4 = 12$
3	$3 \cdot 2^3 = 3 \cdot 8 = 24$

We can see that the function $y = 3 \cdot 2^x$ is bigger than the function $y = 2^x$. In both functions, the value of y doubles every time x increases by one. However, $y = 3 \cdot 2^x$ starts with a value of 3, while $y = 2^x$ starts with a value of 1, so it makes sense that $y = 3 \cdot 2^x$ would be bigger.



Solving Real-World Problems with Exponential Growth

Example: *The population of a town is estimated to increase by 15% per year. The population today is 20,000. Make a graph of the population function and find out what the population will be ten years from now.*

Solution: The population is growing at a rate of 15% each year. When something grows at a percent, this is a clue to use **exponential functions.**

Remember, the general form of an exponential function is $y = a(b)^x$, where a is the beginning value and b is the total growth rate. The beginning value is 20,000. Therefore, $a = 20,000$.

The population is keeping the original number of people and adding 15% more each year.

$$100\% + 15\% = 115\% = 1.15$$

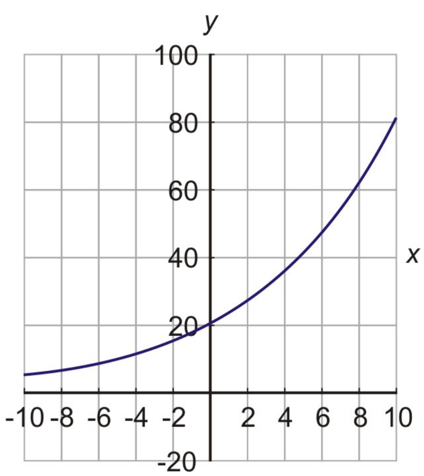
Therefore, the population is growing at a rate of 115% each year. Thus, $b = 1.15$.

The function to represent this situation is $y = 20,000\ (1.15)^x$.

Now make a table of values and graph the function.

Table 3.4:

x	$y = 20 \cdot (1.15)^x$
–10	4.9
–5	9.9
0	20
5	40.2
10	80.9



Notice that we used negative values of x in our table. Does it make sense to think of negative time? In this case $x = -5$ represents what the population was five years ago, so it can be useful information.

The question asked in the problem was *";What will be the population of the town ten years from now?";* To find the population exactly, we use $x = 10$ in the formula. We find $y = 20,000 \cdot (1.15)^{10} = 80,912$.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Exponential Growth Functions (7:41)

1. What is the general equation for an exponential equation? What do the variables represent?
2. How is an exponential growth equation different from a linear equation?
3. What is true about the growth factor of an exponential equation?

Figure 3.5: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/787

4. *True or false?* An exponential growth function has the following form: $f(x) = a(b)^x$, *where* $a > 1$ *and* $b < 1$?
5. What is the $y-$intercept of all exponential growth functions?

Graph the following exponential functions by making a table of values.

6. $y = 3^x$
7. $y = 2^x$
8. $y = 5 \cdot 3^x$
9. $y = \frac{1}{2} \cdot 4^x$
10. $f(x) = \frac{1}{3} \cdot 7^x$
11. $f(x) = 2 \cdot 3^x$
12. $y = 40 \cdot 4^x$
13. $y = 3 \cdot 10^x$

Solve the following problems.

14. The population of a town in 2007 is 113,505 and is increasing at a rate of 1.2% per year. What will the population be in 2012?
15. A set of bacteria begins with 20 and doubles every 2 hours. How many bacteria would be present 15 hours after the experiment began?
16. The cost of manufactured goods is rising at the rate of inflation, about 2.3%. Suppose an item costs \$12 today. How much will it cost five years from now due to inflation?
17. A chain letter is sent out to 10 people telling everyone to make 10 copies of the letter and send each one to a new person. Assume that everyone who receives the letter sends it to 10 new people and that it takes a week for each cycle. How many people receive the letter in the sixth week?
18. Nadia received \$200 for her 10^{th} birthday. If she saves it in a bank with a 7.5% interest rate compounded yearly, how much money will she have in the bank by her 21^{st} birthday?

Mixed Review

19. Suppose a letter is randomly chosen from the alphabet. What is the probability the letter chosen is M, K, or L?
20. Evaluate $t^4 \cdot t^{\frac{1}{2}}$ when $t = 9$.
21. Simplify $28 - (x - 16)$.
22. Graph $y - 1 = \frac{1}{3}(x + 6)$.

3.7 Exponential Decay Functions

In the last lesson, we learned how to solve expressions that modeled exponential growth. In this lesson, we will be learning about exponential decay functions.

General Form of an Exponential Function: $y = a(b)^x$, where $a =$ *initial value* and

$$b = growth\ factor$$

In exponential decay situations, the growth factor must be a fraction between zero and one.

$$0$$

Example: *For her fifth birthday, Nadia's grandmother gave her a full bag of candy. Nadia counted her candy and found out that there were 160 pieces in the bag. Nadia loves candy, so she ate half the bag on the first day. Her mother told her that if she continues to eat at that rate, it will be gone the next day and she will not have any more until her next birthday. Nadia devised a clever plan. She will always eat half of the candy that is left in the bag each day. She thinks that she will get candy every day and her candy will never run out. How much candy does Nadia have at the end of the week? Would the candy really last forever?*

Solution: Make a table of values for this problem.

Table 3.5:

Day	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
# of Candies	160	80	40	20	10	5	2.5	1.25

You can see that if Nadia eats half the candies each day, then by the end of the week she has only 1.25 candies left in her bag.

Write an equation for this exponential function. Nadia began with 160 pieces of candy. In order to get the amount of candy left at the end of each day, we keep multiplying by $\frac{1}{2}$. Because it is an exponential function, the equation is:

$$y = 160 \cdot \frac{1}{2}^x$$

Graphing Exponential Decay Functions

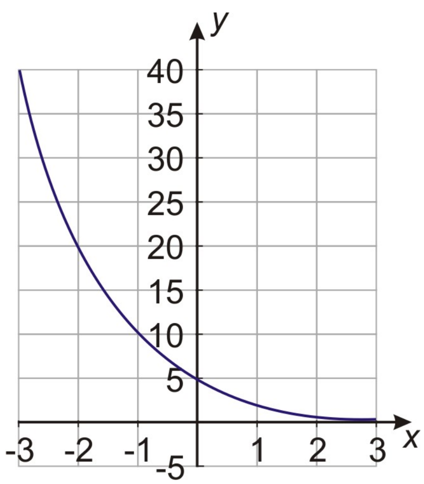
Example: *Graph the exponential function* $y = 5 \cdot \left(\frac{1}{2}\right)^x$.

Solution: Start by making a table of values. Remember when you have a number to the negative power, you are simply taking the reciprocal of that number and taking it to the positive power. Example: $\left(\frac{1}{2}\right)^{-2} = \left(\frac{2}{1}\right)^{2} = 2^2$.

Table 3.6:

x	$y = 5 \cdot \left(\frac{1}{2}\right)^x$
–3	$y = 5\left(\frac{1}{2}\right)^{-3} = 40$
–2	$y = 5\left(\frac{1}{2}\right)^{-2} = 20$
–1	$y = 5\left(\frac{1}{2}\right)^{-1} = 10$
0	$y = 5\left(\frac{1}{2}\right)^{0} = 5$
1	$y = 5\left(\frac{1}{2}\right)^{1} = \frac{5}{2}$
2	$y = 5\left(\frac{1}{2}\right)^{2} = \frac{5}{4}$

Now graph the function.



Using the Property of Negative Exponents, the equation can also be written as $5 \cdot 2^{-x}$.

Comparing Graphs of Exponential Decay Functions

Exponential growth and decay graphs look like opposites and can sometimes be mirror images.

Example: *Graph the functions* $y = 4^x$ *and* 4^{-x} *on the same coordinate axes.*

Solution: Here is the table of values and the graph of the two functions.

Looking at the values in the table, we see that the two functions are ";reverse images"; of each other in the sense that the values for the two functions are reciprocals.

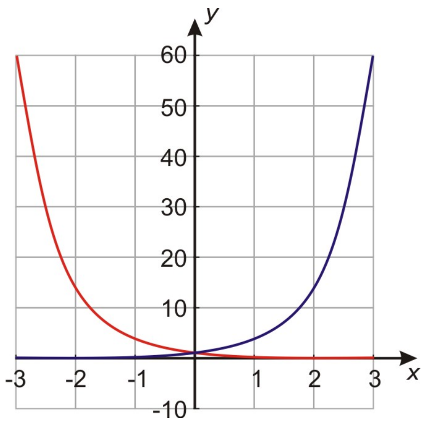
Table 3.7:

x	$y = 4^x$	$y = 4^{-x}$
–3	$y = 4^{-3} = \frac{1}{64}$	$y = 4^{-(-3)} = 64$
–2	$y = 4^{-2} = \frac{1}{16}$	$y = 4^{-(-2)} = 16$

Table 3.7: (continued)

x	$y = 4^x$	$y = 4^{-x}$
-1	$y = 4^{-1} = \frac{1}{4}$	$y = 4^{-(-1)} = 4$
0	$y = 4^0 = 1$	$y = 4^{-(0)} = 1$
1	$y = 4^1 = 4$	$y = 4^{-(1)} = \frac{1}{4}$
2	$y = 4^2 = 16$	$y = 4^{-(2)} = \frac{1}{16}$
3	$y = 4^3 = 64$	$y = 4^{-(3)} = \frac{1}{64}$

Here is the graph of the two functions. Notice that these two functions are mirror images if the mirror is placed vertically on the $y-$axis.



Solving Real-World Problems Involving Exponential Decay Functions

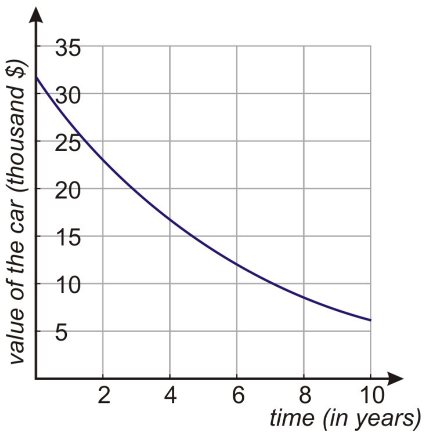
Example: *The cost of a new car is* \$32,000. *It depreciates at a rate of* 15% *per year. This means that it loses* 15% *of its value each year.*

- *Draw the graph of the car's value against time in years.*
- *Find the formula that gives the value of the car in terms of time.*
- *Find the value of the car when it is four years old.*

Solution: Start by making a table of values. To fill in the values we start with 32,000 when $t = 0$. Then we multiply the value of the car by 85% for each passing year. (Since the car loses 15% of its value, it keeps 85% of its value). Remember $85\% = 0.85$.

Table 3.8:

Time	**Value (Thousands)**
0	32
1	27.2
2	23.1
3	19.7
4	16.7
5	14.2



The general formula is $y = a(b)^x$.

In this case: y is the value of the car, x is the time in years, $a = 32,000$ is the starting amount in thousands, and $b = 0.85$ since we multiply the value in any year by this factor to get the value of the car in the following year. The formula for this problem is $y = 32,000(0.85)^x$.

Finally, to find the value of the car when it is four years old, we use $x = 4$ in the formula. Remember the value is in thousands.

$$y = 32,000(0.85)^4 = 16,704.$$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Exponential Decay Functions (10:51)

Figure 3.6: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/788

1. Define *exponential decay.*
2. What is true about *";b";* in an exponential decay function?
3. Suppose $f(x) = a(b)^x$. What is $f(0)$? What does this mean in terms of the y−intercept of an exponential function?

Graph the following exponential decay functions.

4. $y = \frac{1}{5}^x$
5. $y = 4 \cdot \left(\frac{2}{3}\right)^x$
6. $y = 3^{-x}$
7. $y = \frac{3}{4} \cdot 6^{-x}$

Solve the following application problems.

8. The cost of a new ATV (all-terrain vehicle) is $7200. It depreciates at 18% per year.
   (a) Draw the graph of the vehicle's value against time in years.
   (b) Find the formula that gives the value of the ATV in terms of time.
   (c) Find the value of the ATV when it is ten years old.
9. Michigan's population is declining at a rate of 0.5% per year. In 2004, the state had a population of 10,112,620.
   (a) Write a function to express this situation.
   (b) If this rate continues, what will the population be in 2012?
   (c) When will the population of Michigan reach 9,900,000?
   (d) What was the population in the year 2000, according to this model?
10. A certain radioactive substance has a half-life of 27 years. An organism contains 35 grams of this substance on day zero.
    (a) Draw the graph of the amount remaining. Use these values for $x : x = 0, 27, 54, 81, 108, 135$.
    (b) Find the function that describes the amount of this substance remaining after x days.
    (c) Find the amount of radioactive substance after 92 days.
11. The percentage of light visible at d meters is given by the function $V(d) = 0.70^d$.
    (a) What is the growth factor?
    (b) What is the initial value?
    (c) Find the percentage of light visible at 65 meters.
12. A person is infected by a certain bacterial infection. When he goes to the doctor, the population of bacteria is 2 million. The doctor prescribes an antibiotic that reduces the bacteria population to $\frac{1}{4}$ of its size each day.
    (a) Draw the graph of the size of the bacteria population against time in days.
    (b) Find the formula that gives the size of the bacteria population in terms of time.
    (c) Find the size of the bacteria population ten days after the drug was first taken.
    (d) Find the size of the bacteria population after two weeks (14 days).

Mixed Review

13. The population of Kindly, USA is increasing at a rate of 2.14% each year. The population in the year 2010 is 14,578.
    (a) Write an equation to model this situation.
    (b) What would the population of Kindly be in the year 2015?
    (c) When will the population be 45,000?
14. The volume of a sphere is given by the formula $v = \frac{4}{3}\pi r^3$. Find the volume of a sphere with a diameter of 11 inches.
15. Simplify $\frac{6x^2}{14y^3} \cdot \frac{7y}{x^8} \cdot x^0 y$.
16. Simplify $3(x^2y^3x)^2$.
17. Rewrite in standard form: $y - 16 + x = -4x + 6y + 1$.

3.8 Problem-Solving Strategies

We have to deal with problem solving in everyday life. Therefore, it is important to know the steps you must take when problem solving.

Example: *Suppose $4000 is invested at a 6% interest rate compounded annually. How much money will there be in the bank at the end of five years? At the end of 20 years?*

Solution: Read the problem and summarize the information.

$4000 is invested at a 6% interest rate compounded annually. We want to know how much money we have after five years.

- Assign variables. Let $x =$ *time in years* and $y =$ *amount of money in investment account.*
- We start with $4000 and each year we apply a 6% interest rate on the amount in the bank.
- The pattern is that each year we multiply the previous amount by the factor of $100\% + 6\% = 106\% = 1.06$.
- Complete a table of values.

Table 3.9:

Time (years)	**0**	**1**	**2**	**3**	**4**	**5**
Investment Amount ($)	4000	4240	4494.40	4764.06	5049.91	5352.90

Using the table, we see that at the end of five years we have $5352.90 in the investment account.

In the case of five years, we don't need an equation to solve the problem. However, if we want the amount at the end of 20 years, it becomes too difficult to constantly multiply. We can use a formula instead.

Since we take the original investment and keep multiplying by the same factor of 1.06, that means we can use exponential notation.

$$y = 4000 \cdot (1.06)^x$$

To find the amount after five years we use $x = 5$ in the equation.

$$y = 4000 \cdot (1.06)^5 = \$5352.90$$

To find the amount after 20 years we use $x = 20$ in the equation.

$$y = 4000 \cdot (1.06)^{20} = \$12,828.54$$

To check our answers we can plug in some low values of x to see if they match the values in the table:

$$x = 0 \qquad 4000 \cdot (1.06)0 = 4000$$
$$x = 1 \qquad 4000 \cdot (1.06)1 = 4240$$
$$x = 2 \qquad 4000 \cdot (1.06)2 = 4494.40$$

The answers make sense because after the first year, the amount goes up by $240 (6% of $4000). The amount of increase gets larger each year and that makes sense because the interest is 6% of an amount that is larger and larger every year.

Multimedia Link: To learn more about how to use the correct exponential function, visit the http://regentsprep.org/I - algebra lesson page by RegentsPrep.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Word Problem Solving (7:21)

Figure 3.7: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/554

Apply the problem-solving techniques described in this section to solve the following problems.

1. **Half-life** Suppose a radioactive substance decays at a rate of 3.5% per hour. What percent of the substance is left after six hours?
2. **Population decrease** In 1990, a rural area has 1200 bird species. If species of birds are becoming extinct at the rate of 1.5% per decade (10 years), how many bird species will there be left in year 2020?
3. **Growth** Nadia owns a chain of fast food restaurants that operated 200 stores in 1999. If the rate of increase is 8% annually, how many stores does the restaurant operate in 2007?
4. **Investment** Peter invests $360 in an account that pays 7.25% compounded annually. What is the total amount in the account after 12 years?

3.9 Chapter Review

Define the following words.

1. Exponent
2. Geometric Sequence

Use the product properties to simplify the following expressions.

3. $5 \cdot 5 \cdot 5 \cdot 5$

4. $(3x^2y^3) \cdot (4xy^2)$
5. $a^3 \cdot a^5 \cdot a^6$
6. $(y^3)^5$
7. $(x \cdot x^3 \cdot x^5)^{10}$
8. $(2a^3b^3)^2$

Use the quotient properties to simplify the following expressions.

9. $\frac{c^5}{c^3}$
10. $\frac{a^6}{a}$
11. $\frac{a^5b^4}{a^3b^2}$
12. $\frac{x^4y^5z^2}{x^3y^2z}$

Simplify the following expressions.

13. $\frac{6^5}{6^5}$
14. $\frac{y^2}{y^5}$
15. $\frac{7^3}{7^6}$
16. $\frac{2}{x^3}$
17. $\sqrt[4]{a^3}$
18. $\left(a^{\frac{1}{3}}\right)^2$
19. $\left(\frac{x^2}{y^3}\right)^{\frac{1}{3}}$

Write the following in scientific notation.

20. 557,000
21. 600,000
22. 20
23. 0.04
24. 0.0417
25. 0.0000301
26. The distance from the Earth to the moon: 384,403 km
27. The distance from Earth to Jupiter: 483,780,000 miles
28. According to the CDC, the appropriate level of lead in drinking water should not exceed 15 parts per billion (EPA's Lead & Copper Rule).

Write the following in standard notation.

29. 3.53×10^3
30. 89×10^5
31. 2.12×10^6
32. 5.4×10^1
33. 7.9×10^{-3}
34. 4.69×10^{-2}
35. 1.8×10^{-5}
36. 8.41×10^{-3}

Make a graph of the following exponential growth/decay functions.

37. $y = 3 \cdot (6)^x$
38. $y = 2 \cdot \left(\frac{1}{3}\right)^x$
39. Marissa was given 120 pieces of candy for Christmas. She ate one-fourth of them each day. Make a graph to find out in how many days Marissa will run out of candy.
40. Jacoby is given \$1500 for his graduation. He wants to invest it. The bank gives a 12% investment rate each year. Make a graph to find out how much money Jacoby will have in the bank after six years.

3.10 Chapter Test

Simplify the following expressions.

1. $x^3 \cdot x^4 \cdot x^5$
2. $(a^3)^7$
3. $(y^2z^4)^7$
4. $\frac{a^3}{a^5}$
5. $\frac{x^3y^2}{x^6y^4}$
6. $\left(\frac{3x^8y^2}{9x^6y^5}\right)^3$
7. $\frac{3^4}{3^4}$
8. $\frac{2}{x^3}$
9. $\sqrt[3]{5^6}$

Complete the following story problems.

10. The intensity of a guitar amp is 0.00002. Write this in scientific notation.
11. Cole loves turkey hunting. He already has two after his first day of the hunting season. If this number doubles each day, how many turkeys will Cole have after 11 days? Make a table for the geometric sequence.
12. The population of a town increases by 20% each year. It first started with 89 people. What will the population be of the town after 15 years?
13. A radioactive substance decays 2.5% every hour. What percent of the substance will be left after nine hours?
14. After an exterminator comes to a house to exterminate cockroaches, the bugs leave the house at a rate of 16% an hour. How long will it take 55 cockroaches to leave a house after the exterminator comes there?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9618.

Chapter 4

Logarithmic Functions

4.1 Inverse Functions

Learning objectives

- Find the inverse of a function
- Determine if a function is invertible
- State the domain and range for a function and its inverse
- Graph functions and their inverses
- Use composition to verify if two functions are inverses.

Introduction

In chapters 1 and 2 we have examined properties and applications of several families of functions. In this chapter, we will focus on two related functions: exponential functions,and logarithmic functions. These two functions have a special relationship with one another: they are **inverses** of each other. In this first lesson we will develop the idea of inverses, both algebraically and graphically, as background for studying these two types of functions in depth. We will begin with a familiar, every-day example of two functions that are inverses.

Functions and inverses

In the United States, we measure temperature using the Fahrenheit scale. In other countries, people use the Celsius scale. The equation $C = 5/9$ (F - 32) can be used to find C, the Celsius temperature, given F, the Fahrenheit temperature. If we write this equation using function notation, we have $t(x) = \frac{5}{9}(x - 32)$. The input of the function is a Fahrenheit temperature, and the output is a Celsius temperature. For example, the freezing point on the Fahrenheit scale is 32 degrees. We can find the corresponding Celsius temperature using the function:

$$t(32) = \frac{5}{9}(32 - 32) = \frac{5}{9} \cdot 0 = 0$$

This function allows us to convert a Fahrenheit temperature into Celsius, but what if we want to convert from Celsius to Fahrenheit?

Consider again the equation above: $C = \frac{5}{9}(F - 32)$. We can solve this equation to isolate F:

Table 4.1:

	$C = \frac{5}{9}(F - 32)$
	$\frac{9}{5}C = \frac{9}{5} \times \frac{5}{9}(F - 32)$ $\frac{9}{5}C = F - 32$ $\frac{9}{5}C + 32 = F$

If we write this equation using function notation, we get $f(x) = \frac{9}{5}x + 32$. For this function, the input is the Celsius temperature, and the output is the Fahrenheit temperature. For example, if $x = 0, f(0) = \frac{9}{5}(0) + 32 = 0 + 32 = 32$.

Now consider the functions $t(x) = \frac{5}{9}(x - 32)$ and $f(x) = \frac{9}{5}x + 32$ together. The input of one function is the output of the other. This is an informal way of saying that these functions are **inverses.** Formally, the inverse of a function is defined as follows:

Table 4.2:

Inverse functions

Functions $f(x)$ *and* $g(x)$ *are said to inverses if*

$$f(g(x)) = g(f(x)) = x$$

Or, using the composite function notation of Chapter 1:

$$f \circ g = g \circ f = x$$

The following notation is used to indicate inverse functions:
If $f(x)$ and $g(x)$ are inverse functions, then

$$f(x) = g^{-1}(x) \text{ and } g(x) = f^{-1}(x)$$

The following notation is also used: $f = g^{-1}$ and $g = f^{-1}$.

Note that $f^{-1}(x)$ *does* ***not*** *equal* $\frac{}{1}f(x)$.

Informally, we define the inverse of a function as the relation we obtain by switching the domain and range of the function. Because of this definition, you can find an inverse by switching the roles of x and y in an equation. For example, consider the function $g(x) = 2x$. This is the line $y = 2x$. If we switch x and y, we get the equation $x = 2y$. Dividing both sides by 2, we get $y = 1/2\ x$. Therefore the functions $g(x) = 2x$ and $y = 1/2\ x$ are inverses. Using function notation, we can write $y = 1/2\ x$ as $g^{-1}(x) = 1/2\ x$.

Example 1: Find the inverse of each function

Table 4.3:

a. $f(x) = 5x - 8$	b. $f(x) = x^3$.

Solution:

a. First write the function using "; y ="; notation, then interchange x and y:

$f(x) = 5x - 8 \rightarrow y = 5x - 8 \rightarrow x = 5y - 8$

Then isolate y:

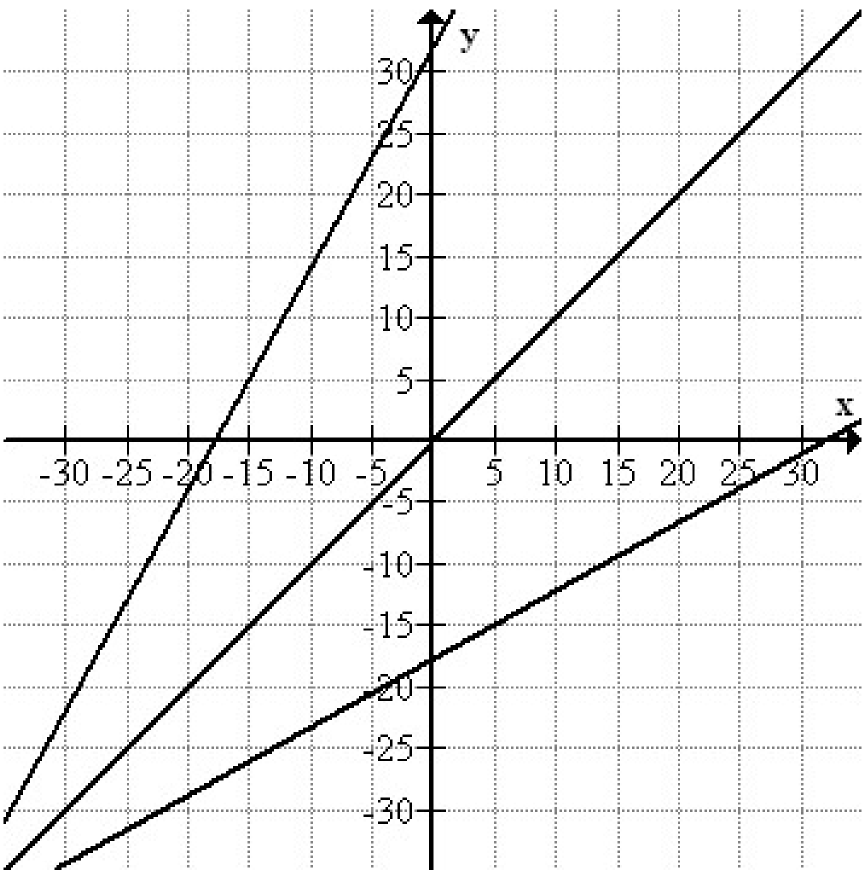
$$x = 5y - 8$$
$$x + 8 = 5y$$
$$y = \frac{1}{5}x + \frac{8}{5}$$

b.

Table 4.4:

First write the function using "; y= ";:	$f(x) = x^3$
	$y = x^3$
Now interchange x and y	$x = y^3$
Now isolate y:	$y = \sqrt[3]{x}$

Because of the definition of inverse, the graphs of inverses are reflections across the line $y = x$. The graph below shows $t(x) = \frac{5}{9}(x - 32)$ and $f(x) = \frac{9}{5}x + 32$ on the same graph, along with the reflection line $y = x$.

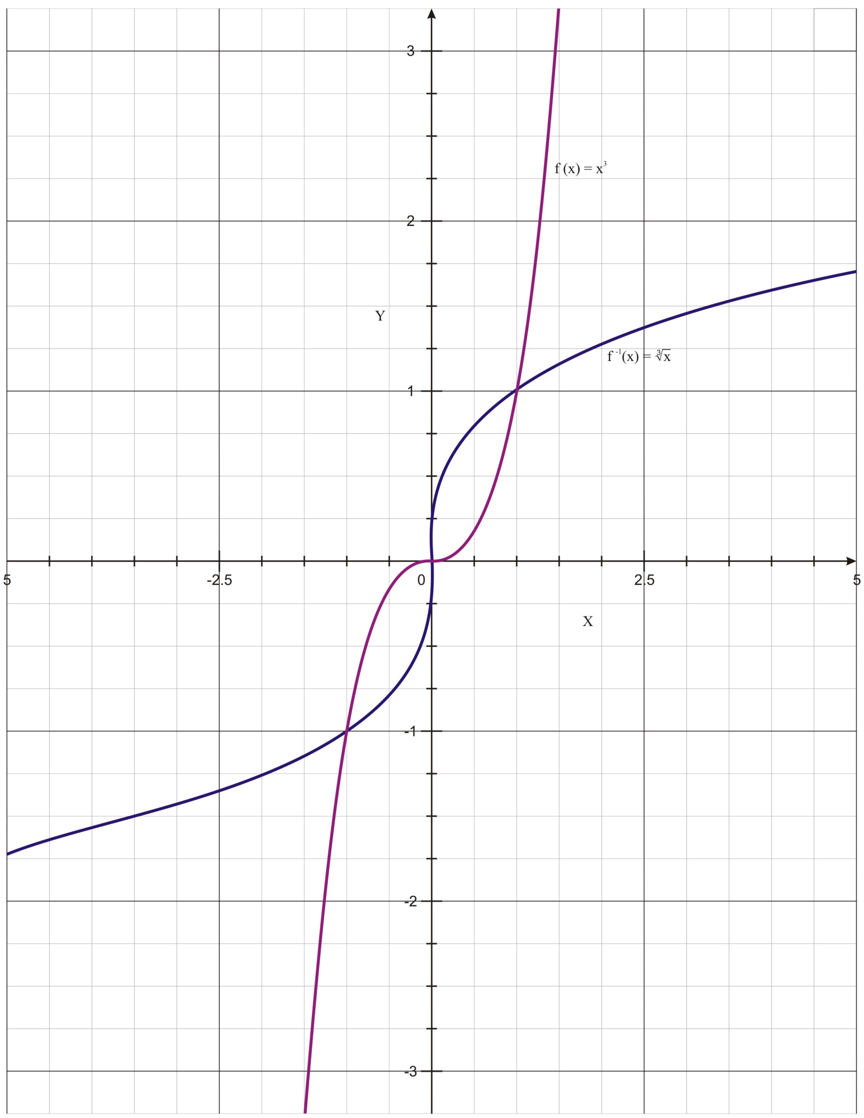


A note about graphing with software or a graphing calculator: if you look at the graph above, you can see that the lines are reflections over the line $y = x$. However, if you do not view the graph in a window that shows equal scales of the x- and y-axes, the graph might not look like this.

Before continuing, there are two other important things to note about inverses. First, remember that the ";' -1'"; is not an exponent, but a symbol that represents an inverse. Second, not every function has an inverse that is a function. In the examples we have considered so far, we inverted a function, and the resulting relation was also a function. However, some functions are not **invertible**; that is, following the process of "inverting" them does not produce a relation that is a function. We will return to this issue below when we examine domain and range of functions and their inverses. First we will look at a set of functions that *are* invertible.

Inverses of 1-to-1 functions

Consider again example 1 above. We began with the function $f(x) = x^3$, and we found the inverse $f^{-1}(x) = \sqrt[3]{x}$. The graphs of these functions are show below.

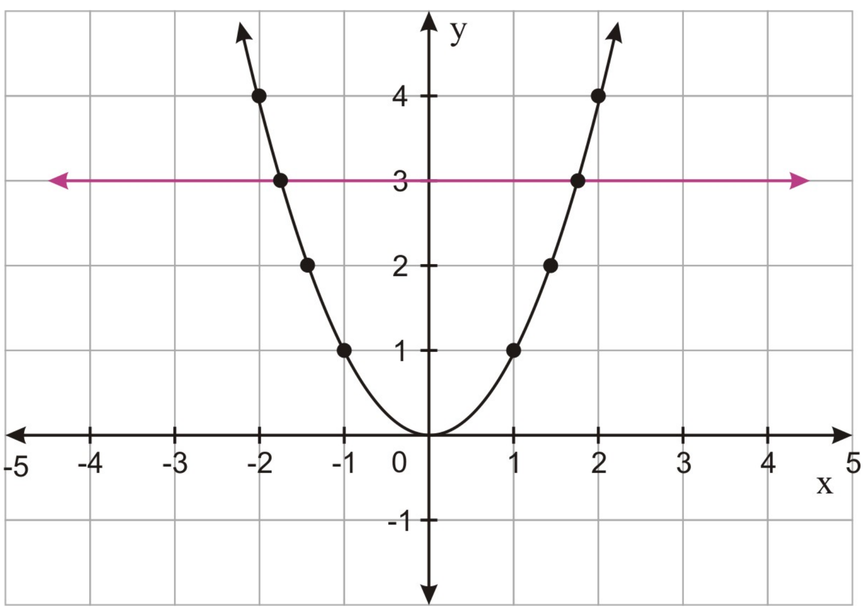


The function $f(x) = x^3$ is an example of a **one-to-one function,** which is defined as follows:

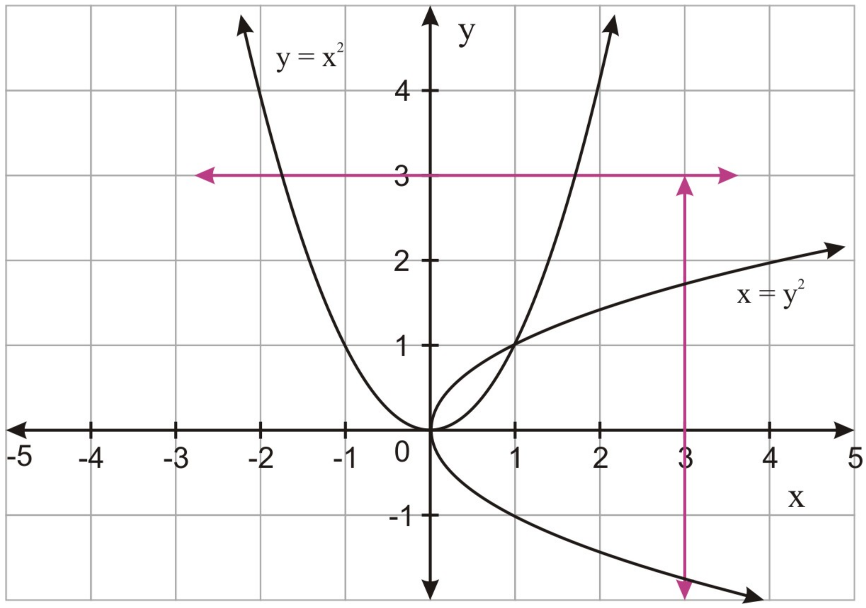
Table 4.5:

One to one A function is **one-to-one** if and only if every element of its domain corresponds to *exactly* one element of its range.

The linear functions we examined above are also one-to-one. The function $y = x^2$, however, is not one-to-one. The graph of this function is shown below.



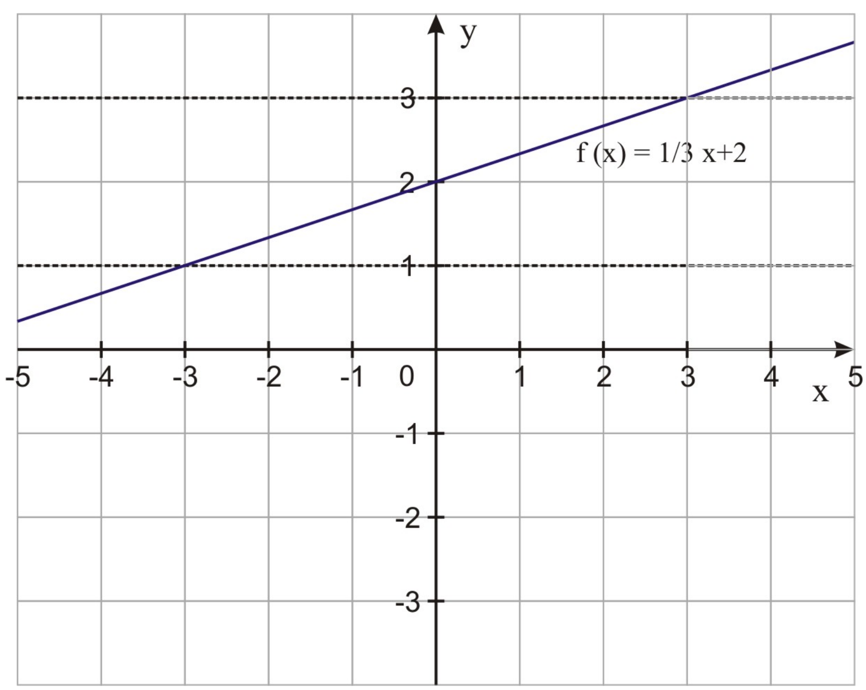
You may recall that you can identify a relation as a function if you draw a vertical line through the graph, and the line touches only one point. Notice then that if we draw a horizontal line through $y = x^2$, the line touches more than one point. Therefore if we inverted the function, the resulting graph would be a reflection over the line $y = x$, and the inverse would not be a function. It fails the vertical line test.



The function $y = x^2$ is therefore *not* a one-to-one function. A function that *is* one-to-one will be invertible. You can determine this graphically by drawing a horizontal line through the graph of the function. For example, if you draw a horizontal line through the graph of $f(x) = \mathrm{x}^3$, the line will only touch one point on the graph, no matter where you draw the line.

Example 2: Graph the function $f(x) = \frac{1}{3}x + 2$. Use a horizontal line test to verify that the function is invertible.

Solution: The graph below shows that this function is invertible. We can draw a horizontal line at any y value, and the line will only cross $f(x) = \frac{1}{3}x + 2$ once.



In sum, a one-to-one function is invertible. That is, if we invert a one-to-one function, its inverse is also a function. Now that we have established what it means for a function to be invertible, we will focus on the domain and range of inverse functions.

Domain and range of functions and their inverses

Because of the definition of inverse, a function's domain is its inverse's range, and the inverse's domain is the function's range. This statement may seem confusing without a specific example.

Example 3: State the domain and range of the function and its inverse:

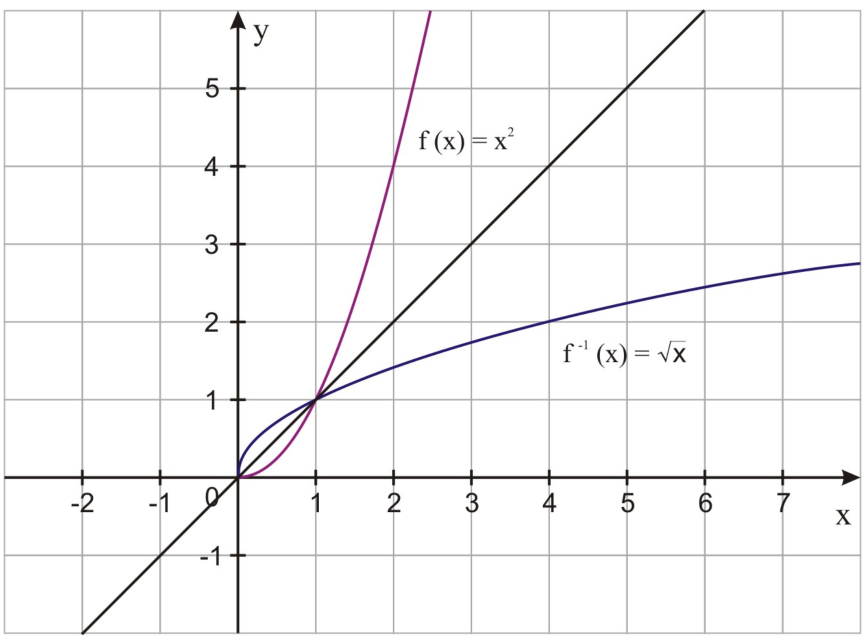
Function: *(1, 2), (2, 5), (3, 7)*

Solution: the inverse of this function is the set of points *(2, 1), (5, 2), (7, 3)*

The domain of the function is *{1, 2, 3}*. This is also the range of the inverse.
The range of the function is *{2, 5, 7}*. This is also the domain of the inverse.

The linear functions we examined previously, as well as $f(x) = x^3$, all had domain and range both equal to the set of all real numbers. Therefore the inverses also had domain and range equal to the set of all real numbers. Because the domain and range were the same for these functions, switching them maintained that relationship.

Also, as we found above, the function $y = x^2$ is not one-to-one, and hence it is not invertible. That is, if we invert it, the resulting relation is not a function. We can change this situation if we define the domain of the function in a more limited way. Let $f(x)$ be a function defined as follows: $f(x) = x^2$, with domain limited to real numbers ≥ 0. Then the inverse of the function is the square root function: $f^{-1}(x) = \sqrt{x}$



Example 4: Define the domain for the function $f(x) = (x - 2)^2$ so that f is invertible.

Solution: The graph of this function is a parabola. We need to limit the domain to one side of the parabola. Conventionally in cases like these we choose the positive side; therefore, the domain is limited to real numbers ≥ 2.

Inverse functions and composition

In the examples we have considered so far, we have taken a function and found its inverse. We can also analyze two functions and determine whether or not they are inverses. Recall the formal definition from above:

Two functions $f(x)$ and $g(x)$ are inverses if and only if $f(g(x)) = g(f(x)) = x$.

This definition is perhaps easier to understand if we look at a specific example. Let's use two functions that we have established as inverses: $f(x) = 2x$ and $g(x) = 1/2\ x$. Let's also consider a specific x value. Let $x = 8$. Then we have $f(\text{g}(8)) = f(1/2 \times 8) = f(4) = 2(4) = 8$. Similarly we could establish that $g(\text{f}(8)) = 8$. Notice that there is nothing special about $x = 8$. For any x value we input into f, the same value will be output by the composed functions:

$$f(g(x)) = f\left(\tfrac{1}{2}x\right) = 2\left(\tfrac{1}{2}x\right) = x$$
$$g(f(x)) = g(2x) = \tfrac{1}{2}(2x) = x$$

Example 5: Use composition of functions to determine if $f(x) = 2x + 3$ and $g(x) = 3x - 2$ are inverses.

Solution: The functions are not inverses.

We only need to check one of the compositions:
$f(g(x)) = f(3x - 2) = 2(3x - 2) + 3 = 6x - 4 + 3 = 6x - 1 \neq x$

Lesson Summary

In this lesson we have defined the concept of inverse, and we have examined functions and their inverses, both algebraically and graphically. We established that functions that are one-to-one are invertible, while other functions are not necessarily invertible. (However, we can redefine the domain of a function such that it is invertible.) In the remainder of the chapter we will examine two families of functions whose members are inverses.

Points to Consider

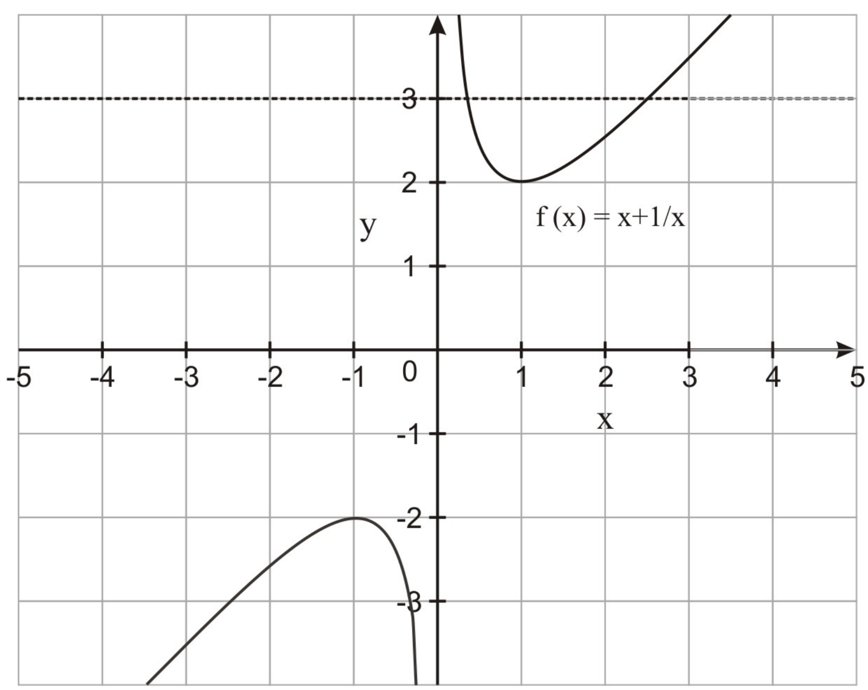
1. Can a function be its own inverse? If so, how?
2. Consider the other function families you learned about in chapter 1. What do their inverses look like?
3. How is the rate of change of a function related to the rate of change of the function's inverse?

Review Questions

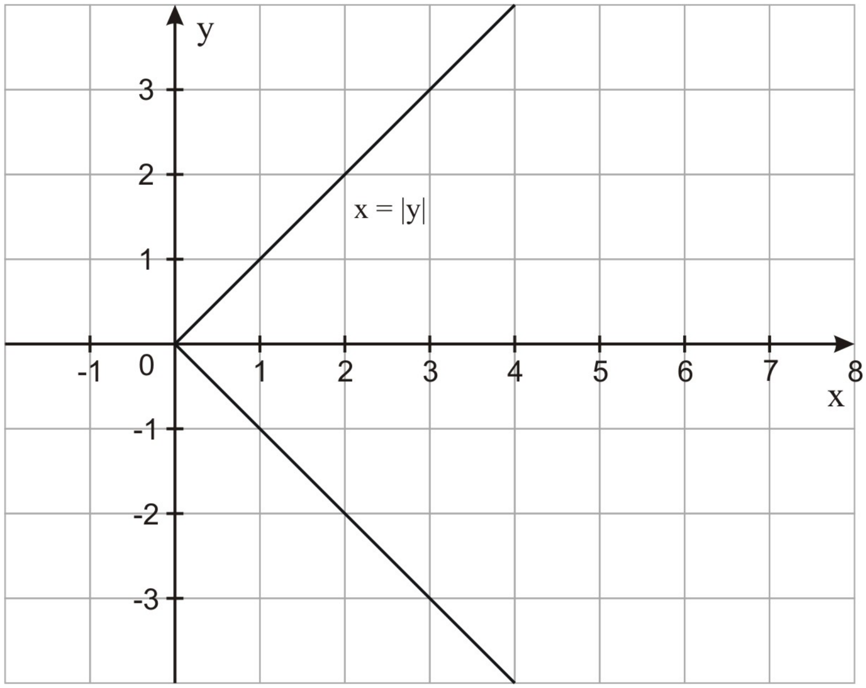
1. Find the inverse of the function $f(x) = \frac{1}{2}x - 7$.
2. Use the horizontal line test to determine if the function $f(x) = x + \frac{1}{x}$ is invertible or not.
3. Use composition of functions to determine if the functions are inverses: $g(x) = 2x - 6$ and $h(x) = \frac{1}{2}x + 3$.
4. Use composition of functions to determine if the functions are inverses: $f(x) = x + 2$ and $p(x) = x - \frac{1}{2}$.
5. Given the function $f(x) = (x+1)^2$, how should the domain be restricted so that the function is invertible?
6. Consider the function $f(x) = \frac{3}{2}x + 4$. a. Find the inverse of the function.
   b. State the slope of the function and its inverse. What do you notice?
7. Given the function (0, 5), (1, 7), (2, 13), (3, 19) a. Find the inverse of the function.
   b. State the domain and range of the function.
   c. State the domain and range of the inverse.

8. Consider the function *a(x)* =
9. Consider the function *f(x)* = *c*, where *c* is a real number. What is the inverse? Is *f* invertible? Explain.
10. A store sells fabric by the length. Red velvet goes on sale after Valentine's day for $4.00 per foot. a. Write a function to model the cost of x feet of red velvet.
    b. What is the inverse of this function?
    c. What does the inverse represent?

Review Answers

1. $y = 2x + 14$
2. 

The function is not invertible.
3. The functions are inverses. $g(h(x)) = g\left(\frac{1}{2}x + 3\right) = 2\left(\frac{1}{2}x + 3\right) - 6 = x + 6 - 6 = x$
   $h(g(x)) = h(2x - 6) = \frac{1}{2}(2x - 6) + 3 = x - 3 + 3 = x$
4. The functions are not inverses. $f(p(x)) = (x - \frac{1}{2}) + 2 = x + \frac{3}{2} \neq x$
5. $x \geq -1$
6. a. $y = \frac{2}{3}x - \frac{8}{3}$b. The slope of the function is 3/2 and the slope of the inverse is 2/3. The slopes are reciprocals.
7. a. (5, 0), (7, 1), (13, 2), (19, 3) b. Domain: {0, 1, 2, 3} and Range: {5, 7, 13, 19}
   c. Domain: {5, 7, 13, 19} and Range: {0, 1, 2, 3
8. }
9. a.



b. The function is not invertible. Several ways to justify: the inverse fails the vertical line test; the original function fails the horizontal line test.

10. The function f is a horizontal line with equation $y = c$. The domain is the set of all real numbers, and the range is the single value c. Therefore the inverse would be a function whose domain is c and the range is all real numbers. This is the vertical line $x = c$. This is not a function. So $f(x) = c$ is not invertible.
11. a. $C(x) = 4x$b. $C^{-1}x = \frac{1}{4}x$
    c. The inverse function tells you the number of feet you bought, given the amount of money you spent.

Vocabulary

Inverse The inverse of a function is the relation obtained by interchanging the domain and range of a function.

Invertible A function is invertible if its inverse is a function.

One-to-one A function is one-to-one if every element of its domain is paired with exactly one element of its range.

4.2 Logarithmic Functions

Learning objectives

- Translate numerical and algebraic expressions between exponential and logarithmic form.
- Evaluate logarithmic functions.
- Determine the domain of logarithmic functions.
- Graph logarithmic functions.
- Solve logarithmic equations.

Introduction

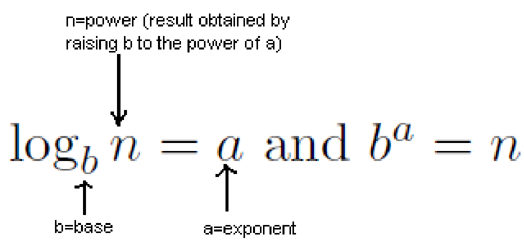
In the previous lesson we examined exponential expressions and functions. Now we will consider another representation for the same relationships involved in exponential expressions and functions.

Consider the function $y= 2^x$. Every x-value of this function is an **exponent**. Every y-value is a power of 2. As you learned in lesson 1, functions that are one-to-one have inverses that are functions. This is the case with exponential functions. If we take the inverse of $y = 2^x$ (by interchanging the domain and range) we obtain this equation: $x = 2^y$. In order to write this equation such that y is expressed as a function of x, we need a different notation.

The solution to this problem is found in the **logarithm**. John Napier originally introduced the logarithm to 17$^{\text{th}}$ century mathematicians as a technique for simplifying complicated calculations. While today's technology allows us to do most any calculations we could imagine, logarithmic functions continue to be a focus of study in mathematics, as a useful way to work with exponential expressions and functions.

Changing Between Exponential and Logarithmic Expressions

Every exponential expression can be written in logarithmic form. For example, the equation $x=2^y$ is written as follows: $y=log_2x$. In general, the equation $log_bn = a$ is equivalent to the equation $b^a=n$. That is, b is the **base**, a is the **exponent**, and n is the **power,** or the result you obtain by raising b to the power of a. Notice that the exponential form of an expression emphasizes the power, while the logarithmic form emphasizes the exponent. More simply put, a logarithm (or ";log"; for short) *is* an exponent.



We can write any exponential expression in logarithmic form.

Example 1: Rewrite each exponential expression as a log expression.

Table 4.6:

a.3^4=81	b.b^{4x}=*52*

Solution:

a. In order to rewrite an expression, you must identify its base, its exponent, and its power. The 3 is the base, so it is placed as the subscript in the log expression. The 81 is the power, and so it is placed after the ";log";. Thus we have: 3^4=81 is the same as $log_3 81=4$.

To read this expression, we say ";the logarithm base 3 of 81 equals 4."; This is equivalent to saying "; 3 to the 4$^{\text{th}}$ power equals 81.";

b. The b is the base, and the expression 4x is the exponent, so we have: $log_b 52 = 4x$. We say, ";log base b of 52, equals $4x$.";

We can also express a logarithmic statement in exponential form.

Example 2: Rewrite the logarithmic expressions in exponential form.

Table 4.7:

a. $\log_{10} 100 = 2$	b. $\log_b w = 5$

Solution:

a. The base is 10, and the exponent is 2, so we have: $10^2 = 100$
b. The base is b, and the exponent is 5, so we have: $b^5 = w$.

Perhaps the most common example of a logarithm is the Richter scale, which measures the magnitude of an earthquake. The magnitude is actually the logarithm base 10 of the amplitude of the quake. That is, $m = \log_{10} A$. This means that, for example, an earthquake of magnitude 4 is 10 *times* as strong as an earthquake with magnitude 3. We can see why this is true of we look at the logarithmic and exponential forms of the expressions: An earthquake of magnitude 3 means $3 = \log_{10} A$. The exponential form of this expression is $10^3 = A$. Thus the amplitude of the quake is 1,000. Similarly, a quake with magnitude 4 has amplitude 10^4=10,000. We will return to this example in lesson 3.8.

Evaluating Logarithmic Functions

As noted above, a logarithmic function is the inverse of an exponential function. Consider again the function $y = 2_x$ and its inverse $x = 2^y$. Above, we rewrote the inverse as $y = \log_2 x$. If we want to emphasize the fact that the log equation represents a **function**, we can write the equation as $f(x) = \log_2 x$. To **evaluate** this function, we choose values of x and then determine the corresponding y values, or function values.

Example 3: Evaluate the function $f(x) = \log_2 x$ for the values:

Table 4.8:

a. $x = 2$	b. $x = 1$	c. $x = -2$

Solution:

a. If $x = 2$, we have:

Table 4.9:

	$f(x) = \log_2 x$
	$f(2) = \log_2 2$

To determine the value of $\log_2 2$, you can ask yourself: ";2 to what power equals 2?"; Answering this question is often easy if you consider the exponential form: $2^? = 2$

The missing exponent is 1. So we have $f(2)$-$\log_2 2 = 1$

b. If $x=1$, we have:

Table 4.10:

$f(x)=\log_2 x$
$f(1)=\log_2 1$

As we did in (a), we can consider the exponential form: $2^? = 1$. The missing exponent is 0. So we have $f(1)=\log^2 1=0$.

c. If x=-2, we have:

Table 4.11:

$f(x)$	$=\log_2 x$
$f(-2)$	$= \log_2 -2$

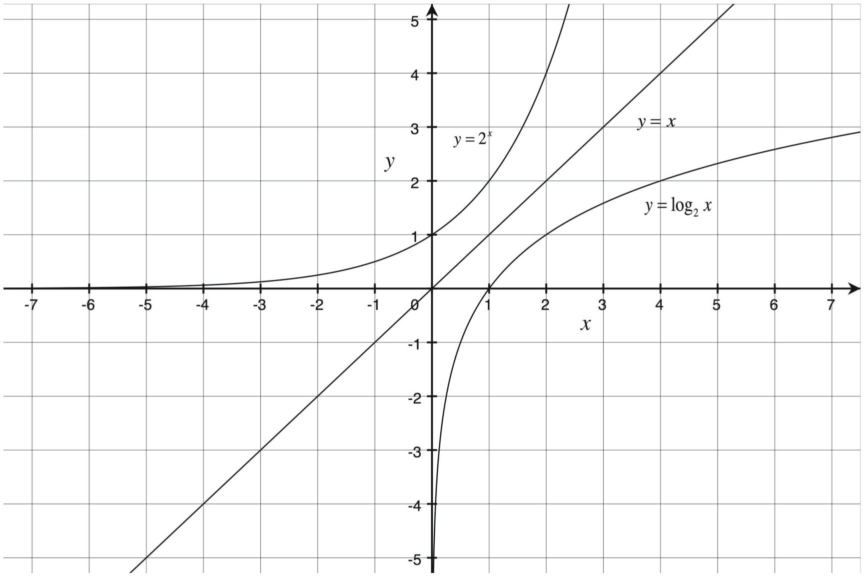
Again, consider the exponential form: $2^? = -2$. There is no such exponent. Therefore $f(-2)=\log_2 -2$ does not exist.

Example 3c illustrates an important point: there are restrictions on the **domain** of a logarithmic function. For the function $f(x)=\log_2 x$, x cannot be a negative number. Therefore we can state the domain of this function as: ";the set of all real numbers greater than 0."; Formally, we can write it as a set: $\{x \in |x > 0\}$. In general, the domain of a logarithmic function is restricted to those values that will make the argument of the logarithm non-negative.

For example, consider the function $f(x)=\log_3(x\text{-}4)$. If you attempt to evaluate the function for x values of 4 or less, you will find that the function values do not exist. Therefore the domain of the function is $\{x \in |x > 4\}$. The domain of a logarithmic function is one of several key issues to consider when graphing.

Graphing Logarithmic Functions

Because the function $f(x)=\log_2 x$ is the inverse of the function $g(x)=2^x$, the graphs of these functions are reflections over the line $y = x$. The figure below shows the graphs of these two functions:



We can see that the functions are inverses by looking at the graph. For example, the graph of $g(x)=2^x$ contains the point (1, 2), while the graph of $f(x)=\log_2 x$ contains the point (2, 1).

Also, note that while that the graph of $g(x)=2^x$ is asymptotic to the x-axis, the graph of $f(x)=\log_2 x$ is asymptotic to the y-axis. This behavior of the graphs gives us a visual interpretation of the restricted range of g and the restricted domain of f.

When graphing log functions, it is important to consider x- values across the domain of the function. In particular, we should look at the behavior of the graph as it gets closer and closer to the asymptote. Consider $f(x)=\log_2 x$ for values of x between 0 and 1.

If $x=1/2$, then $f(1/2)=\log_2(1/2)=-1$ because $2^{-1}=1/2$
If $x=1/4$, then $f(1/4)$=log_2(1/4)=-2 because $2^{-2}=1/4$
If $x=1/8\}$, then $f(1/8)=\log_2(1/8)=-3$ because $2^{-3}=1/8$

From these values you can see that if we choose x values that are closer and closer to 0, the y values decrease (heading towards $-\infty$!). In terms of the graph, these values show us that the graph gets closer and closer to the y-axis. Formally we say that the vertical asymptote of the graph is $x = 0$.

Example 4: Graph the function $f(x)=\log_4 x$ and state the domain and range of the function.

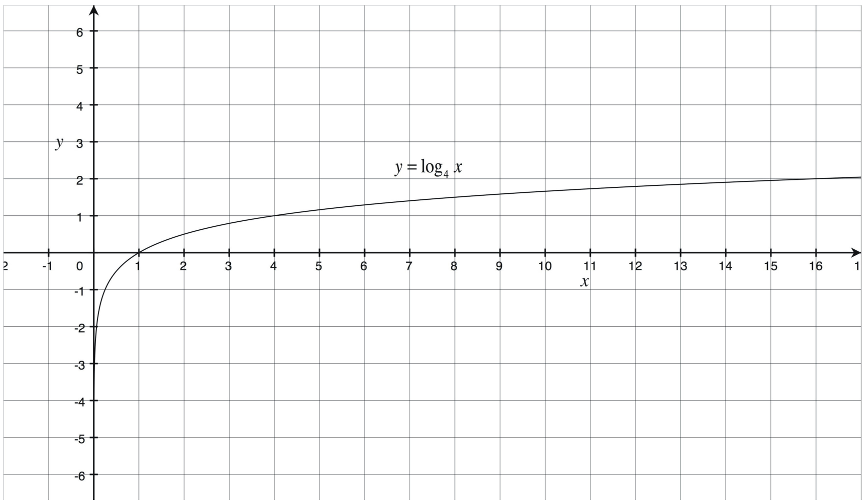
Solution: The function $f(x)=\log_4 x$ is the inverse of the function $g(x) = 4^x$. We can sketch a graph of $f(x)$ by evaluating the function for several values of x, or by reflecting the graph of g over the line $y = x$.

If we choose to plot points, it is helpful to organize the points in a table:

Table 4.12:

x	$y=\log_4 x$
1/4	
1	0
4	1
16	2

The graph is asymptotic to the y-axis, so the domain of f is the set of all real numbers that are greater than 0. We can write this as a set:$\{x \in |x > 0\}$. While the graph might look as if it has a horizontal asymptote, it does in fact continue to rise. The range is .



A note about graphing calculators: You can use a graphing calculator to graph logarithmic functions, but many calculators will only allow you to use base 10 or base e. However, after the next lesson you will be able to rewrite *any* log as a log with base 10 or base e.

In this section we have looked at graphs of logarithmic functions of the form $f(x)=\log_b x$. Now we will consider the graphs of other forms of logarithmic equations.

Graphing Logarithmic Functions Using Transformations

As you saw in the previous lesson, you can graph exponential functions by considering the relationships between equations. For example, you can use the graph of $f(x)=2^x$ to sketch a graph of $g(x)=2^x + 3$. Every y value of $g(x)$ is the same as a y value of $f(x)$, plus 3. Therefore we can shift the graph of $f(x)$ up 3 units to obtain a graph of $g(x)$.

We can use the same relationships to efficiently graph log functions. Consider again the log function $f(x)=\log_2 x$. The table below summarizes how we can use the graph of this function to graph other related function.

Table 4.13:

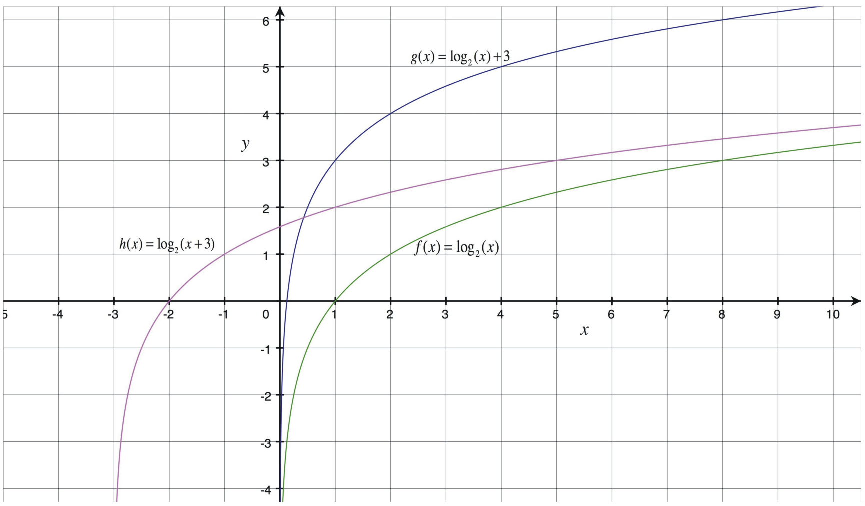
Equation	Relationship to $f(x)=\log_2 x$	Domain
$g(x)=\log_2(x - a)$, for $a > 0$	Obtain a graph of g by shifting the graph of f a units to the right.	$x > a$
$g(x) = \log_2(x+a)$ for $a > 0$	Obtain a graph of g by shifting the graph of f a units to the left.	$x > -a$
$g(x)=\log_2(x)$ + a for $a > 0$	Obtain a graph of g by shifting the graph of f up a units.	$x > 0$
$g(x)=\log_2(x)$-a for $a > 0$	Obtain a graph of g by shifting the graph of f down a units.	$x > 0$
$g(x)=a\log_2(x)$ for $a > 0$	Obtain a graph of g by vertically stretching the graph of f *by a factor of a.*	$x > 0$

Table 4.13: (continued)

Equation	Relationship to $f(x)=\log_2 x$	Domain
$g(x)=-a\log_2(x)$, for $a > 0$	Obtain a graph of g by vertically stretching the graph of f *by a factor of a*, and by reflecting the graph over the x-axis.	$x > 0$
$g(x)=\log_2(-x)$	Obtain a graph of g by reflecting the graph of f over the y-axis.	$x < 0$

Example 5: Graph the functions $f(x)=\log_2(x)$,$g(x) = \log_2(x) + 3$, and $h(x) = \log_2(x + 3)$

Solution: The graph below shows these three functions together:



Notice that the location of the *3* in the equation makes a difference! When the 3 is added to $\log_2 x$, the shift is vertical. When the 3 is added to the x, the shift is horizontal. It is also important to remember that adding 3 to the x is a horizontal shift to the left. This makes sense if you consider the function value when x = -3:

$h(-3)=\log_2(-3 + 3)=\log_2 0$ = undefined

This is the vertical asymptote! To graph these functions, we evaluated them for certain values of x. But what if we want to know what the x value is for a particular y value? This means that we need to solve a logarithmic equation.

Solving Logarithmic Equations

In general, to solve an equation means to find the value(s) of the variable that makes the equation a true statement. To solve log equations, we have to think about what ";log"; *means.*

Consider the equation $\log_2 x=5$. What is the exponential form of this equation?
The equation $\log_2 x=5$ means that $2^5 = x$. So the solution to the equation is $x = 2^5 = 32$.

We can use this strategy to solve many logarithmic equations.

Example 6: Solve each equation for x:

Table 4.14:

a.$\log_4 x = 3$	b. $\log_5(x + 1) = 2$	c. $1 + 2\log_3(x$ - $5)$ = 7

Solution: a. Writing the equation in exponential form gives us the solution: $x = 4^3 = 64$.

b. Writing the equation in exponential form gives us a new equation: $5^2 = x + 1$.
We can solve this equation for x:

Table 4.15:

	5^2	$= x + 1$
	25 x	$= x + 1$ $= 24$

c. First we have to isolate the log expression:

Table 4.16:

	$1 + 2\log_3(x\text{-}5)$	= 7
	$2\log_3(x$ - $5)$ $\log_3$(x-5)	= 6 = 3

Now we can solve the equation by rewriting it in exponential form:

Table 4.17:

	$\log_3$(x-5)	= 3
	3^3 27 x	$= x$ - 5 $= x$ - 5 = 32

We can also solve equations in which both sides of the equation contain logs. For example, consider the equation $\log_2(3x\text{-}1)=\log_2(5x$ - $7)$. Because the logarithms have the same base (2), the arguments of the log (the expressions $3x$ - 1 and $5x$ - 7) *must be equal.* So we can solve as follows:

Table 4.18:

	$\log_3(3x\text{-}1)$	$= \log_2(5x$ - $7)$
	$3x$ - 1 +7 |+7 $3x + 6$ -$3x$	$= 5x$ - 7 $= 5x$

Table 4.18: (continued)

	$\log_3(3x\text{-}1)$	$= \log_2(5x - 7)$
	6	$= 2x$
	x	$= 3$

Example 7: Solve for x: $\log_2(9x)=\log_2(3x+8)$

Solution: The log equation implies that the expressions $9x$ and $3x+8$ are equal:

Table 4.19:

	$log_2(9x)$	$= log_2(3x+8)$
	$9x$	$= 3x+8$
	$-3x$	$-3x$
	$6x$	$= 8$
	x	$= \frac{8}{6}$
	x	$= \frac{4}{3}$

Lesson Summary

In this lesson we have defined the logarithmic function as the inverse of the exponential function. When working with logarithms, it helps to keep in mind that a *logarithm is an exponent.* For example, $3 = \log_2 8$ and $2^3 = 8$ are two forms of the same numerical relationship among the three numbers 2, 3, and 8. The 2 is the base, the 3 is the exponent, and 8 is the 3^{rd} power of 2.

Because logarithmic functions are the inverses of exponential functions, we can use our knowledge of exponential functions to graph logarithmic functions. You can graph a log function either by reflecting an exponential function over the line $y = x$, or by evaluating the function and plotting points. In this lesson you learned how to graph ";parent"; graphs such as $y = \log_2 x$ and $y = \log_4 x$, as well as how to use these parent graphs to graph more complicated log functions. When graphing, it is important to keep in mind that logarithmic functions have restricted domains. Each graph will have a vertical asymptote.

We can also use our knowledge of exponential relationships to solve logarithmic equations. In this lesson we solved 2 kinds of logarithmic equations. First, we solved equations by rewriting the equations in exponential form. Second, we solved equations in which both sides of the equation contained a log. To solve these equations, we used the following rule:

$$\log_b f(x)=\log_b g(x) \rightarrow f(x) = g(x) .$$

Points to Consider

1. What methods can you use to graph logarithmic functions?
2. What methods can you use to solve logarithmic equations?
3. What forms of log equations can you solve using the methods in this lesson? Can you write an equation that cannot be solved using these methods?

Review Questions

Write the exponential statement in logarithmic form.

1. $3^2 = 9$
2. $z^4 = 10$
3. Write the logarithmic statement in exponential form.
4. $\log_5 25 = 2$
5. $log_4 \frac{1}{6} = -1$
6. Complete the table of values for the function $f(x) = \log_3 x$

Table 4.20:

x	y = f(x)
1/9	
1/3	
1	
3	
9	

7.
8. Use the table above to graph $f(x)$=$\log_3 x$. State the domain and range of the function.
9. Consider $g(x)$ = -$\log_3(x$ - 2)a. How is the graph of $g(x)$ related to the graph of $f(x) = \log_3 x$?
   b. Graph $g(x)$ by transforming the graph of $f(x)$.
10. Solve each logarithmic equation: a. $\log_3 9x = 4$
    b. 7 + $\log_2 x = 11$ (Hint: subtract 7 from both sides first.)
11. Solve each logarithmic equation:

Table 4.21:

a. $\log_5 6x$ = -1	b. $\log_5 6x = \log_5(2x + 16)$	c. $\log_5 6x = \log_5(3x$ - 10)

12.
13. Explain why the equation in 9c has no solution.

Review Answer

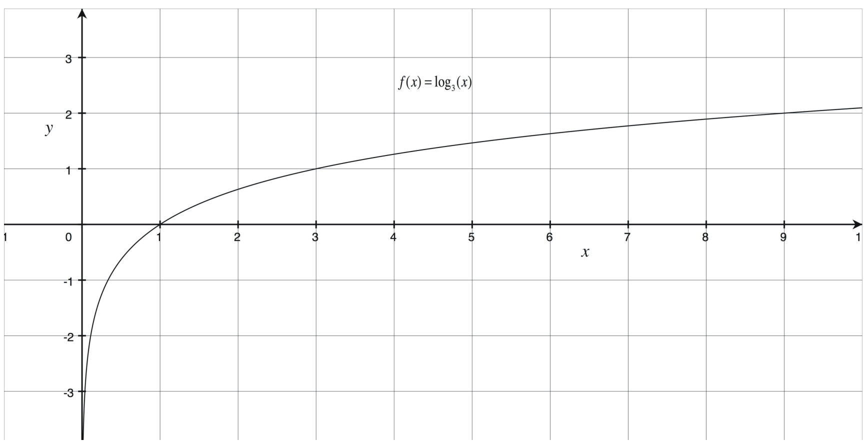
1. $3^2 = 9$
2. z^4 =10
3. $5^2 = 25$
4. $6^{-1} = \frac{1}{6}$

Table 4.22:

x	y = f(x)
1/9	
1/3	
1	0

Table 4.22: (continued)

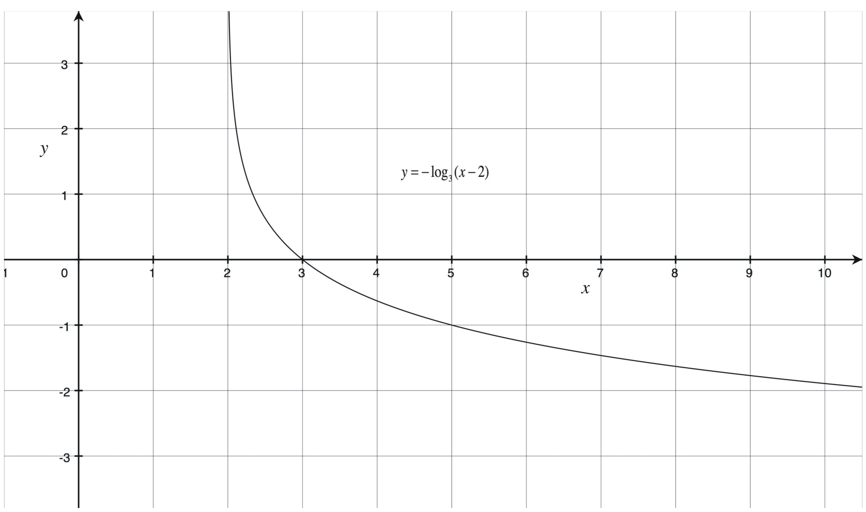
x	y = f(x)
3	1
9	2

5.
6. 

D: All real numbers > 0

R: All real numbers.

7. a. The graph of g(x) can be obtained by shifting the graph of f(x) 2 units to the right, and reflecting it over the x- axis.b.



8. The solutions are:

Table 4.23:

a. x = 9	b. x = 16

9.
10. The solutions are:

Table 4.24:

a. x = 1/30	b. x = 4	c. no solution

11.
12. When we solve $6x$=$3x$-10 we find that x=-10/3, a value outside of the domain. Because there is no other x value that satisfies the equation, there is no solution.

Vocabulary

Argument The expression ";inside"; a logarithmic expression. The argument represents the ";power"; in the exponential relationship.

Asymptote An asymptote is line whose distance to a given curve tends to zero. An asymptote may or may not intersect its associated curve.

Domain The domain of a function is the set of all values of the independent variable (x) for which the function is defined.

Evaluate To evaluate a function is to identify a function value (y) for a given value of the independent variable (x).

Function A function is a relation between a domain (set of x values) and range (set of y values) in which every element of the domain is paired with one and only one element of the range. A function that is ";one to one"; is a function in which every element of the domain is paired with exactly one y value.

Logarithm The exponent of the power to which a base number must be raised to equal a given number.

Range The range of a function is the set of all function values, or values of the dependent variable (y).

4.3 Properties of Logarithms

Learning objectives

- Use properties of logarithms to write logarithmic expressions in different forms.
- Evaluate common logarithms and natural logarithms.
- Use the change of base formula and a scientific calculator to find the values of logs with any bases.

Introduction

In the previous lesson we defined the logarithmic function as the inverse of an exponential function, and we evaluated log expressions in order to identify values of these functions. In this lesson we will work with more complicated log expressions. We will develop properties of logs that we can use to write a log expression as the sum or difference of several expressions, or to write several expressions as a single log expression. We will also work with logs with base 10 and base e, which are the bases most often used in applications of logarithmic functions.

Properties of Logarithms

Because a logarithm is an exponent, the properties of logs are the same as the properties of exponents. Here we will prove several important properties of logarithms.

Property 1: $\log_{\mathrm{b}}(xy) = \log_b x + \log_b y$

Proof: Let $\log_b x = n$ and $\log_b y = m$.

Rewrite both log expressions in exponential form:
$\log_b x = n \rightarrow b^n = x$
$\log_b y = m \rightarrow b^{\mathrm{m}} = y$
Now multiply x and y: $xy = b^n \times b^m = b^{n\,+\,m}$
Therefore we have an exponential statement: $b^{n\,+\,m} = xy$.
The log form of the statement is: $\log_b xy = n + m$.
Now recall how we defined n and m:
$\log_b xy = n + m = \log_b \mathrm{x} + log_b\ y$.

Property 2: $log_b\left(\frac{x}{y}\right) = log_b x - log_b y$

We can prove property 2 analogously to the way we proved property 1.

Proof: Let $\log_b x = n$ and $\log_b y = m$.

Rewrite both log expressions in exponential form:
$\log_b x = n \rightarrow b^n = x$
$\log_b y = m \rightarrow b^m = y$
Now divide x by y: $\frac{x}{y} = \frac{b^n}{b^m} = b^{n-m}$
Therefore we have an exponential statement: $b^{n-m} = \frac{x}{y}$.
The log form of the statement is: $log_b\left(\frac{x}{y}\right) = n - m$.
Now recall how we defined n and m:
$log_b\left(\frac{x}{y}\right) = n - m = log_b x - log_b y$.

Property 3: $\log_b x^n = n \log_{\mathrm{b}} x$

The proof of the third property relies on another property of logs that we can derive by thinking about the definition of a log. Consider the expression $\log_2 2^{13}$. What does this expression *mean?*

The exponential form of $\log_2 2^{13} = ?$ is $2^? = 2^{13}$. Looking at the exponential form should convince you that the missing exponent is 13. That is, $\log_2 2^{13} = 13$. In general, $\log_{\mathrm{b}} b^n = n$. This property will be used in the proof of property 3.

Proof (of Property 3):

Let $\log_b x = w$.
The exponential form of this log statement is $b^w = x$.
If we raise both sides of this equation to the power of n, we have $(b^w)^{\mathrm{n}} = x^{\mathrm{n}}$.

Using the power property of exponents, this equation simplifies to $b^{wn} = x^n$
If two expressions are equal, then the logs of both expressions are equal:

$\log_b \mathrm{b}^{wn} = \log_b x^n$
Now consider the value of the left side of the equation: $\log_b b^{\mathrm{wn}} = \mathrm{wn}$.
Above, we showed that $b^{wn} = x^n$. By substitution, we have $\log_b x^n = wn$.
Above, we defined w: $\log_b x = w$. By substitution, we have
$\log_b x^n = (\log_b x)\ n = n\log_b x$.

We can use these properties to rewrite log expressions.

Expanding expressions

Using the properties we have derived above, we can write a log expression as the sum or difference of simpler expressions. Consider the following examples:

1. $\log_2 8x = \log_2 8 + \log_2 x = 3 + \log_2 x$
2. $log_3\left(\frac{x^2}{3}\right) = log_3 x^2 - log_3 3 = 2log_3 x - 1$

Using the log properties in this way is often referred to as "expanding". In the first example, expanding the log allowed us to simplify, as log2 8 = 3. Similarly, in the second example, we simplified using the log properties, and the fact that $\log_3 3 = 1$.

Example 1: Expand each expression:

Table 4.25:

a. $log_5 25x^2y$	b. $log_{10}\left(\frac{100x}{9b}\right)$

Solution:

a. $\log_5 25x^2 y = \log_5 25 + \log_5 x^2 + \log_5 y = 2 + 2\log_5 x + \log_5 y$
b.

Table 4.26:

$log_{10}\left(\frac{100x}{9b}\right)$	$= log_{10}100x - log_{10}9b$
	$= log_{10}100 + log_{10}x - [log_{10}9 + log_{10}b]$ $= 2 + log_{10}x - log_{10}9 - log_{10}b$

Just as we can write a single log expression as a sum and difference of expressions, we can also write expanded expressions as a single expression.

Condensing expressions

To condense a log expression, we will use the same properties we used to expand expressions. Consider the expression $\log_6 8 + \log_6 27$. Alone, each of these expressions does not have an integer value. The value of $\log_6 8$ is between 1 and 2; the value of $\log_6 27$ is also between 1 and 2. If we condense the expression, we get:

$$\log_6 8 + \log_6 27 = \log_6 (8 \times 27) = \log_6 216 = 3$$

We can also condense algebraic expressions. This will be useful later for solving logarithmic equations.

Example 2: Condense each expression:

Table 4.27:

a. $2\log_3 x + \log_3 5x - \log_3 (x + 1)$	b. $\log_2 (x^2 - 4) - \log_2 (x + 2)$

Solution:

a. $2log_3x + log_35x - log_3(x + 1) = log_3x_2 + log_35x - log_3(x + 1)$

$$= log_3(x^2(5x)) - log_3(x + 1)$$
$$= log_3\left(\frac{5x^3}{x+1}\right)$$

b. $log_2(x^2 - 4) - log_2(x + 2) = log_2\left(\frac{x^2-4}{x+2}\right)$

$$= log_2\left(\frac{(x+2)(x-2)}{x+2}\right)$$
$$= log_2(x - 2)$$

It is important to keep in mind that a log expression may not be defined for certain values of x. First, the argument of the log must be positive. For example, the expressions in example 2b above are not defined for $x \leq 2$ (which allows us to "cancel" $(x+2)$ without worrying about the condition $x \neq -2$).

Second, the argument must be defined. For example, in example 2a above, the expression $\left(\frac{5x^3}{x+1}\right)$ is undefined if $x = -1$.

The log properties apply to logs with any real base. Next we will examine logs with base 10 and base e, which are the most common bases for logs (though only one is actually called ";common";).

Common logarithms and natural logs

A **common logarithm** is a log with base 10. We can evaluate a common log just as we evaluate any other log. A common log is usually written without a base.

Example 3: Evaluate each log

Table 4.28:

a. log 1	b. log 10	c. $log\sqrt{10}$

Solution:

a. log 1 = 0, as $10^0 = 1$.
b. log 10 = 1, as $10^1 = 10$
c. $log\sqrt{10} = \frac{1}{2}$ because $\sqrt{10} = 10^{1/2}$

As noted in lesson 3, logarithms were introduced in order to simplify calculations. After Napier introduced the logarithm, another mathematician, Henry Briggs, proposed that the base of a logarithm be standardized as 10. Just as Napier had labored to compile tables of log values (though his version of the logarithm is somewhat different from what we use today), Briggs was the first person to publish a table of common logs. This was in 1617!

Until recently, tables of common logs were included in the back of math textbooks. Publishers discontinued this practice when scientific calculators became readily available. A scientific calculator will calculate the value of a common log to 8 or 9 digits. Most calculators have a button that says LOG. For example, if you have TI graphing calculator, you can simply press LOG, and then a number, and the calculator will give you a log value up to 8 or 9 decimal places. For example, if you enter LOG(7), the calculator returns .84509804. This means that $10^{.84509804} \approx 7$. If we want to judge the reasonableness of this value, we need to think about powers of 10. Because $10^1 = 10$, log(7) should be less than 1.

Example 4: For each log value, determine two integers between which the log value should lie. Then use a calculator to find the value of the log.

Table 4.29:

a. log 50	b. log 818

Solution:

a. log 50
The value of this log should be between 1 and 2, as $10^1 = 10$, and $10^2 = 100$.
Using a calculator, you should find that log 50 ≈ 1.698970004.
b. log 818
The value of this log should be between 2 and 3, as $10^2 = 100$, and $10^3 = 1000$.
Using a calculator, you should find that log 818 ≈ 2.912753304.

The calculator's ability to produce log values is an example of the huge benefit that technology can provide. Only a few years ago, the calculations in the previous example would have each taken several minutes, while now they only take several seconds. While most people might not calculate log values in their every day lives, scientists and engineers are grateful to have such tools to make their work faster and more efficient.

Along with the LOG key on your calculator, you will find another logarithm key that says LN. This is the abbreviation for the **natural log**, the log with base e. Natural logs are written using ";ln"; instead of ";log."; That is, we write the expression $\log_e x$ as In x. How you evaluate a natural log depends on the argument of the log. You can evaluate some natural log expressions without a calculator. For example, ln $e = 1$, as $e^1 = e$. To evaluate other natural log expressions requires a calculator. Consider for example ln 7. Recall that $e \approx 2.7$. This tells us that ln 7 should be slightly less than 2, as $(2.7)^2 = 7.29$. Using a calculator, you should find that ln 7 ≈ 1.945910149.

Example 5: Find the value of each natural log.

Table 4.30:

a. $ln100$	b. $ln\sqrt{e}$

Solution:

a. ln 100 is between 4 and 5. You can estimate this by considering powers of 2.7, or powers of 3: $3^4 = 81$, and $3^5 = 243$.
Using a calculator, you should find that ln 100 ≈ 4.605171086.

b. Recall that a square root is the same as an exponent of 1/2. Therefore $ln\sqrt{e} = ln(e^{1/2}) = 1/2$

You may have noticed that the common log and the natural log are the only log buttons on your calculator. We can use either the common log or the natural log to find the values of logs with other bases.

Change of Base

Consider the log expression $\log_3 35$. The value of this expression is approximately 3 because $3^3 = 27$. In order to find a more exact value of $\log_3 35$, we can rewrite this expression in terms of a common log or natural log. Then we can use a calculator.

Let's consider a general log expression, $\log_b x = y$. This means that $b^y = x$. Recall that if two expressions are equal, then the logs of the expressions are equal. We can use this fact, and the power property of logs, to write $\log_b x$ in terms of common logs.

Table 4.31:

$b^y = x \Rightarrow logb^y = logx$	The logs of the expressions are equal

Table 4.32:

$\Rightarrow ylogb = logx$	Use the power property of logs
$\Rightarrow y = \frac{logx}{logb}$ $\Rightarrow log_b x = \frac{logx}{logb}$	Divide both sides by $logb$ Substitute $log_b x = y$

The final equation, $log_b x = \frac{logx}{logb}$, is called the change of base formula. Notice that the proof did not rely on the fact that the base of the log is 10. We could have used a natural log. Thus another form of the change of base formula is $\log_b x = \frac{lnx}{lnb}$.

Note that we could have used a log with any base, but we use the common log and the natural log so that we can use a calculator to find the value of an expression. Consider again $\log_3 35$. If we use the change of base formula, and then a calculator, we find that

$$log_3 35 = \frac{log35}{log3} = 3.23621727.$$

Example 6: Estimate the value, and then use the change of base formula to find the value of $\log_2 17$.

Solution: $\log_2 17$ is close to 4 because $2^4 = 16$ and $2^5 = 32$. Using the change of base formula, we have $log_2 17 = \frac{log17}{log2}$. Using a calculator, you should find that the approximate value of this expression is 4.087462841.

Lesson Summary

In this lesson we have developed and used properties of logarithms, including a formula that allows us to calculate the value of a log expression with any base. Out of context, it may seem difficult to understand the value of these kind of calculations. However, as you will see in later lessons in this chapter, we can use exponential and logarithmic functions to model a variety of phenomena.

Points to Consider

1. Why is the common log called common? Why 10?
2. Why would you want to estimate the value of a log before using a calculator to find its exact value?
3. What kind of situations might be modeled with a logarithmic function?

Review Questions

1. Expand the expression: $\log_b 5x^2$
2. Expand the expression: $\log_3 81x^5$
3. Condense the expression: $\log(x + 1) + \log(x - 1)$
4. Condense the expression: $3\ln(x) + 2\ln(y) - \ln(5x - 2)$
5. Evaluate the expressions: a. $\log 1000$
   b. $\log 0.01$
6. Evaluate the expressions: a. $\ln e^4$
   b. $ln\left(\frac{1}{e^9}\right)$
7. Use the change of base formula to find the value of $\log_5 100$.
8. What is the difference between $\log_b x^n$ and $(\log_b x)^n$?
9. Condense the expression in order to simplify: $3 \log 2 + \log 125$
10. Is this equation true for any values of x and y? $\log_2 (x + y) = \log_2 x + \log_2 y$If so, give the values. If not, explain why not.

Review Answers

1. $\log_b 5 + 2 \log_b x$
2. $4 + 5 \log_3 x$
3. $\log (x^2 - 1)$
4. $ln\left(\frac{x^3y^2}{5x-2}\right)$
5. a. 3 b. -2
6. a. 4 b. -9
7. $\frac{log100}{log5} \approx 2.86$
8. The first expression is equivalent to $n \log_b x$. The second expression is the n^{th} power of the log.
9. $\log 1000 = 3$
10. $\log_2 (x + y) = \log_2 x + \log_2 y$ if and only if $x + y = xy$. The solutions to this equation are the possible values of x and y. For example, $x = 3$ and $y = 1.5$

Vocabulary

Common logarithm A common logarithm is a log with base $10k$. The log is usually written without the base.

Natural logarithm A natural log is a log with base e. The natural log is written as ln.

Scientific calculator A scientific calculator is an electronic, handheld calculator that will do calculations beyond the four operations $(+, -, \times, \div)$, such as square roots and logarithms. Graphing calculators will do scientific operations, as well as graphing and equation solving operations.

4.4 Growth and Decay

Learning objectives

- Model situations using exponential and logistic functions.
- Solve problems involving these models, using your knowledge of properties of logarithms, and using a graphing calculator.

Introduction

In lesson 5 you learned about modeling phenomena with exponential and logarithmic functions. In the examples in lesson 5, you used a graphing calculator to find a line that fits a given set of data. Here we will use algebraic techniques to develop models, and you will learn about another kind of function, the logistic function, that can be used to model growth.

Exponential growth

In general, if you have enough information about a situation, you can write an exponential function to model growth in the situation. Let's start with a straightforward example:

Example 1: A social networking website is started by a group of 10 friends. They advertise their site before they launch, and membership grows fast: the membership doubles every day. At this rate, what will the membership be in a week? When will the membership reach 100,000?

Solution: To model this situation, let's look at how the membership changes each day:

Table 4.33:

Time (in days)	Membership
0	10
1	$2 \times 10 = 20$
2	$2 \times 2 \times 10 = 40$
3	$2 \times 2 \times 2 \times 10 = 80$
4	$2 \times 2 \times 2 \times 2 \times 10 = 160$

Notice that the membership on day x is $10(2^{\mathrm{x}})$. Therefore we can model membership with the function $M(x) = 10(2^{\mathrm{x}})$. In seven days, the membership will be $M(7) = 10(2^7) = 1280$.

We can solve an exponential equation to find out when the membership will reach 100,000:

$$\begin{aligned}
10(2^x) &= 100,000 \\
2^x &= 10,000 \\
log2^x &= log10,000 \\
xlog2 &= 4 \\
x &= \frac{4}{log2} \approx 13.3
\end{aligned}$$

At this rate, the membership will reach 100,000 in less than two weeks. This result may seem unreasonable. That's very fast growth!

So let's consider a slower rate of doubling. Let's say that the membership doubles every 7 days.

Table 4.34:

Time (in days)	Membership
0	10
7	$2 \times 10 = 20$
14	$2 \times 2 \times 10 = 40$
21	$2 \times 2 \times 2 \times 10 = 80$
28	$2 \times 2 \times 2 \times 2 \times 10 = 160$

We can no longer use the function $M(x) = 10(2^x)$. However, we *can* use this function to find another function to model this new situation. Looking at one data point will help. Consider for example the fact that $M(21) = 10(2^3)$. This is the case because 21 days results in 3 periods of doubling. In order for $x = 21$ to produce 2^3 in the equation, the exponent in the function must be $x/7$. So we have $M(x) = 10\left(2^{\frac{x}{7}}\right)$. Let's verify that this equation makes sense for the data in the table:

$$M(0) = 10\left(2^{\frac{0}{7}}\right) = 10(1) = 10$$
$$M(7) = 10\left(2^{\frac{7}{7}}\right) = 10(2) = 20$$
$$M(14) = 10\left(2^{\frac{14}{7}}\right) = 10(2^2) = 10(4) = 40$$
$$M(21) = 10\left(2^{\frac{21}{7}}\right) = 10(2^3) = 10(8) = 80$$
$$M(28) = 10\left(2^{\frac{28}{7}}\right) = 10(2^4) = 10(16) = 160$$

Notice that each x value represents one more event of doubling, and in order for the function to have the correct power of 2, the exponent must be $(x/7)$.

With the new function $M(x) = 10\left(2^{\frac{x}{7}}\right)$, the membership doubles to 20 in one week, and reaches 100,000 in about 3 months:

$$10\left(2^{\frac{x}{7}}\right) = 100,000$$
$$2^{\frac{x}{7}} = 100,000$$
$$log2^{\frac{x}{7}} = log100,000$$
$$\frac{x}{7}log2 = 4$$
$$xlog2 = 28$$
$$x = \frac{28}{log2} \approx 93$$

The previous two examples of exponential growth have specifically been about doubling. We can also model a more general growth pattern with a more general growth model. While the graphing calculator produces a function of the form $y = a(b^x)$, population growth is often modeled with a function in which e is the base. Let's look at this kind of example:

The population of a town was 20,000 in 1990. Because of its proximity to technology companies, the population grew to 35,000 by the year 2000. If the growth continues at this rate, how long will it take for the population to reach 1 million?

The general form of the exponential growth model is much like the continuous compounding function you learned in the previous lesson. We can model exponential growth with a function of the form $P(\mathrm{t}) = P_0e^{kt}$. The expression $P(t)$ represents the population after t years, the coefficient P_0 represents the initial population, and k is a growth constant that depends on the particular situation.

In the situation above, we know that $P_0 = 20{,}000$ and that $P(10) = 35,000$. We can use this information to find the value of k:

$P(t) = P_0 e^{kt}$
$P(10) = 35000 = 20000e^{k \cdot 10}$
$\frac{35,000}{20000} = e^{10k}$
$1.75 = e^{10k}$
$ln1.75 = lne^{10k}$
$ln1.75 = 10klne$
$ln1.75 = 10k(1)$
$ln1.75 = 10k$
$k = \frac{ln1.75}{10} \approx 0.056$

Therefore we can model the population growth with the function $P(t) = 20000e^{\frac{ln1.75}{10}t}$. We can determine when the population will reach 1,000,000 by solving an equation, or using a graph.

Here is a solution using an equation:
$1000000 = 20000e^{\frac{ln1.75}{10}t}$
$50 = e^{\frac{ln1.75}{10}t}$
$ln50 = ln\left(e^{\frac{ln1.75}{10}t}\right)$
$ln50 = \frac{ln1.75}{10}t(lne)$
$ln50 = \frac{ln1.75}{10}t(1)$
$10ln50 = ln1.75t$
$t = \frac{10ln50}{ln1.75} \approx 70$

At this rate, it would take about 70 years for the population to reach 1 million. Like the initial doubling example, the growth rate may seem very fast. In reality, a population that grows exponentially may not sustain its growth rate over time. Next we will look at a different kind of function that can be used to model growth of this kind.

Logistic models

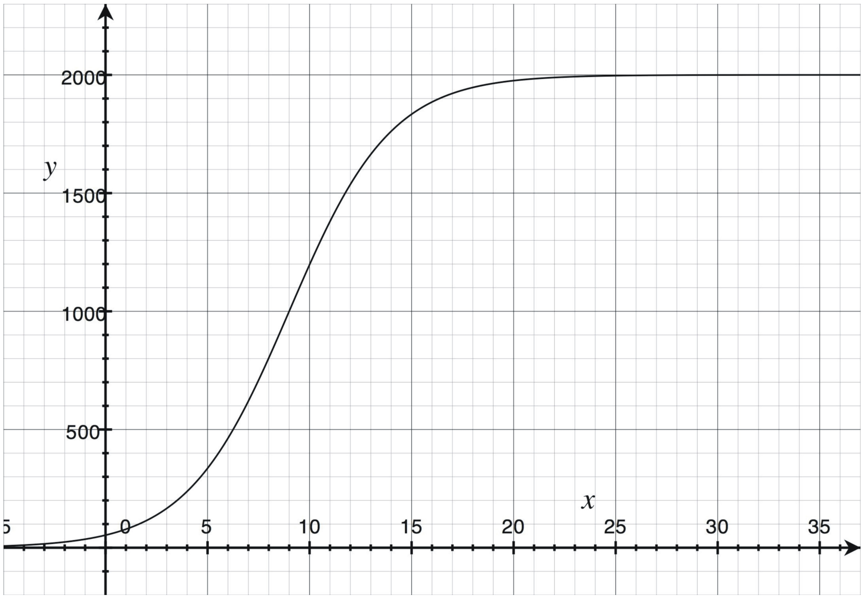
Given that resources are limited, a population may slow down in its growth over time. Consider the last example, the town whose population exploded in the 1990s. If there are no more houses to be bought, or tracts of land to be developed, the population will not continue to grow exponentially. The table below shows the population of this town slowing down, though still growing:

Table 4.35:

t (1990=0)	Population
0	20,000
10	35,000
15	38,000
20	40,000

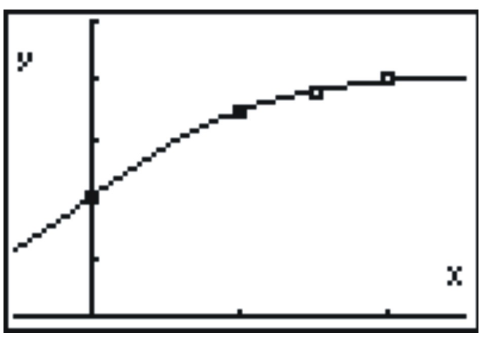
As the population growth slows down, the population may approach what is called a **carrying capacity,** or an upper bound of the population. We can model this kind of growth using a logistic function, which is a function of the form $f(x) = \frac{c}{1+a(e^{-bx})}$.

The graph below shows an example of a logistic function. This kind of graph is often called an ";s curve"; because of its shape.



Notice that the graph shows slow growth, then fast growth, and then slow growth again, as the population or quantity in question approaches the carrying capacity. Logistics functions are used to model population growth, as well as other situations, such as the amount of medicine in a person's system

Given the population data above, we can use a graphing calculator to find a logistic function to model this situation. The details of this process are explained in the Technology Note in Lesson 3.5. As shown there, enter the data into L_1 and L_2. Then run a logistic regression. (Press <TI font_STAT>, scroll right to CALC, and scroll down to B. Logistic.) An approximation of the logistic model for this data is: $f(x) = \frac{41042.38}{1+1.050e^{-.178x}}$. A graph of this function and the data is shown here.



Notice that the graph has a horizontal asymptote around 40,000. Looking at the equation, you should notice that the numerator is about 41,042. This value is in fact the horizontal asymptote, which represents the carrying capacity. We can understand why this is the carrying capacity if we consider the limit of the function as x approaches infinity. As x gets larger and larger, $e^{-.178x}$ will get smaller and smaller. So 1.05 $e^{-.178x}$ will get smaller. This means that the denominator of the function will get closer and closer to 1:

$$\lim_{x\to\infty}(1 + 1.050e^{-.178x}) = 1.$$

Therefore the limit of the function is (approximately) (41042/1) = 41042. This means that given the current growth, the model predicts that the population will not go beyond 41,042. This kind of growth is seen in population, as well as other situations in which some quantity grows very fast and then slows down, or when a quantity steeply decreases, and then levels off. You will see work with more examples of logistic functions in the review questions.

Exponential decay

Just as a quantity can grow, or increase exponentially, we can model a decreasing quantity with an exponential function. This kind of situation is referred to as exponential decay. Perhaps the most common example of exponential decay is that of radioactive decay, which refers to the transformation of an atom of one type into an atom of a different type, when the nucleus of the atom loses energy. The rate of radioactive decay is usually measured in terms of ";half-life,"; or the time it takes for half of the atoms in a sample to decay. For example Carbon-14 is a radioactive isotope that is used in ";carbon dating,"; a method of determining the age of organic materials. The half-life of Carbon-14 is 5730 years. This means that if we have a sample of Carbon-14, it will take 5730 years for half of the sample to decay. Then it will take another 5730 years for half of the remaining sample to decay, and so on.

We can model decay using the same form of equation we use to model growth, except that the exponent in the equation is negative: $A(t) = A_0\ e^{-kt}$. For example, say we have a sample of Carbon-14. How much time will pass before 75% of the original sample remains?

We can use the half-life of 5730 years to determine the value of k:

Table 4.36:

$A(t) = A_0 e^{-kt}$	
$\frac{1}{2} = 1e^{-k \cdot 5730}$	We do not know the value of A_0, *so we use ";1"; as 100%. (1/2) of the sample remains when* t = *5730 years*
$ln\frac{1}{2} = lne^{-k \cdot 5730}$	Take the ln of both sides
$ln\frac{1}{2} = -5730klne$	Use the power property of logs
$ln\frac{1}{2} = -5730k$	$ln(e) = 1$
$-ln2 = -5730k$	$ln(1/2) = ln(2^{-1}) = -ln2$
$ln2 = 5730k$	
$k = \frac{ln2}{5730}$	Isolate k

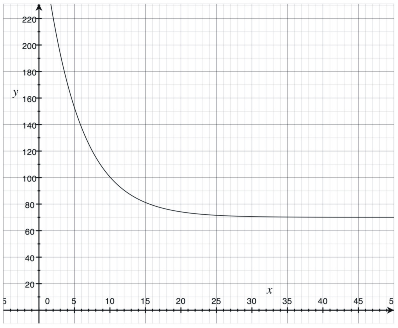
Now we can determine when the amount of Carbon-14 remaining is 75% of the original:

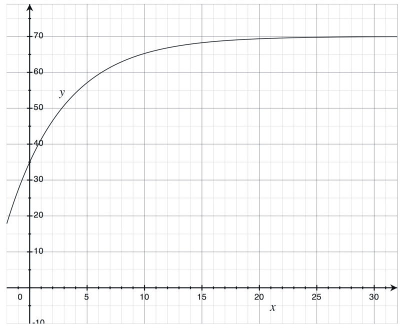
$$0.75 = 1e^{\frac{-ln2}{5730}t}$$

$$0.75 = 1e^{\frac{-ln2}{5730}t}$$
$$ln(0.75) = lne^{\frac{-ln2}{5730}t}$$
$$ln(0.75) = \frac{-ln2}{5730}t$$
$$t = \frac{5730ln(0.75)}{-ln2} \approx 2378$$

Therefore it would take about 2,378 years for 75% of the original sample to be remaining. In practice, scientists can approximate the age of an artifact using a process that relies on their knowledge of the half-life of Carbon-14, as well as the ratio of Carbon-14 to Carbon-12 (the most abundant, stable form of carbon) in an object. While the concept of half-life often is used in the context of radioactive decay, it is also used in other situations. In the review questions, you will see another common example, that of medicine in a person's system.

Related to exponential decay is Newton's Law of Cooling. The Law of Cooling allows us to determine the temperature of a cooling (or warming) object, based on the temperature of the surroundings and the time since the object entered the surroundings. The general form of the cooling function is $T(x) = T_5 + (T_0$ - $T_5)$ e^{-kx}, where T_5, is the surrounding temperature, T_0 is the initial temperature, and x represents the time since the object began cooling or warming.





The first graph shows a situation in which an object is cooling. The graph has a horizontal asymptote at $y = 70$. This tells us that the object is cooling to 70°F. The second graph has a horizontal asymptote at y = 70 as well, but in this situation, the object is warming up to 70°F.

We can use the general form of the function to answer questions about cooling (or warming) situations. Consider the following example: you are baking a casserole in a dish, and the oven is set to 325°F. You take the pan out of the oven and put it on a cooling rack in your kitchen which is 70°F, and after 10 minutes the pan has cooled to 300°F. How long will it take for the pan to cool to 200^0F?

We can use the general form of the equation and the information given in the problem to find the value of k:

$$T(x) = T_s + (T_0 - T_s)e^{-kx}$$
$$T(x) = 70 + (325 - 70)e^{-kx}$$
$$T(x) = 70 + (255)e^{-kx}$$
$$T(10) = 70 + 255e^{-10k} = 300$$
$$255e^{-10k} = 230$$
$$e^{-10k} = \frac{230}{255}$$
$$lne^{-10k} = ln\left(\frac{230}{255}\right)$$
$$-10k = ln\left(\frac{230}{255}\right)$$
$$k = \frac{ln\left(\frac{230}{255}\right)}{-10} \approx 0.0103$$

Now we can determine the amount of time it takes for the pan to cool to 200 degrees:

$$T(x) = 70 + (255)e^{-.0103}x$$
$$T(x) = 70 + (255)e^{-.0103}x$$
$$200 = 70 + (255)e^{-.0103}x$$
$$130 = (255)e^{-.0103}x$$
$$\frac{130}{255} = e^{-.0103}x$$
$$ln\left(\frac{130}{255}\right) = lne^{-.0103}x$$
$$ln\left(\frac{130}{255}\right) = -.0103x$$
$$x = \frac{ln\left(\frac{130}{255}\right)}{-.0103} \approx 65$$

Therefore, in the given surroundings, it would take about an hour for the pan to cool to 200 degrees.

Lesson Summary

In this lesson we have developed exponential and logistic models to represent different phenomena. We have considered exponential growth, logistic growth, and exponential decay. After reading the examples in this lesson, you should be able to write a function to represent a given situation, to evaluate the function for a given value of x, and to solve exponential equations in order to find values of x, given values of the function. For example, in a situation of exponential population growth as a function of time, you should be able to determine the population at a particular time, and to determine the time it takes for the population to reach a given amount. You should be able to solve these kinds of problems by solving exponential equations, and by using graphing utilities, as we have done throughout the chapter.

Points to Consider

1. How can we use the same equation for exponential growth and decay?
2. What are the restrictions on domain and range for the examples in this lesson?
3. How can we use different equations to model the same situations?

Review Questions

1. The population of a town was 50,000 in 1980, and it grew to 70,000 by 1995. a. Write an exponential function to model the growth of the population.
   b. Use the function to estimate the population in 2010.
   c. What if the population growth was linear? Write a linear equation to model the population growth, and use it to estimate the population in 2010.
2. A telecommunications company began providing wireless service in 1994, and during that year the company had 1000 subscribers. By 2004, the company had 12,000 subscribers. a. Write an exponential function to model the situation
   b. Use the model to determine how long it will take for the company to reach 50,000 subscribers.
3. The population of a particular strain of bacteria triples every 8 hours. a. Write a general exponential function to model the bacteria growth.
   b. Use the model to determine how long it will take for a sample of bacteria to be 100 times its original population.
   c. Use a graph to verify your solution to part b.
4. The half-life of acetaminophen is about 2 hours. a. If you take 650 mg of acetaminophen, how much will be left in your system after 7 hours?
   b. How long before there is less than 25 mg in your system?
5. The population of a city was 200,000 in 1991, and it decreased to 170,000 by 2001. a. Write an exponential function to model the decreasing population, and use the model to predict the population in 2008.
   b. Under what circumstances might the function cease to model the situation after a certain point in time?
6. Consider the following situation: you buy a large box of pens for the start of the school year, and after six weeks, (1/3) of the pens remain. After another six weeks, (1/3) of the remaining pens were remaining. If you continue this pattern, when will you only have 5% of the pens left?
7. Use Newton's law of cooling to answer the question: you pour hot water into a mug to make tea. The temperature of the water is about 200 degrees. The surrounding temperature is about 75 degrees. You let the water cool for 5 minutes, and the temperature decreases to 160 degrees. What will the temperature be after 15 minutes?
8. The spread of a particular virus can be modeled with the logistic function $f(x) = \frac{2000}{1+600e^{-.75x}}$, where

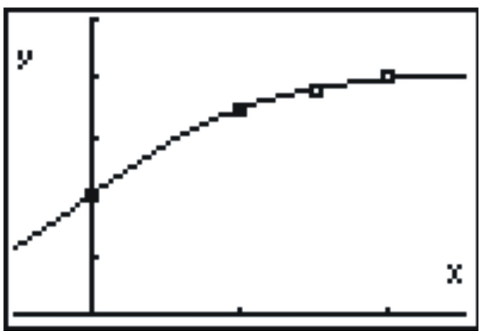
x is the number of days the virus has been spreading, and f(x) represents the number of people who have the virus.a. How many people will be affected after 7 days?

b. How many days will it take for the spread to be within one person of carrying capacity?

9. Consider again the situation in problem #2: A telecommunications company began providing wireless service in 1994, and during that year the company had 1000 subscribers. By 2004, the company had 12,000 subscribers. If the company has 15,000 subscribers in 2005, and 16,000 in 2007, what type of model do you think should be used to model the situation? Use a graphing calculator to find a regression equation, and use the equation to predict the number of subscribers in 2010.
10. Compare exponential and logistic functions as tools for modeling growth. What do they have in common, and how do they differ?

Review Answers

1. a. $A(T) = 50,000e^{\frac{ln(\frac{7}{5})}{15}t}$b. $98,000$
   c. $f(t) = \frac{4000}{3}t + 50000$. The population would be 90,000, which is different by about 9%.
2. a. $S(t) = 1000e^{\frac{ln(12)}{10}t}$b. $t = \frac{10ln50}{ln12} \approx 15.74$
3. a. $A(t) = A_0\ (3^{t/8})$b. $t = \frac{16}{log3} \approx 33.53$
   c. The graph below shows $y = 100$ and $y = 3^{x/8}$, which intersect at approximately $x = 33.53$



4. a. About 57.45 mg b. About 9.4 hours
5. a. $P(t) = 200000e^{\frac{-ln.85}{-10}t}, P(17) \approx 151720$b. If the economy or other factors change, the population might begin to increase, or the rate of decrease could change as well.
6. $t = \frac{6log(0.05)}{log(\frac{1}{3})} \approx 16$ weeks
7. About 114 degrees.
8. a. About 482 people. b. After 19 days, over 1999 people have the virus.
9. The graph indicates a logistic model. $f(x) \approx \frac{18872}{1+21.45e^{-377x}}$ gives 17952 subscribers in 2010.
10. Both types of functions model fast increase in growth, but the logistic model shows the growth slowing down after some point, with some upper bound on the quantity in question. (Many people argue that logistic growth is more realistic.)

Vocabulary

Carrying capacity The supportable population of an organism, given the food, habitat, water and other necessities available within an ecosystem is known as the ecosystem's carrying capacity for that organism.

Radioactive decay Radioactive decay is the process in which an unstable atomic nucleus loses energy by emitting radiation in the form of particles or electromagnetic waves. This decay, or loss of energy, results in an atom of one type transforming to an atom of a different type. For example, Carbon-14 transforms into Nitrogen-14

Half-life The amount of time it takes for half of a given amount of a substance to decay. The half-life remains the same, no matter how much of the substance there is.

Isotope Isotopes are any of two or more forms of a chemical element, having the same number of protons in the nucleus, or the same atomic number, but having different numbers of neutrons in the nucleus, or different atomic weights.

4.5 Applications

Learning objectives

- Work with the decibel system for measuring loudness of sound.
- Work with the Richter scale, which measures the magnitude of earthquakes.
- Work with pH values and concentrations of hydrogen ions.

Introduction

Because logarithms are related to exponential relationships, logarithms are useful for measuring phenomena that involve very large numbers or very small numbers. In this lesson you will learn about three situations in which a quantity is measured using logarithms. In each situation, a logarithm is used to simplify measurements of either very small numbers or very large numbers. We begin with measuring the intensity of sound.

Intensity of sound

Sound intensity is measured using a logarithmic scale. The intensity of a sound wave is measured in Watts per square meter, or W/m^2. Our hearing threshold (or the minimum intensity we can hear at a frequency of 1000 Hz), is $2.5 \times 10^{12}\ W/m^2$. The intensity of sound is often measured using the decibel (dB) system. We can think of this system as a function. The input of the function is the intensity of the sound, and the output is some number of decibels. The decibel is a dimensionless unit; however, because decibels are used in common and scientific discussions of sound, the values of the scale have become familiar to people.

We can calculate the decibel measure as follows:

$$\text{Intensity level } (dB) = 10log\left[\frac{\text{intensity of sound in } W/m^2}{.937\times10^{-12}W/m^2}\right]$$

An intensity of $.937 \times 10^{-12}\ W/m^2$ corresponds to 0 decibels:

$$10log\left[\frac{.937\times10^{-12}W/m^2}{.937\times10^{-12}W/m^2}\right] = 10log1 = 10(0) = 0.$$

Note: The sound equivalent to 0 decibels is approximately the lowest sound that humans can hear. If the intensity is ten times as large, the decibel level is 10:

$10log\left[\frac{.937\times10^{-11}W/m^2}{.937\times10^{-12}W/m^2}\right] = 10log10 = 10(1) = 10$

If the intensity is 100 times as large, the decibel level is 20, and if the intensity is 1000 times as large, the decibel level is 30. (The scale is created this way in order to correspond to human hearing. We tend to underestimate intensity.) The threshold for pain caused by sound is 1 W/m^2. This intensity corresponds to about 120 decibels:

$10log\left[\frac{1W/m^2}{.937\times10^{-12}}W/m^2\right] \approx 10:12 = 120$

Many common phenomena are louder than this. For example, a jet can reach about 140 decibels, and concert can reach about 150 decibels.

(Source: Ohanian, H.C. (1989) Physics. New York: W.W. Norton & Company.)

For ease of calculation, the equation is often simplified: .937 is rounded to 1:

Table 4.37:

Intensity level (dB)	$= 10log\left[\frac{\text{intensity of sound in } W/m^2}{1\times10^{-12}W/m^2}\right]$
	$= 10log\left[\frac{\text{intensity of sound in } W/m^2}{10^{-12}W/m^2}\right]$

In the example below we will use this simplified equation to answer a question about decibels. (In the review exercises, you can also use this simplified equation).

Example 1: Verify that a sound of intensity 100 times that of a sound of 0 dB corresponds to 20 dB.

Solution: $dB = 10log\left(\frac{100\times10^{-12}}{10^{-12}}\right) = 10log(100) = 10(2) = 20.$

Intensity and magnitude of earthquakes

An earthquake occurs when energy is released from within the earth, often caused by movement along fault lines. An earthquake can be measured in terms of its intensity, or its magnitude. Intensity refers to the effect of the earthquake, which depends on location with respect to the epicenter of the quake. Intensity and magnitude are not the same thing.

As mentioned in lesson 3, the magnitude of an earthquake is measured using logarithms. In 1935, scientist Charles Richter developed this scale in order to compare the ";size"; of earthquakes. You can think of Richter scale as a function in which the input is the amplitude of a seismic wave, as measured by a seismograph, and the output is a magnitude. However, there is more than one way to calculate the magnitude of an earthquake because earthquakes produce two different kinds of waves that can be measured for amplitude. The calculations are further complicated by the need for a correction factor, which is a function of the distance between the epicenter and the location of the seismograph.

Given these complexities, seismologists may use different formulas, depending on the conditions of a specific earthquake. This is done so that the measurement of the magnitude of a specific earthquake is consistent with Richter's original definition. (Source: http://earthquake.usgs.gov/learning/topics/richter.php)

Even without a specific formula, we can use the Richter scale to compare the size of earthquakes. For example, the 1906 San Francisco earthquake had a magnitude of about 7.7. The 1989 Loma Prieta earthquake had a magnitude of about 6.9. (The epicenter of the quake was near Loma Prieta peak in the

Santa Cruz mountains, south of San Francisco.) Because the Richter scale is logarithmic, this means that the 1906 quake was six times as strong as the 1989 quake:

$$\frac{10^{7.7}}{10^{6.9}} = 10^{7.7-6.9} = 10^{.8} \approx 6.3$$

This kind of calculation explains why magnitudes are reported using a whole number and a decimal. In fact, a decimal difference makes a big difference in the size of the earthquake, as shown below and in the review exercises

Example 2: An earthquake has a magnitude of 3.5. A second earthquake is 100 times as strong. What is the magnitude of the second earthquake?

Solution: The second earthquake is 100 times as strong as the earthquake of magnitude 3.5. This means that if the magnitude of the second earthquake is x, then:

$$\frac{10^x}{10^{3.5}} = 100$$
$$10^{x-3.5} = 100 = 10^2$$
$$x - 3.5 = 2$$
$$x = 5.5$$

So the magnitude of the second earthquake is 5.5.

The pH scale

If you have studied chemistry, you may have learned about acids and bases. An acid is a substance that produces hydrogen ions when added to water. A hydrogen ion is a positively charged atom of hydrogen, written as H^+. A base is a substance that produces hydroxide ions (OH^-) when added to water. Acids and bases play important roles in everyday life, including within the human body. For example, our stomachs produce acids in order to breakdown foods. However, for people who suffer from gastric reflux, acids travel up to and can damage the esophagus. Substances that are bases are often used in cleaners, but a strong base is dangerous: it can burn your skin.

To measure the concentration of an acid or a base in a substance, we use the pH scale, which was invented in the early 1900's by a Danish scientist named Soren Sorenson. The pH of a substance depends on the concentration of H^+, which is written with the symbol $[H^+]$.

pH = - log $[H^+]$

(Note: concentration is usually measured in moles per liter. A mole is 6.02×10^{23} units. Here, it would be 6.02×10^{23} hydrogen ions.)

For example, the concentration of H^+ in stomach acid is about 1×10^{-1}. So the pH of stomach acid is -log (10^{-1}) = -(-1) = 1. The pH scale ranges from 0 to 14. A substance with a low pH is an acid. A substance with a high pH is a base. A substance with a pH in the middle of the scale is considered to be neutral.

Example 3: The pH of ammonia is 11. What is the concentration of H^+?

Solution: pH = - log $[H^+]$. If we substitute 11 for pH we can solve for H^+:

$$11 = -log[H^+]$$
$$-11 = log[H^+]$$
$$10^{-11} = 10^{log[H^+]}$$
$$10^{-11} = H^+$$

Lesson Summary

In this lesson we have looked at three examples of logarithmic scales. In the case of the decibel system, using a logarithm has produced a simple way of categorizing the intensity of sound. The Richter scale allows us to compare earthquakes. And, the pH scale allows us to categorize acids and bases. In each case, a logarithm helps us work with large or small numbers, in order to more easily understand the quantities involved in certain real world phenomena.

Points to Consider

1. How are the decibel system and the Richter scale the same, and how are they different?
2. What other phenomena might be modeled using a logarithmic scale?

Review Questions

1. Verify that a sound of intensity 1000 times that of a sound of 0 dB corresponds to 30 dB.
2. Calculate the decibel level of a sound with intensity 10^{-8} W/m^2.
3. Calculate the intensity of a sound if the decibel level is 25.
4. The 2004 Indian Ocean earthquake was recorded to have a magnitude of about 9.5. In 1960, an earthquake in Chile was recorded to have a magnitude of 9.1. How much stronger was the 2004 Indian Ocean quake?
5. Two earthquakes of the same magnitude do not necessarily cause the same amount of destruction. How is that possible?
6. The concentration of H^+ in pure water is 1×10^{-7}. What is the pH?
7. The pH of normal human blood is 7.4. What is the concentration of H^+

Review Answers

1. $dB = 10log\left(\frac{100\times10^{-12}}{10^{-12}}\right) = 10log(100) = 10(3) = 30$
2. $dB = 10log\left(\frac{10^{-8}}{10^{-12}}\right) = 10log(10000) = 10(4) = 40$
3. $10^{-9.5}$ or 3.16×10^{-10}
4. $10^{0.4} \approx 2.5$
5. According to the USGS, the damage depends on the strength of shaking, the length of shaking, the type of soil in the area, and the types of buildings. Many buildings in the San Francisco Bay Area are undergoing ";seismic retrofitting,"; in anticipation of ";the big one.";
6. The pH is 7.
7. $10^{-7.4} \approx 3.98 \times 10^{-8}$

Vocabulary

Acid An acid is a substance that produces hydrogen ions when added to water.

Amplitude The amplitude of a wave is the distance from its highest (or lowest) point to its center.

Base A base is a substance that produces hydroxide ions (OH^-) when added to water

Decibel A decibel is a unitless measure of the intensity of sound.

Mole 6.02×10^{23} units of a substance.

Seismograph A seismograph is a device used to measure the amplitude of earthquakes.

Chapter 5

Quadratic Equations and Functions

5.1 Quadratics

As you saw in Chapter 2, algebraic functions not only produce straight lines but curved ones too. A special type of curved function is called a parabola. Perhaps you have seen the shape of a parabola before:

- The shape of the water from a drinking fountain
- The path a football takes when thrown
- The shape of an exploding firework
- The shape of a satellite dish
- The path a diver takes into the water
- The shape of a mirror in a car's headlamp

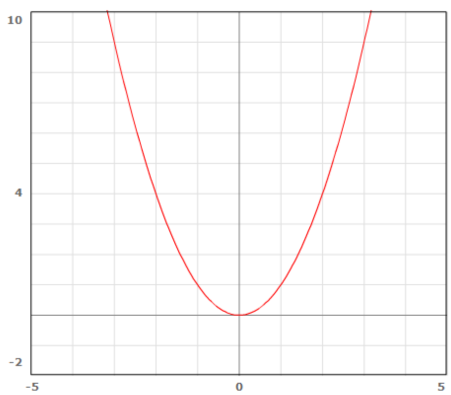
Many real life situations model a quadratic equation. This chapter will explore the graph of a quadratic equation and how to solve such equations using various methods.

5.2 Graphs of Quadratic Functions

Chapter 9 introduced the concept of factoring quadratic trinomials of the form $0 = ax^2 + bx + c$. This is also called the **standard form for a quadratic equation.** The most basic quadratic equation is $y = x^2$. The word quadratic comes from the Latin word *quadrare,* meaning ";to square."; By creating a table of values and graphing the ordered pairs, you find that a quadratic equation makes a *U*-shaped figure called a **parabola.**

Table 5.1:

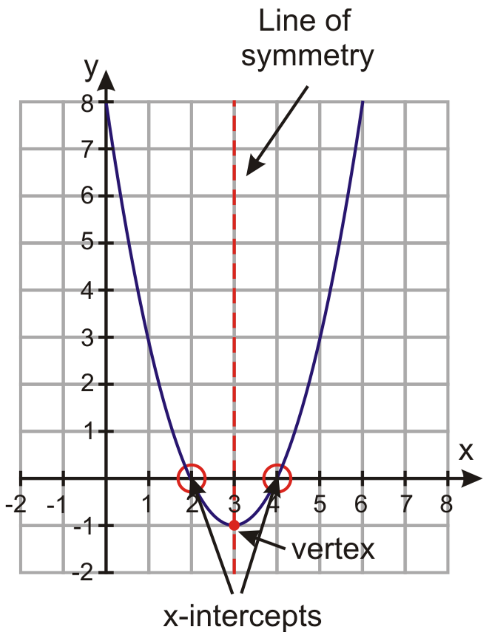
x	y
–2	4
–1	1
0	0
1	1
2	4



The Anatomy of a Parabola

A parabola can be divided in half by a vertical line. Because of this, parabolas have **symmetry.** The vertical line dividing the parabola into two equal portions is called the **line of symmetry.** All parabolas have a **vertex,** the ordered pair that represents the bottom (or the top) of the curve.

The vertex of a parabola has an ordered pair (h, k).



Because the line of symmetry is a vertical line, its equation has the form $y = h$, where $h = \textit{the } x\textit{-coordinate of the vertex.}$

An equation of the form $y = ax^2$ forms a parabola.

If a is positive, the parabola will open **upward.** The vertex will be a **minimum.**

If a is negative, the parabola will open **downward.** The vertex will be a **maximum.**

The variable a in the equation above is called the **leading coefficient** of the quadratic equation. Not only will it tell you if the parabola opens up or down, but it will also tell you the width.

If $a > 1$ or $a < -1$, the parabola will be **narrow** about the line of symmetry.

If -1, the parabola will be **wide** about the line of symmetry.

Example 1: *Determine the direction and shape of the parabola formed by* $y = -\frac{1}{2}x^2$.

Solution: The value of a in the quadratic equation is –1.

- Because a is negative, the parabola opens downward.
- Because a is between –1 and 1, the parabola is wide about its line of symmetry.

Domain and Range

Several times throughout this textbook, you have experienced the terms **domain** and **range.** Remember:

- Domain is the set of all inputs ($x-$coordinates).
- Range is the set of all outputs ($y-$coordinates).

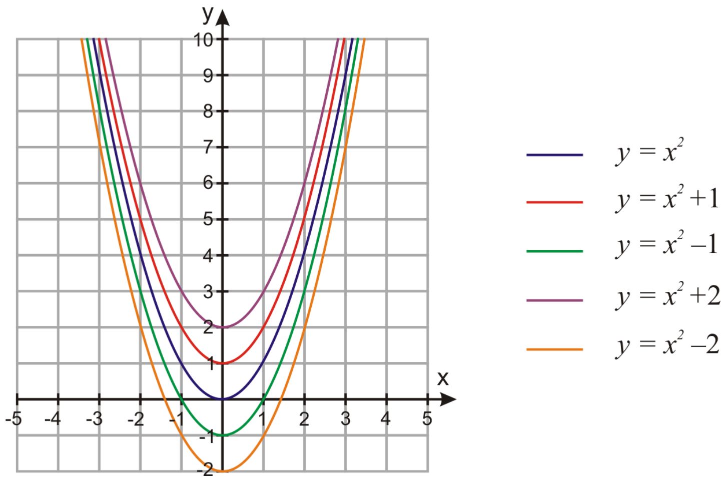
The domain of every quadratic equation is all real numbers ($\mathbb{R}$). The range of a parabola depends upon whether the parabola opens up or down.

If a is positive, the range will be $y \geq k$.

If a is negative, the range will be $y \leq k$, where $k = y\textit{-coordinate of the vertex.}$

Vertical Shifts

Compare the five parabolas to the right. What do you notice?



The five different parabolas are congruent with different $y-$intercepts. Each parabola has an equation of the form $y = ax^2 + c$, where $a = 1$ and $c = y-intercept$. In general, the value of c will tell you where the parabola will intersect the $y-$axis.

The equation $y = ax^2 + c$ is a parabola with a $y-$intercept of $(0, c)$.

The vertical movement along a parabola's line of symmetry is called a **vertical shift.**

Example 1: *Determine the direction, shape, and y−intercept of the parabola formed by* $y = \frac{3}{2}x^2 - 4$.

Solution: The value of a in the quadratic equation is $\frac{3}{2}$.

- Because a is positive, the parabola opens upward.
- Because a is greater than 1, the parabola is narrow about its line of symmetry.
- The value of c is –4, so the $y-$intercept is (0, –4).

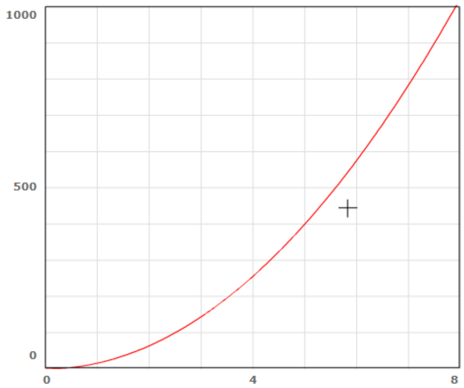
Projectiles are often described by quadratic equations. When an object is dropped from a tall building or cliff, it does not travel at a constant speed. The longer it travels, the faster it goes. Galileo described this relationship between distance fallen and time. It is known as his kinematical law. It states the ";distance traveled varies directly with the square of time."; As an algebraic equation, this law is:

$$d = 16t^2$$

Use this information to graph the distance an object travels during the first six seconds.

Table 5.2:

t	d
0	0
1	16
2	64
3	144
4	256
5	400
6	576



The parabola opens upward and its vertex is located at the origin. The value of $a > 1$, so the graph is narrow about its line of symmetry. However, because the values of the dependent variable d are very large, the graph is misleading.

Example 2: *Anne is playing golf. On the fourth tee, she hits a slow shot down the level fairway. The ball follows a parabolic path described by the equation, $y = x - 0.04x^2$, where $x =$ distance in feet from the tee and $y =$ height of the golf ball, in feet.*

Describe the shape of this parabola. What is its $y-$intercept?

Solution: The value of a in the quadratic equation is –0.04.

- Because a is negative, the parabola opens downward.
- Because a is between –1 and 1, the parabola is wide about its line of symmetry.
- The value of c is 0, so the $y-$intercept is (0, 0).

The distance it takes a car to stop (in feet) given its speed (in miles per hour) is given by the function $d(s) = \frac{1}{20}\ s^2 + s$. This equation is in standard form $f(x) = ax^2 + bx + c$, where $a = \frac{1}{20}, b = 1$, and $c = 0$.

Graph the function by making a table of speed values.

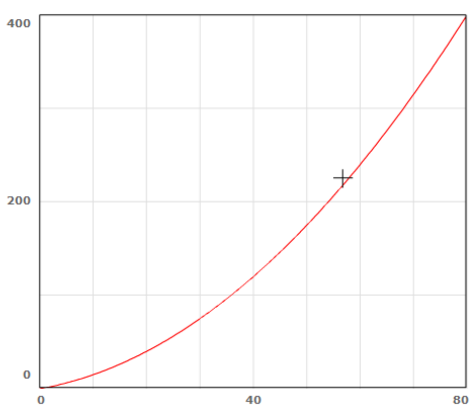


Table 5.3:

s	d
0	0
10	15
20	40

Table 5.3: (continued)

s	d
30	75
40	120
50	175
60	240

- The parabola opens upward with a vertex at (0, 0).
- The line of symmetry is $x = 0$.
- The parabola is wide about its line of symmetry.

Using the function to find the stopping distance of a car travelling 65 miles per hour yields:

$$d(65) = \frac{1}{20}(65)^2 + 65 = 276.25 \ feet$$

Multimedia Link: For more information regarding stopping distance, watch this CK-12 Basic Algebra: Algebra Applications: Quadratic Functions - YouTube video.

Figure 5.1: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/258

Practice Set

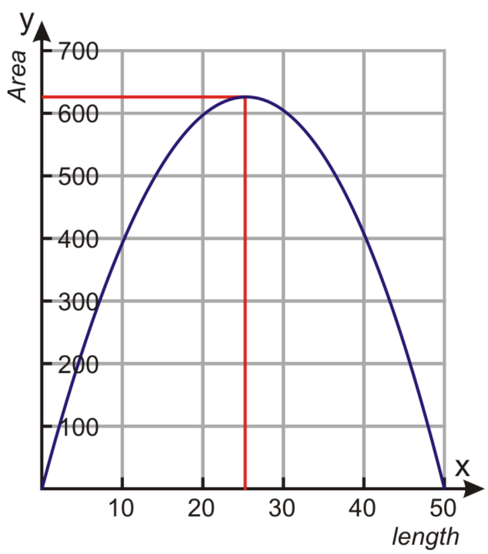
Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Graphs of Quadratic Functions (16:05)

1. Define the following terms in your own words.
   (a) Vertex
   (b) Line of symmetry
   (c) Parabola
   (d) Minimum
   (e) Maximum
2. Without graphing, how can you tell if $y = ax^2 + bx + c$ opens up or down?
3. Using the parabola below, identify the following:

Figure 5.2: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/801

(a) Vertex
(b) $y-$intercept
(c) $x-$intercepts
(d) Domain
(e) Range
(f) Line of symmetry
(g) Is a positive or negative?
(h) Is a -1 or $a < -1$ or $a > 1$?



4. Use the stopping distance function from the lesson to find:

   (a) $d(45)$
   (b) What speed has a stopping distance of about 96 feet?

5. Using Galileo's law from the lesson, find:

   (a) The distance an object has fallen at 3.5 seconds
   (b) The total distance the object has fallen in 3.5 seconds

Graph the following equations by making a table. Let $-3 \le x \le 3$. Determine the range of each equation.

6. $y = 2x^2$
7. $y = -x^2$
8. $y = x^2 - 2x + 3$

9. $y = 2x^2 + 4x + 1$
10. $y = -x^2 + 3$
11. $y = x^2 - 8x + 3$
12. $y = x^2 - 4$

Which has a more positive $y-$intercept?

13. $y = x^2$ or $y = 4x^2$
14. $y = 2x^2 + 4$ or $y = \frac{1}{2}x^2 + 4$
15. $y = -2x^2 - 2$ or $y = -x^2 - 2$

Identify the vertex and $y-$intercept. Is the vertex a maximum or a minimum?

16. $y = x^2 - 2x - 8$
17. $y = -x^2 + 10x - 21$
18. $y = 2x^2 + 6x + 4$

Does the graph of the parabola open up or down?

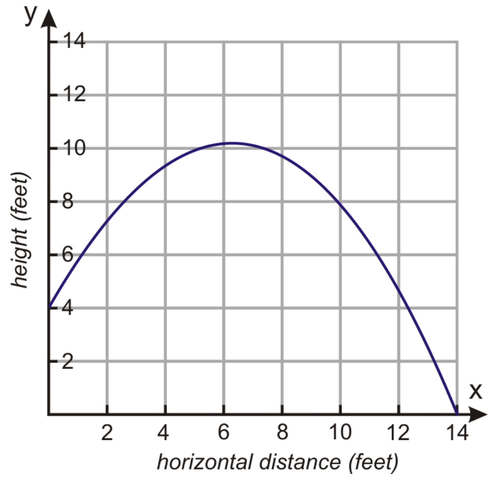
19. $y = -2x^2 - 2x - 3$
20. $y = 3x^2$
21. $y = 16 - 4x^2$

Which equation has a larger vertex?

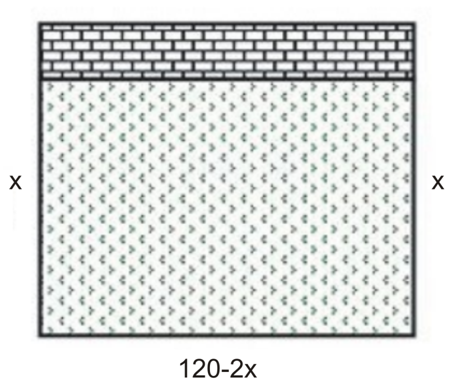
22. $y = x^2$ or $y = 4x^2$
23. $y = -2x^2$ or $y = -2x^2 - 2$
24. $y = 3x^2 - 3$ or $y = 3x^2 - 6$

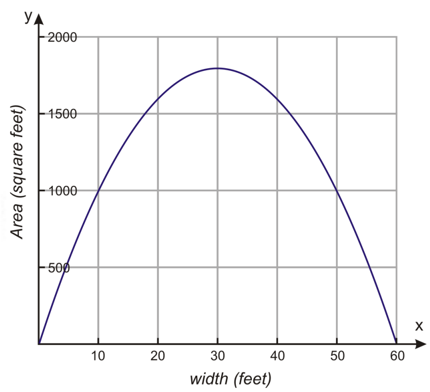
Graph the following functions by making a table of values. Use the vertex and $x-$intercepts to help you pick values for the table.

25. $y = 4x^2 - 4$
26. $y = -x^2 + x + 12$
27. $y = 2x^2 + 10x + 8$
28. $y = \frac{1}{2}x^2 - 2x$
29. $y = x - 2x^2$
30. $y = 4x^2 - 8x + 4$
31. Nadia is throwing a ball to Peter. Peter does not catch the ball and it hits the ground. The graph shows the path of the ball as it flies through the air. The equation that describes the path of the ball is $y = 4 + 2x - 0.16x^2$. Here, y is the height of the ball and x is the horizontal distance from Nadia. Both distances are measured in feet. How far from Nadia does the ball hit the ground? At what distance, x, from Nadia, does the ball attain its maximum height? What is the maximum height?



32. Peter wants to enclose a vegetable patch with 120 feet of fencing. He wants to put the vegetable patch against an existing wall, so he needs fence for only three of the sides. The equation for the area is given by $a = 120x - x^2$. From the graph, find what dimensions of the rectangle would give him the greatest area.





Mixed Review

33. Factor $6u^2v - 11u^2v^2 - 10u^2v^3$ using its GCF.
34. Factor into primes: $3x^2 + 11x + 10$.
35. Simplify $-\frac{1}{9}(63)\left(-\frac{3}{7}\right)$.
36. Solve for $b : |b + 2| = 9$.
37. Simplify $(4x^3y^2z)^3$.
38. What is the slope and $y-$intercept of $7x + 4y = 9$?

5.3 Solving Quadratic Equations by Graphing

Isaac Newton's theory for projectile motion is represented by the equation:

$$h(t) = -\frac{1}{2}(g)t^2 + v_0t + h_0$$

- $t =$*time* (*usually in seconds*)
- $g =$*gravity due to acceleration; either* $9.8m/s^2$ *or* $32ft/s^2$

- $v_0 =$*initial velocity*
- $h_0 =$*initial height of object*

Consider the following situation: ";A quarterback throws a football at an initial height of 5.5 feet with an initial velocity of 35 feet per second.";

By substituting the appropriate information:

- $g = 32$ because the information is given in feet
- $v_0 = 35$
- $h_0 = 5.5$

The equation becomes $h(t) = -\frac{1}{2}(32)t^2 + 35t + 5.5 \rightarrow h(t) = -16t^2 + 35t + 5.5$.

Using the concepts from the previous lesson, we know

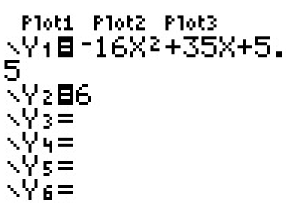
- The value of a is negative, so the parabola opens downward.
- The vertex is a maximum point.
- The $y-$intercept is (0, 5.5).
- The graph is narrow about its line of symmetry.

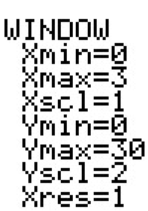
At what time will the football be 6 feet high? This equation can be solved by graphing the quadratic equation.

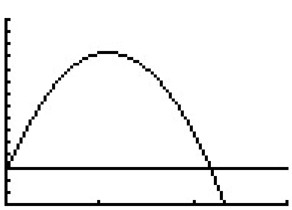
Solving a Quadratic Using a Calculator

Chapter 7 focused on how to solve systems by graphing. You can think of this situation as a system: $\begin{cases} y = -16t^2 + 35t + 5.5 \\ y = 6 \end{cases}$. You are looking for the appropriate $x-$coordinates that give a $y-$coordinate of 6 feet. Therefore, you are looking for the **intersection** of the two equations.

Begin by typing the equations into the $[Y =]$ menu of your calculator. Adjust the window until you see the vertex, $y-$intercept, $x-$intercepts, and the horizontal line of 6 units.







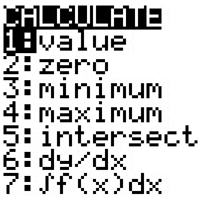
By looking at the graph, you can see there are **two** points of intersection. Using the methods from chapter 7, find both points of intersection.

$$(0.014, 6) \textit{ and } (2.172, 6)$$

At 0.014 seconds and again at 2.17 seconds, the football is six feet from the ground.

Using a Calculator to Find the Vertex

You can also use a graphing calculator to determine the **vertex** of the parabola. The vertex of this equation is a maximum point, so in the [**CALCULATE**] menu of the graphing option, look for [**MAXIMUM**].



Choose option #4. The calculator will ask you, ";LEFT BOUND?"; Move the cursor to the left of the vertex and hit [**ENTER**].

The calculator will ask, ";RIGHT BOUND?"; Move the cursor to the right of the vertex and hit [**ENTER**].

Hit [**ENTER**] to guess.

The maximum point on this parabola is (1.09, 24.64).

Example 1: *Will the football reach 25 feet high?*

Solution: The vertex represents the maximum point of this quadratic equation. Since its height is 24.64 feet, we can safely say the football will **not** reach 25 feet.

Example 2: *When will the football hit the ground, assuming no one will catch it?*

Solution: We know want to know at what time the height is zero. $\begin{cases} y = -16t^2 + 35t + 5.5 \\ y = 0 \end{cases}$. By repeating the process above and finding the intersection of the two lines, the solution is (2.33, 0). At 2.33 seconds, the ball will hit the ground.

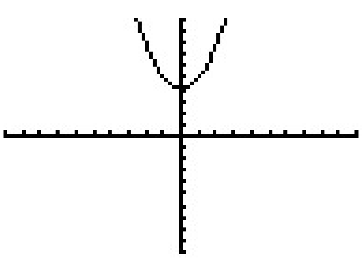
The point at which the ball reaches the ground ($y = 0$) represents the $x-$**intercept** of the graph.

The $x-$intercept of a quadratic equation is also called a **root, solution,** or **zero.**

Example: *Determine the number of solutions to* $y = x^2 + 4$.

Solution: Graph this quadratic equation, either by hand or with a graphing calculator. Adjust the calculator's window to see both halves of the parabola, the vertex, the $x-$axis, and the $y-$intercept.

The solutions to a quadratic equation are also known as its $x-$**intercepts.** This parabola does not cross the $x-$axis. Therefore, this quadratic equation has **no real solutions.**



Example: *Andrew has 100 feet of fence to enclose a rectangular tomato patch. He wants to find the dimensions of the rectangle that encloses the most area.*

Solution: The perimeter of a rectangle is the sum of all four sides. Let $w = \mathit{width}$ and $l = \mathit{length}$. The perimeter of the tomato patch is $100 = l + l + w + w \rightarrow 100 = 2l + 2w$.

The area of a rectangle is found by the formula $A = l(w)$. We are looking for the intersection between the area and perimeter of the rectangular tomato patch. This is a system.

$$\begin{cases} 100 = 2l + 2w \\ A = l(w) \end{cases}$$

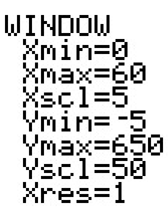
Before we can graph this system, we need to rewrite the first equation for either l or w. We will then use the Substitution Property.

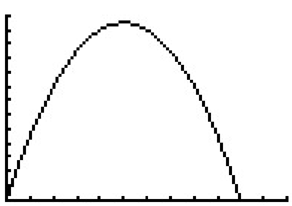
$$100 = 2l + 2w \rightarrow 100 - 2l = 2w$$
$$\frac{100 - 2l}{2} = w \rightarrow 50 - l = w$$

Use the Substitution Property to replace the variable w in the second equation with the express $50 - l$.

$$A = l(50 - l) = 50l - l^2$$

Graph this equation to visualize it.





The parabola opens downward so the vertex is a maximum. The maximum value is (25, 625). The length of the tomato patch should be 25 feet long to achieve a maximum area of 625 square feet.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Solving Quadratic Equations by Graphing (10:51)

1. What are the alternate names for the solution to a parabola?
2. Define the following variables in the function $h(t) = -\frac{1}{2}(g)t^2 + v_0 t + h_0$.

Figure 5.3: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/804

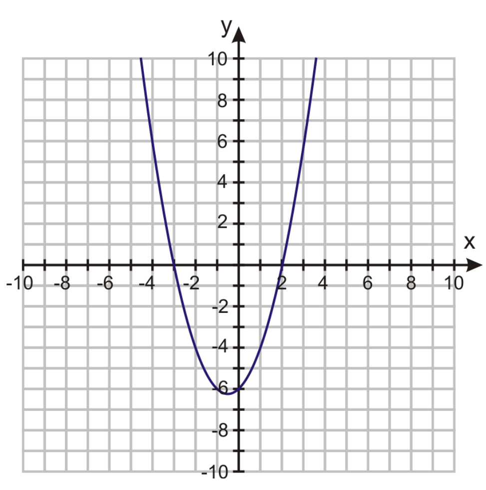
(a) h_0
(b) t
(c) v_0
(d) g
(e) $h(t)$

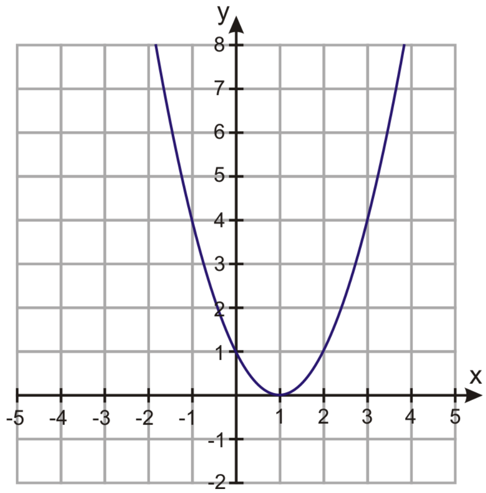
3. A rocket is launched from a height of 3 meters with an initial velocity of 15 meters per second.

(a) Model the situation with a quadratic equation.
(b) What is the maximum height of the rocket? When will this occur?
(c) What is the height of the rocket after four seconds? What does this mean?
(d) When will the rocket hit the ground?
(e) At what time will the rocket be 13 meters from the ground?

How many solutions does the quadratic equation have?

4.



5. 

6. $-x^2 + 3 = 0$
7. $2x^2 + 5x - 7 = 0$
8. $-x^2 + x - 3 = 0$

Find the zeros of the quadratic equations below. If necessary, round your answers to the nearest hundredth.

9. $y = -x^2 + 4x - 4$
10. $y = 3x^2 - 5x$
11. $x^2 + 3x + 6 = 0$
12. $-2x^2 + x + 4 = 0$
13. $x^2 - 9 = 0$
14. $x^2 + 6x + 9 = 0$
15. $10x^2 - 3x^2 = 0$
16. $\frac{1}{2}x^2 - 2x + 3 = 0$
17. $y = -3x^2 + 4x - 1$
18. $y = 9 - 4x^2$
19. $y = x^2 + 7x + 2$
20. $y = -x^2 - 10x - 25$
21. $y = 2x^2 - 3x$
22. $y = x^2 - 2x + 5$
23. Andrew is an avid archer. He launches an arrow that takes a parabolic path, modeled by the equation $y = -4.9t^2 + 48t$. Find how long it takes the arrow to come back to the ground.

For questions 24 – 26,

(a) Find the roots of the quadratic polynomial.

(b) Find the vertex of the quadratic polynomial.

24. $y = x^2 + 12x + 5$
25. $y = x^2 + 3x + 6$
26. $y = -x^2 - 3x + 9$
27. Sharon needs to create a fence for her new puppy. She purchased 40 feet of fencing to enclose three sides of a fence. What dimensions will produce the greatest area for her puppy to play?
28. An object is dropped from the top of a 100-foot-tall building.

    (a) Write an equation to model this situation.

(b) What is the height of the object after 1 second?
(c) What is the maximum height of the object?
(d) At what time will the object be 50 feet from the ground?
(e) When will the object hit the ground?

Mixed Review

29. Factor $3r^2 - 4r + 1$.
30. Simplify $(2 + \sqrt{3})(4 + \sqrt{3})$.
31. Write the equation in slope-intercept form and identify the slope and $y-$intercept: $9 - 3x + 18y = 0$.
32. The half life of a particular substance is 16 days. An organism has 100% of the substance on day zero. What is the percentage remaining after 44 days?
33. Multiply and write your answer in scientific notation: $0.00000009865 \times 123564.21$
34. A mixture of 12% chlorine is mixed with a second mixture containing 30% chlorine. How much of the 12% mixture is needed to mix with 80 mL to make a final solution of 150 mL with a 20% chlorine concentration?

5.4 Solving Quadratic Equations Using Square Roots

Suppose you needed to find the value of x such that $x^2 = 81$. How could you solve this equation?

- Make a table of values.
- Graph this equation as a system.
- Cancel the square using its inverse operation.

The inverse of a square is a square root.

By applying the square root to each side of the equation, you get:

$$x = \pm\sqrt{81}$$
$$x = 9 \text{ or } x = -9$$

In general, the solution to a quadratic equation of the form $0 = ax^2 - c$ is:

$$x = \sqrt{\frac{c}{a}} \text{ or } x = -\sqrt{\frac{c}{a}}$$

Example 1: *Solve* $(x - 4)^2 - 9 = 0$.

Solution: Begin by adding 9 to each side of the equation.

$$(x - 4)^2 = 9$$

Cancel square by applying square root.

$$x - 4 = 3 \text{ or } x - 4 = -3$$

Solve both equations for $x : x = 7 \text{ or } x = 1$

In the previous lesson, you learned Newton's formula for projectile motion. Let's examine a situation in which there is no initial velocity. When a ball is dropped, there is no outward force placed on its path; therefore, there is no initial velocity.

A ball is dropped from a 40-foot building. When does the ball reach the ground?

Using the equation from the previous lesson, $h(t) = -\frac{1}{2}(g)t^2 + v_0t + h_0$, and substituting the appropriate information, you get:

$$\begin{aligned} & & 0 &= -\frac{1}{2}(32)t^2 + (0)t + 40 \\ &\text{Simplify} & 0 &= -16t^2 + 40 \\ &\text{Solve for } x: & -40 &= -16t^2 \\ & & 2.5 &= t^2 \\ & & t &\approx 1.58 \text{ and } t \approx -1.58 \end{aligned}$$

Because t is in seconds, it does not make much sense for the answer to be negative. So the ball will reach the ground at approximately 1.58 seconds.

Example: *A rock is dropped from the top of a cliff and strikes the ground 7.2 seconds later. How high is the cliff in meters?*

Solution: Using Newton's formula, substitute the appropriate information.

$$\begin{aligned} & & 0 &= -\frac{1}{2}(9.8)(7.2)^2 + (0)(7.2) + h_0 \\ &\text{Simplify:} & 0 &= -254.016 + h_0 \\ &\text{Solve for } h_0: & h_0 &= 254.016 \end{aligned}$$

The cliff is approximately 254 meters tall.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Solving Quadratic Equations by Square Roots (11:03)

Figure 5.4: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/805

Solve each quadratic equation.

1. $x^2 = 196$
2. $x^2 - 1 = 0$
3. $x^2 - 100 = 0$
4. $x^2 + 16 = 0$
5. $9x^2 - 1 = 0$
6. $4x^2 - 49 = 0$
7. $64x^2 - 9 = 0$
8. $x^2 - 81 = 0$
9. $25x^2 - 36 = 0$
10. $x^2 + 9 = 0$
11. $x^2 - 16 = 0$
12. $x^2 - 36 = 0$
13. $16x^2 - 49 = 0$
14. $(x - 2)^2 = 1$
15. $(x + 5)^2 = 16$
16. $(2x - 1)^2 - 4 = 0$
17. $(3x + 4)^2 = 9$
18. $(x - 3)^2 + 25 = 0$
19. $x^2 - 6 = 0$
20. $x^2 - 20 = 0$
21. $3x^2 + 14 = 0$
22. $(x - 6)^2 = 5$
23. $(4x + 1)^2 - 8 = 0$
24. $(x + 10)^2 = 2$
25. $2(x + 3)^2 = 8$
26. How long does it take a ball to fall from a roof to the ground 25 feet below?
27. Susan drops her camera in the river from a bridge that is 400 feet high. How long is it before she hears the splash?
28. It takes a rock 5.3 seconds to splash in the water when it is dropped from the top of a cliff. How high is the cliff in meters?
29. Nisha drops a rock from the roof of a building 50 feet high. Ashaan drops a quarter from the top-story window, which is 40 feet high, exactly half a second after Nisha drops the rock. Which hits the ground first?
30. Victor drops an apple out of a window on the 10^{th} floor, which is 120 feet above ground. One second later, Juan drops an orange out of a 6^{th}-floor window, which is 72 feet above the ground. Which fruit reaches the ground first? What is the time difference between the fruits' arrival to the ground?

Mixed Review

31. Graph $y = 2x^2 + 6x + 4$. Identify the following:
    (a) Vertex
    (b) $x-$intercepts
    (c) $y-$intercepts
    (d) axis of symmetry
32. What is the difference between $y = x + 3$ and $y = x^2 + 3$?
33. Determine the domain and range of $y = -(x - 2)^2 + 7$.
34. The Glee Club is selling hot dogs and sodas for a fundraiser. On Friday the club sold 112 hot dogs and 70 sodas and made \$154.00. On Saturday the club sold 240 hot dogs and 120 sodas and made \$300.00. How much is each soda? Each hot dog?

5.5 Solving Quadratic Equations by Completing the Square

There are several ways to write an equation for a parabola:

- Standard form: $y = ax^2 + bx + c$
- Factored form: $y = (x + m)(x + n)$
- **Vertex form:** $y = a(x - h)^2 + k$

Vertex form of a quadratic equation: $y = a(x-h)^2 + k$, where $(h, k) =$ *vertex of the parabola* and $a =$ *leading coefficient*

Example 1: *Determine the vertex of* $y = -\frac{1}{2}(x - 4)^2 - 7$. *Is this a minimum or a maximum point of the parabola?*

Solution: Using the definition of vertex form, $h = 4, k = -7$.

- The vertex is (4, –7).
- Because a is negative, the parabola opens downward.
- Therefore, the vertex (4, –7) is a maximum point of the parabola.

Once you know the vertex, you can use symmetry to graph the parabola.

Table 5.4:

x	y
2	
3	
4	–7
5	
6	

Example 2: *Write the equation for a parabola with* $a = 3$ *and vertex (–4, 5) in vertex form.*

Solution: Using the definition of vertex form $y = a(x - h)^2 + k, h = -4$ and $k = 5$.

$$y = 3(x - (-4))^2 + 5$$
$$y = 3(x + 4)^2 + 5$$

Consider the quadratic equation $y = x^2 + 4x - 2$. What is its vertex? You could graph this using your calculator and determine the vertex or you could **complete the square.**

Completing the Square

Completing the square is a method used to create a **perfect square trinomial,** as you learned in the previous chapter.

A **perfect square trinomial** has the form $a^2 + 2(ab) + b^2$, which factors into $(a + b)^2$.

Example: Find the missing value to create a perfect square trinomial: $x^2 + 8x + ?$.

Solution: The value of a is x. To find b, use the definition of the middle term of the perfect square trinomial.

$$\begin{aligned} & 2(ab) = 8x \\ a \text{ is } x, \quad & 2(xb) = 8x \\ \text{Solve for } b: \quad & \frac{2xb}{2x} = \frac{8x}{2x} \rightarrow b = 4 \end{aligned}$$

To complete the square you need the value of b^2.

$$b^2 = 4^2 = 16$$

The missing value is 16.

To complete the square, the equation must be in the form: $y = x^2 + \left(\frac{1}{2}b\right)x + b^2$.

Looking at the above example, $\frac{1}{2}(8) = 4$ and $4^2 = 16$.

Example 3: *Find the missing value to complete the square of* $x^2 + 22x + ?$. *Then factor.*

Solution: Use the definition of the middle term to complete the square.

$$\frac{1}{2}(b) = \frac{1}{2}(22) = 11$$

Therefore, $11^2 = 121$ and the perfect square trinomial is $x^2 + 22x + 121$. Rewriting in its factored form, the equation becomes $(x + 11)^2$.

Solve Using Completing the Square

Once you have the equation written in vertex form, you can solve using the method learned in the last lesson.

Example: Solve $x^2 + 22x + 121 = 0$.

Solution: By completing the square and factoring, the equation becomes:

$$\begin{aligned} & (x + 11)^2 = 0 \\ \text{Solve by taking the square root:} \quad & x + 11 = \pm 0 \\ \text{Separate into two equations:} \quad & x + 11 = 0 \text{ or } x + 11 = 0 \\ \text{Solve for } x: \quad & x = -11 \end{aligned}$$

Example: Solve $x^2 + 10x + 9 = 0$.

Solution: Using the definition to complete the square, $\frac{1}{2}(b) = \frac{1}{2}(10) = 5$. Therefore, the last value of the perfect square trinomial is $5^2 = 25$. The equation given is

$$x^2 + 10x + 9 = 0, \text{ and } 9 \neq 25$$

Therefore, to complete the square, we must rewrite the standard form of this equation into vertex form.

Subtract 9:

$$x^2 + 10x = -9$$

Complete the square: Remember to use the Addition Property of Equality.

$$x^2 + 10x + 25 = -9 + 25$$

Factor the left side. $$(x + 5)^2 = 16$$

Solve using square roots. $$\sqrt{(x + 5)^2} = \pm\sqrt{16}$$

$$x + 5 = 4 \text{ or } x + 5 = -4$$

$$x = -1 \text{ or } x = -9$$

Example: *An arrow is shot straight up from a height of 2 meters with a velocity of 50 m/s.* What is the maximum height that the arrow will reach and at what time will that happen?

Solution: The maximum height is the vertex of the parabola. Therefore, we need to rewrite the equation in vertex form.

We rewrite the equation in vertext form. $$y = -4.9t^2 + 50t + 2$$

$$y - 2 = -4.9t^2 + 50t$$

$$y - 2 = -4.9(t^2 - 10.2t)$$

Complete the square inside the parentheses. $$y - 2 - 4.9(5.1)^2 = -4.9(t^2 - 10.2t + (5.1)^2)$$

$$y - 129.45 = -4.9(t - 5.1)^2$$

The maximum height is 129.45 meters.

Multimedia Link: Visit the http://www.mathsisfun.com/algebra/completing-square.html - mathisfun webpage for more explanation on completing the square.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Solving Quadratic Equations by Completing the Square (14:06)

Figure 5.5: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/806

1. What does it mean to ";complete the square";?
2. Describe the process used to solve a quadratic equation by completing the square.
3. Using the equation from the arrow in the lesson,

(a) How high will an arrow be four seconds after being shot? After eight seconds?
(b) At what time will the arrow hit the ground again?

Write the equation for the parabola with the given information.

4. $a = a$, $vertex = (h, k)$
5. $a = \frac{1}{3}$, $vertex = (1, 1)$
6. $a = -2$, $vertex = (-5, 0)$
7. Containing (5, 2) and vertex (1, –2)
8. $a = 1$, $vertex = (-3, 6)$

Complete the square for each expression.

9. $x^2 + 5x$
10. $x^2 - 2x$
11. $x^2 + 3x$
12. $x^2 - 4x$
13. $3x^2 + 18x$
14. $2x^2 - 22x$
15. $8x^2 - 10x$
16. $5x^2 + 12x$

Solve each quadratic equation by completing the square.

17. $x^2 - 4x = 5$
18. $x^2 - 5x = 10$
19. $x^2 + 10x + 15 = 0$
20. $x^2 + 15x + 20 = 0$
21. $2x^2 - 18x = 0$
22. $4x^2 + 5x = -1$
23. $10x^2 - 30x - 8 = 0$
24. $5x^2 + 15x - 40 = 0$

Rewrite each quadratic function in vertex form.

25. $y = x^2 - 6x$
26. $y + 1 = -2x^2 - x$
27. $y = 9x^2 + 3x - 10$
28. $y = 32x^2 + 60x + 10$

For each parabola, find:

1. The vertex
2. $x-$intercepts
3. $y-$intercept
4. If it opens up or down
5. The graph the parabola

29. $y - 4 = x^2 + 8x$
30. $y = -4x^2 + 20x - 24$
31. $y = 3x^2 + 15x$
32. $y + 6 = -x^2 + x$
33. $x^2 - 10x + 25 = 9$
34. $x^2 + 18x + 81 = 1$
35. $4x^2 - 12x + 9 = 16$
36. $x^2 + 14x + 49 = 3$
37. $4x^2 - 20x + 25 = 9$
38. $x^2 + 8x + 16 = 25$
39. Sam throws an egg straight down from a height of 25 feet. The initial velocity of the egg is 16 ft/sec. How long does it take the egg to reach the ground?
40. Amanda and Dolvin leave their house at the same time. Amanda walks south and Dolvin bikes east. Half an hour later they are 5.5 miles away from each other and Dolvin has covered three miles more than the distance that Amanda covered. How far did Amanda walk and how far did Dolvin bike?
41. Two cars leave an intersection. One car travels north; the other travels east. When the car traveling north had gone 30 miles, the distance between the cars was 10 miles more than twice the distance traveled by the car heading east. Find the distance between the cars at that time.

Mixed Review

42. A ball dropped from a height of four feet bounces 70% of its previous height. Write the first five terms of this sequence. How high will the ball reach on its 8^{th} bounce?
43. Rewrite in standard form: $y = \frac{2}{7}x - 11$.
44. Graph $y = 5\left(\frac{1}{2}\right)^x$. Is this exponential growth or decay? What is the growth factor?
45. Solve for $r : |3r - 4| \le 2$.
46. Solve for $m : -2m + 6 = -8(5m + 4)$.
47. Factor $4a^2 + 36a - 40$.

Quick Quiz

1. Graph $y = -3x^2 - 12x - 13$ and identify:
   (a) The vertex
   (b) The axis of symmetry
   (c) The domain and range
   (d) The $y-$intercept
   (e) The $x-$intercepts estimated to the nearest tenth
2. Solve $y = x^2 + 9x + 20$ by graphing.
3. Solve for $x : 74 = x^2 - 7$.
4. A baseball is thrown from an initial height of 5 feet with an initial velocity of 100 *ft/sec.*
   (a) What is the maximum height of the ball?
   (b) When will the ball reach the ground?
   (c) When is the ball 90 feet in the air?
5. Solve by completing the square: $v^2 - 20v + 25 = 6$

5.6 Solving Quadratic Equations Using the Quadratic Formula

This chapter has presented three methods to solve a quadratic equation:

- By graphing to find the zeros;
- By solving using square roots; and
- By using completing the square to find the solutions

This lesson will present a fourth way to solve a quadratic equation: using the Quadratic Formula.

History of the Quadratic Formula

As early as 1200 BC, people were interested in solving quadratic equations. The Babylonians solved simultaneous equations involving quadratics. In 628 AD, Brahmagupta, an Indian mathematician, gave the first explicit formula to solve a quadratic equation. The Quadratic Formula was written as it is today by the Arabic mathematician Al-Khwarizmi. It is his name upon which the word ";Algebra"; is based.

The solution to any quadratic equation in standard form $0 = ax^2 + bx + c$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: Solve $x^2 + 10x + 9 = 0$ using the Quadratic Formula.

Solution: We know from the last lesson the answers are $x = -1$ *or* $x = -9$.

By applying the Quadratic Formula and $a = 1, b = 10$, and $c = 9$, we get:

$$\begin{aligned}
x &= \frac{-10 \pm \sqrt{(10)^2 - 4(1)(9)}}{2(1)} \\
x &= \frac{-10 \pm \sqrt{100 - 36}}{2} \\
x &= \frac{-10 \pm \sqrt{64}}{2} \\
x &= \frac{-10 \pm 8}{2} \\
x &= \frac{-10 + 8}{2} \ or \ x = \frac{-10 - 8}{2} \\
x &= -1 \ or \ x = -9
\end{aligned}$$

Example 1: *Solve* $-4x^2 + x + 1 = 0$ *using the Quadratic Formula.*

Solution:

Quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Plug in the values $a = -4, b = 1, c = 1$. $x = \frac{-1 \pm \sqrt{(1)^2 - 4(-4)(1)}}{2(-4)}$

Simplify. $x = \frac{-1 \pm \sqrt{1 + 16}}{-8} = \frac{-1 \pm \sqrt{17}}{-8}$

Separate the two options. $x = \frac{-1 + \sqrt{17}}{-8}$ and $x = \frac{-1 - \sqrt{17}}{-8}$

Solve. $x \approx -.39$ and $x \approx .64$

Multimedia Link For more examples of solving quadratic equations using the Quadratic Formula, see Khan Academy Equation Part 2 (9:14).

Figure 5.6: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/94

Figure 2 provides more examples of solving equations using the quadratic equation. This video is not necessarily different from the examples above, but it does help reinforce the procedure of using the Quadratic Formula to solve equations.

Finding the Vertex of a Quadratic Equation in Standard Form

The $x-$coordinate of the vertex of $0 = ax^2 + bx + c$ is $x = -\frac{b}{a}$

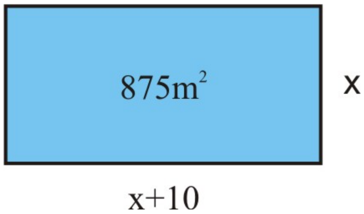
Which Method to Use?

Usually you will not be told which method to use. You will have to make that decision yourself. However, here are some guidelines to which methods are better in different situations.

- **Graphing** – a good method to visualize the parabola and easily see the intersections. Not always precise.
- **Factoring** – best if the quadratic expression is easily factorable
- **Taking the square root** – is best used of the form $0 = ax^2 - c$
- **Completing the square** – can be used to solve any quadratic equation. It is a very important method for rewriting a quadratic function in vertex form.
- **Quadratic Formula** – is the method that is used most often for solving a quadratic equation. If you are using factoring or the Quadratic Formula, make sure that the equation is in standard form.

Example: *The length of a rectangular pool is 10 meters more than its width. The area of the pool is 875 square meters. Find the dimensions of the pool.*

Solution: Begin by drawing a sketch. The formula for the area of a rectangle is $A = l(w)$.



$$\begin{aligned} A &= (x+10)(x) \\ 875 &= x^2 + 10x \end{aligned}$$

Now solve for x using any method you prefer.

The result is $x = 25$. So, the length of the pool is 35 meters and the width is 25 meters.

Practice Set

The following video will guide you through a proof of the Quadratic Formula. CK-12 Basic Algebra: Proof of Quadratic Formula (7:44)

Figure 5.7: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/807

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra: Using the Quadratic Formula (16:32)

Figure 5.8: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/808

1. What is the Quadratic Formula? When is the most appropriate situation to use this formula?

2. When was the first known solution of a quadratic equation recorded?

Find the $x-$coordinate of the vertex of the following equations.

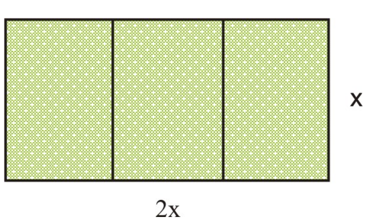
3. $x^2 - 14x + 45 = 0$
4. $8x^2 - 16x - 42 = 0$
5. $4x^2 + 16x + 12 = 0$
6. $x^2 + 2x - 15 = 0$

Solve the following quadratic equations using the Quadratic Formula.

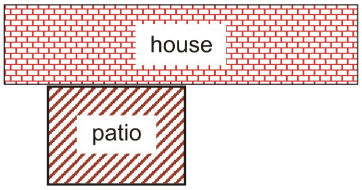
7. $x^2 + 4x - 21 = 0$
8. $x^2 - 6x = 12$
9. $3x^2 - \frac{1}{2}x = \frac{3}{8}$
10. $2x^2 + x - 3 = 0$
11. $-x^2 - 7x + 12 = 0$
12. $-3x^2 + 5x = 0$
13. $4x^2 = 0$
14. $x^2 + 2x + 6 = 0$

Solve the following quadratic equations using the method of your choice.

15. $x^2 - x = 6$
16. $x^2 - 12 = 0$
17. $-2x^2 + 5x - 3 = 0$
18. $x^2 + 7x - 18 = 0$
19. $3x^2 + 6x = -10$
20. $-4x^2 + 4000x = 0$
21. $-3x^2 + 12x + 1 = 0$
22. $x^2 + 6x + 9 = 0$
23. $81x^2 + 1 = 0$
24. $-4x^2 + 4x = 9$
25. $36x^2 - 21 = 0$
26. $x^2 + 2x - 3 = 0$
27. The product of two consecutive integers is 72. Find the two numbers.
28. The product of two consecutive odd integers is 11 less than 3 times their sum. Find the integers.
29. The length of a rectangle exceeds its width by 3 inches. The area of the rectangle is 70 square inches. Find its dimensions.
30. Suzie wants to build a garden that has three separate rectangular sections. She wants to fence around the whole garden and between each section as shown. The plot is twice as long as it is wide and the total area is 200 square feet. How much fencing does Suzie need?



31. Angel wants to cut off a square piece from the corner of a rectangular piece of plywood. The larger piece of wood is 4 feet $\times$ 8 feet and the cut off part is $\frac{1}{3}$ of the total area of the plywood sheet. What is the length of the side of the square?

32. Mike wants to fence three sides of a rectangular patio that is adjacent the back of his house. The area of the patio is 192 ft^2 and the length is 4 feet longer than the width. Find how much fencing Mike will need.
33. 

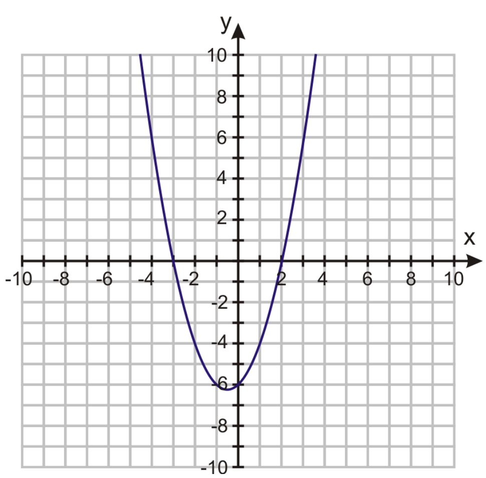
Mixed Review

33. The theatre has three types of seating: balcony, box, and floor. There are four times as many floor seats as balcony. There are 200 more box seats than balcony seats. The theatre has a total of 1,100 seats. Determine the number of balcony, box, and floor seats in the theatre.
34. Write an equation in slope-intercept form containing (10, 65) and (5, 30).
35. 120% of what number is 60?
36. Name the set() of numbers to which $\sqrt{16}$ belongs.
37. Divide $6\frac{1}{7} \div -2\frac{3}{4}$.
38. The set is the number of books in a library. Which of the following is the most appropriate domain for this set: all real numbers; positive real numbers; integers; or whole numbers? Explain your reasoning.

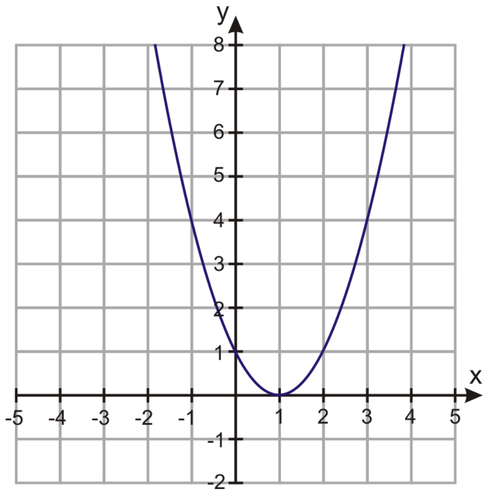
5.7 The Discriminant

You have seen parabolas that intersect the $x-$axis twice, once, or not at all. There is a relationship between the number of real $x-$intercepts and the Quadratic Formula.

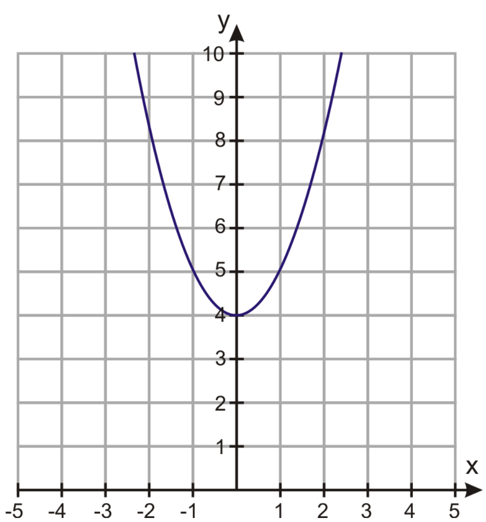
Case 1: The parabola has two $x-$intercepts. This situation has two possible solutions for x, because the value inside the square root is positive. Using the Quadratic Formula, the solutions are $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$.



Case 2: The parabola has one $x-$intercept. This situation occurs when the vertex of the parabola just touches the $x-$axis. This is called a repeated root, or **double root.** The value inside the square root is zero. Using the Quadratic Formula, the solution is $x = \frac{-b}{2a}$.



Case 3: The parabola has no $x-$intercept. This situation occurs when the parabola does not cross the $x-$axis. The value inside the square root is negative, therefore there are no real roots. The solutions to this type of situation are **imaginary,** which you will learn more about in a later textbook.



The value inside the square root of the Quadratic Formula is called the **discriminant.** It is symbolized by D. It dictates the number of real solutions the quadratic equation has. This can be summarized with the Discriminant Theorem.

- If $D > 0$, the parabola will have two $x-$intercepts. The quadratic equation will have two real solutions.
- If $D = 0$, the parabola will have one $x-$intercept. The quadratic equation will have one real solution.
- If $D < 0$, the parabola will have no $x-$intercepts. The quadratic equation will have zero real solutions.

Example 1: *Determine the number of real solutions to* $-3x^2 + 4x + 1 = 0$.

Solution: By finding the value of its discriminant, you can determine the number of $x-$intercepts the parabola has and thus the number of real solutions.

$$D = b^2 - 4(a)(c)$$
$$D = (4)^2 - 4(-3)(1)$$
$$D = 16 + 12 = 28$$

Because the discriminant is positive, the parabola has two real $x-$intercepts and thus two real solutions.

Example: *Determine the number of solutions to* $-2x^2 + x = 4$.

Solution: Before we can find its discriminant, we must write the equation in standard form $ax^2+bx+c=0$.

Subtract 4 from each side of the equation: $-2x^2 + x - 4 = 0$.

Find the discriminant.
$$D = (1)^2 - 4(-2)(-4)$$
$$D = 1 - 32 = -31$$

The value of the discriminant is negative; there are no real solutions to this quadratic equation. The parabola does not cross the $x-$axis.

Example 2: *Emma and Bradon own a factory that produces bike helmets. Their accountant says that their profit per year is given by the function* $P = 0.003x^2 + 12x + 27,760$*, where x represents the number of helmets produced. Their goal is to make a profit of $40,000 this year. Is this possible?*

Solution: The equation we are using is $40,000 = 0.003x^2 + 12x + 27,760$. By finding the value of its discriminant, you can determine if the profit is possible.

Begin by writing this equation in standard form:

$$\begin{aligned} 0 &= 0.003x^2 + 12x - 12,240 \\ D &= b^2 - 4(a)(c) \\ D &= (12)^2 - 4(0.003)(-12,240) \\ D &= 144 + 146.88 = 290.88 \end{aligned}$$

Because the discriminant is positive, the parabola has two real solutions. Yes, the profit of $40,000 is possible.

Multimedia Link: This http://sciencestage.com/v/20592/a-level-maths-:-roots-of-a-quadratic-equation-:-discriminant-:-examsolutions.html - video, presented by Science Stage, helps further explain the discriminant using the Quadratic Formula.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Discriminant of Quadratic Equations (10:14)

Figure 5.9: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/809

1. What is a discriminant? What does it do?
2. What is the formula for the discriminant?
3. Can you find the discriminant of a linear equation? Explain your reasoning.
4. Suppose $D = 0$. Draw a sketch of this graph and determine the number of real solutions.
5. $D = -2.85$. Draw a possible sketch of this parabola. What is the number of real solutions to this quadratic equation.
6. $D > 0$. Draw a sketch of this parabola and determine the number of real solutions.

Find the discriminant of each quadratic equation.

7. $2x^2 - 4x + 5 = 0$
8. $x^2 - 5x = 8$
9. $4x^2 - 12x + 9 = 0$
10. $x^2 + 3x + 2 = 0$
11. $x^2 - 16x = 32$
12. $-5x^2 + 5x - 6 = 0$

Determine the nature of the solutions of each quadratic equation.

13. $-x^2 + 3x - 6 = 0$
14. $5x^2 = 6x$
15. $41x^2 - 31x - 52 = 0$
16. $x^2 - 8x + 16 = 0$
17. $-x^2 + 3x - 10 = 0$
18. $x^2 - 64 = 0$

A solution to a quadratic equation will be irrational if the discriminant is **not a perfect square.** If the discriminant is a perfect square, then the solutions will be rational numbers. Using the discriminant, determine whether the solutions will be rational or irrational.

19. $x^2 = -4x + 20$
20. $x^2 + 2x - 3 = 0$
21. $3x^2 - 11x = 10$
22. $\frac{1}{2}x^2 + 2x + \frac{2}{3} = 0$
23. $x^2 - 10x + 25 = 0$
24. $x^2 = 5x$
25. Marty is outside his apartment building. He needs to give Yolanda her cell phone but he does not have time to run upstairs to the third floor to give it to her. He throws it straight up with a vertical velocity of 55 feet/second. Will the phone reach her if she is 36 feet up? (Hint: The equation for the height is given by $y = -32t^2 + 55t + 4$.)
26. Bryson owns a business that manufactures and sells tires. The revenue from selling the tires in the month of July is given by the function $R = x(200 - 0.4x)$ where x is the number of tires sold. Can Bryson's business generate revenue of \$20,000 in the month of July?
27. Marcus kicks a football in order to score a field goal. The height of the ball is given by the equation $y = -\frac{32}{6400}x^2 + x$, where y is the height and x is the horizontal distance the ball travels. We want to know if Marcus kicked the ball hard enough to go over the goal post, which is 10 feet high.

Mixed Review

28. Factor $6x^2 - x - 12$.
29. Find the vertex of $y = -\frac{1}{4}x^2 - 3x - 12 = y$ by completing the square.
30. Solve using the Quadratic Formula: $-4x^2 - 15 = -4x$.
31. How many centimeters are in four fathoms? (Hint: 1 $fathom = 6$ $feet$)
32. Graph the solution to $\begin{cases} 3x + 2y \le -4 \\ x - y > -3 \end{cases}$.
33. How many ways can 3 toppings be chosen from 7 options?

5.8 Linear, Exponential, and Quadratic Models

So far in this text you have learned how to graph three very important types of equations.

- Linear equations in slope-intercept form $y = mx + b$
- Exponential equations of the form $y = a(b)^x$
- Quadratic equations in standard form $y = ax^2 + bx + c$

In real-world applications, the function that describes some physical situation is not given. Finding the function is an important part of solving problems. For example, scientific data such as observations of planetary motion are often collected as a set of measurements given in a table. One job for the scientist is to figure out which function best fits the data. In this lesson, you will learn some methods that are used to identify which function describes the relationship between the dependent and independent variables in a problem.

Using Differences to Determine the Model

By finding the differences between the dependent values, we can determine the **degree** of the model for the data.

- If the first difference is the same value, the model will be **linear.**
- If the second difference is the same value, the model will be **quadratic.**
- If the number of times the difference has been taken exceeds five, the model may be exponential or some other special equation.

x	y	difference of $y-$ values
-2	-4	$-1 + 4 = 3$
-1	-1	$2 + 1 = 3$
0	2	$5 - 2 = 3$
1	5	$8 - 5 = 3$
2	8	

Example: The first difference is the same value (3). This data can be modeled using a **linear regression line.**

The equation to represent this data is $y = 3x + 2$

When we look at the difference of the $y-$values, we must make sure that we examine entries for which the $x-$values increase by the same amount.

For example, examine the values in the following table.

At first glance, this function might not look linear because the difference in the $y-$values is not always the same.

difference of $x-$ values	x	y	difference of $y-$ values
	0	5	
1 - 0 = 1			-1 + 4 = 3
	1	10	
3 - 1 = 2			2 + 1 = 3
	3	20	
4 - 3 = 1			5 − 2 = 3
	4	25	
6 - 4 = 2			8 − 5 = 3
	6	35	

However, we see that the difference in $y-$values is 5 when we increase the $x-$values by 1, and it is 10 when we increase the $x-$values by 2. This means that the difference in $y-$values is always 5 when we increase the $x-$values by 1. Therefore, the function is linear.

The equation is modeled by $y = 5x + 5$.

An example of a quadratic model would have the following look when taking the second difference.

x	$y = 2x^2\ 3x + 1$	difference of $y-$ values	difference of differences
0	0		
		0 − 1 = -1	
			3 + 1 = 4
1	1		
		3 − 0 = 3	
			7 −3 = 4
2	3		
		10 − 3 = 7	
			11−7 = 4
3	10		
		21 − 10 = 11	
			15−11= 4
4	21		
		36 − 21 = 15	
			19 − 15= 4
5	36		
		55 − 36 = 19	
6	55		

Using Ratios to Determine the Model

Finding the difference involves subtracting the dependent values leading to a degree of the model. By taking the **ratio** of the values, one can obtain whether the model is exponential.

If the ratio of dependent values is the same, then the data is modeled by an **exponential** equation, as in the example below.

x	y
0	4
1	12
2	36
3	108
4	324

ratio of *y* - values

$\frac{12}{4} = 3$

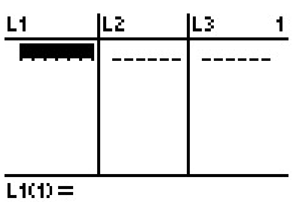
$\frac{36}{12} = 3$

$\frac{108}{36} = 3$

$\frac{324}{108} = 3$

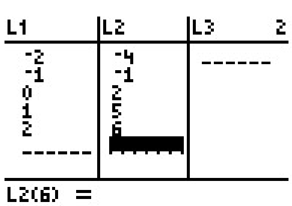
Determine the Model Using a Graphing Calculator

To enter data into your graphing calculator, find the [**STAT**] button. Choose [**EDIT**].

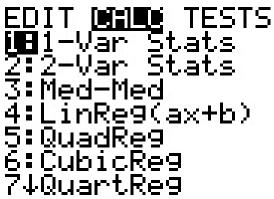


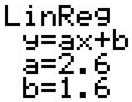
- [**L1**] represents your independent variable, your x.
- [**L2**] represents your dependent variable, your y.

Enter the data into the appropriate list. Using the first set of data to illustrate yields:



You already know this data is best modeled by a linear regression line. Using the [**CALCULATE**] menu of your calculator, find the linear regression line, ***linreg.***





Look at the screen above. This is where you can find the ***quadratic regression line [QUADREG]***, the ***cubic regression line [CUBICREG]***, and the exponential regression line, ***[EXPREG].***

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Linear, Quadratic, and Exponential Models (8:15)

Figure 5.10: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/811

1. The second set of differences have the same value. What can be concluded?
2. Suppose you find the difference five different times and still don't come to a common value. What can you safely assume?
3. Why would you test the ratio of differences?
4. If you had a cubic (3^{rd}-degree) function, what could you conclude about the differences?

Determine whether the data can be modeled by a linear equation, a quadratic equation, or neither.

5.

x	-4	-3	-2	-1	0	1
y	10	7	4	1	-2	-5

6.

x	-2	-1	0	1	2	3
y	4	3	2	3	6	11

7.

x	0	1	2	3	4	5
y	50	75	100	125	150	175

8.

x	-10	-5	0	5	10	15
y	10	2.5	0	2.5	10	22.5

9.

x	1	2	3	4	5	6
y	4	6	6	4	0	-6

10.

x	-3	-2	-1	0	1	2
y	-27	-8	-1	0	1	8

Can the following data be modeled with an exponential function?

11.

x	0	1	2	3	4	5
y	200	300	1800	8300	25,800	62,700

12.

x	0	1	2	3	4	5
y	120	180	270	405	607.5	911.25

13.

x	0	1	2	3	4	5
y	4000	2400	1440	864	518.4	311.04

Determine whether the data is best represented by a quadratic, linear, or exponential function. Find the function that best models the data.

14.

x	0	1	2	3	4
y	400	500	625	781.25	976.5625

15.

x	-9	-7	-5	-3	-1	1
y	-3	-2	-1	0	1	2

16.

x	-3	-2	-1	0	1	2	3
y	14	4	-2	-4	-2	4	14

17. As a ball bounces up and down, the maximum height it reaches continually decreases. The table below shows the height of the bounce with regard to time.
    (a) Using a graphing calculator, create a scatter plot of this data.
    (b) Find the quadratic function of best fit.
    (c) Draw the quadratic function of best fit on top of the scatter plot.
    (d) Find the maximum height the ball reaches.
    (e) Predict how high the ball is at 2.5 seconds.

Table 5.5:

Time (seconds)	**Height (inches)**
2	2
2.2	16
2.4	24
2.6	33
2.8	38
3.0	42
3.2	36
3.4	30
3.6	28
3.8	14
4.0	6

18. A chemist has a 250-gram sample of a radioactive material. She records the amount remaining in the sample every day for a week and obtains the following data.
    (a) Draw a scatter plot of the data.
    (b) Which function best suits the data: exponential, linear, or quadratic?
    (c) Find the function of best fit and draw it through the scatter plot.
    (d) Predict the amount of material present after 10 days.

Table 5.6:

Day	**Weight(grams)**
0	250
1	208
2	158
3	130
4	102
5	80
6	65
7	50

19. The following table show the pregnancy rate (per 1000) for U.S. women aged 15 – 19 (source: US Census Bureau). Make a scatter plot with the rate as the dependent variable and the number of years since 1990 as the independent variable. Find which model fits the data best. Use this model to predict the rate of teen pregnancy in the year 2010.

Table 5.7:

Year	**Rate of Pregnancy (per 1000)**
1990	116.9
1991	115.3
1992	111.0
1993	108.0

Table 5.7: (continued)

Year	Rate of Pregnancy (per 1000)
1994	104.6
1995	99.6
1996	95.6
1997	91.4
1998	88.7
1999	85.7
2000	83.6
2001	79.5
2002	75.4

Mixed Review

20. Cam bought a bag containing 16 cups of flour. He needs $2\frac{1}{2}$ cups for each loaf of bread. Write this as an equation in slope-intercept form. When will Cam run out of flour?
21. A basketball is shot from an initial height of 7 feet with an velocity of 10 *ft/sec.*
    (a) Write an equation to model this situation.
    (b) What is the maximum height the ball reaches?
    (c) What is the $y-$intercept? What does it mean?
    (d) When will the ball hit the ground?
    (e) Using the discriminant, determine whether the ball will reach 11 feet. If so, how many times?
22. Graph $y = |x - 2| + 3$. Identify the domain and range of the graph.
23. Solve $6 \geq -5(c + 4) + 10$.
24. Is this relation a function? $\{(-6, 5), (-5, -3), (-2, -1), (0, -3), (2, 5)\}$. If so, identify its domain and range.
25. Name and describe five problem-solving strategies you have learned so far in this chapter.

5.9 Problem-Solving Strategies: Choose a Function Model

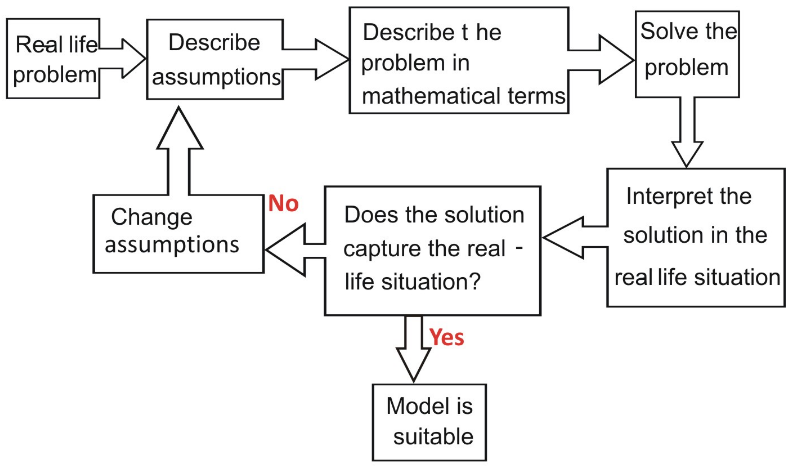
As you learn more and more mathematical methods and skills, it is important to think about the purpose of mathematics and how it works as part of a bigger picture. Mathematics is used to solve problems that often arise from real-life situations. **Mathematical modeling** is a process by which we start with a real-life situation and arrive at a quantitative solution.

Modeling involves creating a set of mathematical equations that describes a situation, solving those equations, and using them to understand the real-life problem.

Often the model needs to be adjusted because it does not describe the situation as well as we wish.

A mathematical model can be used to gain understanding of a real-life situation by learning how the system works, which variables are important in the system, and how they are related to each other. Models can also be used to predict and forecast what a system will do in the future or for different values of a parameter. Lastly, a model can be used to estimate quantities that are difficult to evaluate exactly.

Mathematical models are like other types of models. The goal is not to produce an exact copy of the ";real"; object but rather to give a representation of some aspect of the real thing. The modeling process can be summarized as a flow chart:



Notice that the modeling process is very similar to the problem-solving format we have been using throughout this book. One of the most difficult parts of the modeling process is determining which function best describes a situation. We often find that the function we choose is not appropriate. Then we must choose a different one.

Consider an experiment regarding the elasticity of a spring.

Example: *A spring is stretched as you attach more weight at its bottom. The following table shows the length of the spring in inches for different weights in ounces.*

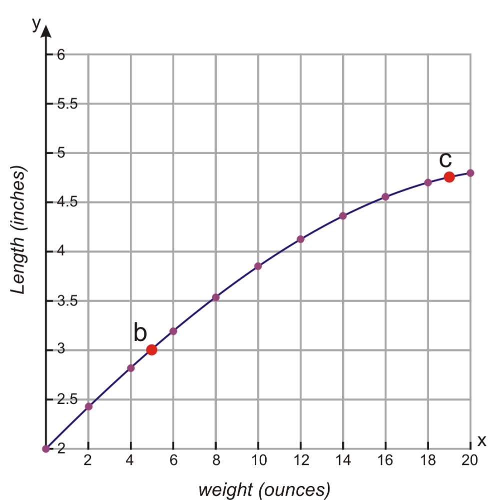
Weight (oz)	0	2	4	6	8	10	12	14	16	18	20
Length (in)	2	2.4	2.8	3.2	3.5	3.9	4.1	4.4	4.6	4.7	4.8

a) Find the length of the spring as a function of the weight attached to it.

b) Find the length of the spring when you attach 5 ounces.

c) Find the length of the spring when you attach 19 ounces.

Solution: Begin by graphing the data to get a visual of what the model may look like.



- This data clearly does not fit a linear equation. It has a distinct curve that will not be modeled well by a straight line.
- Nor does this graph seem to fit a parabolic shape; thus it is not modeled by a quadratic equation.

- The curve does not fit the exponential curves studied in Chapter 8.
- By taking the third set of differences, the value is approximately equal. Use the methods learned in the previous lesson to find a cubic regression equation. Check by graphing to see if this model is a good fit.

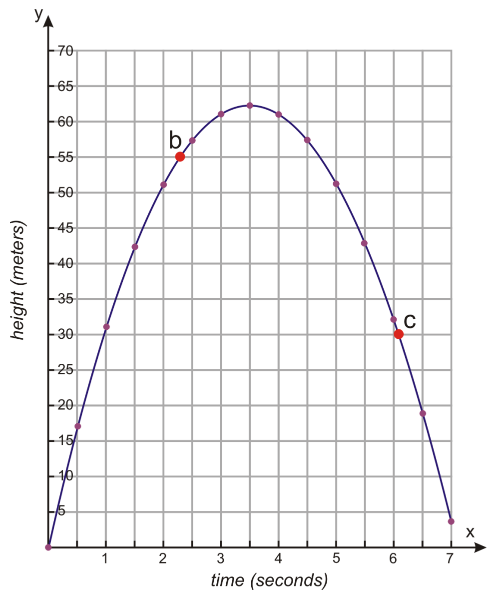
Example: *A golf ball is hit down a straight fairway. The following table shows the height of the ball with respect to time. The ball is hit at an angle of* 70° *with the horizontal with a speed of 40 meters/sec.*

Time (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0
Height (meters)	0	17.2	31.5	42.9	51.6	57.7	61.2	62.3	61.0	57.2	51.0	42.6	31.9	19.0	4.1

a) *Find the height of the ball as a function of time.*

b) *Find the height of the ball when* $t = 2.4$ *seconds.*

c) *Find the height of the ball when* $t = 6.2$ *seconds.*



Solution: Begin by graphing the data to visualize the model.

This data fits a parabolic curve quite well. We can therefore conclude the best model for this data is a quadratic equation.

To solve part a), use the graphing calculator to determine the **quadratic regression line.**

$$y = -4.92x^2 + 34.7x + 1.2$$

b) The height of the ball when $t = 2.4$ *seconds* is:

$$y = -4.92(2.4)^2 + 34.7(2.4) + 1.2 = 56.1 \textit{ meters}$$

c) The height of the ball when $t = 6.2$ *seconds* is

$$y = -4.92(6.2)^2 + 34.7(6.2) + 1.2 = 27.2 \textit{ meters}$$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following videos. Note that there is not always a match between the number of the practice exercise in the videos and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Identifying Quadratic Models (8:05)

Figure 5.11: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/812

CK-12 Basic Algebra: Identifying Exponential Models (4:00)

Figure 5.12: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/813

CK-12 Basic Algebra: Quadratic Regression (9:17)

Figure 5.13: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/814

1. A thin cylinder is filled with water to a height of 50 centimeters. The cylinder has a hole at the bottom that is covered with a stopper. The stopper is released at time $t = 0$ seconds and allowed to empty. The following data shows the height of the water in the cylinder at different times.

Time (sec)	0	2	4	6	8	10	12	14	16	18	20	22	24
Height (cm)	50	42.5	35.7	29.5	23.8	18.8	14.3	10.5	7.2	4.6	2.5	1.1	0.2

(a) What seems to be the best model for this situation?
(b) Find the linear regression line and determine the height of the water at 4.2 seconds.
(c) Find a quadratic equation and determine the height of the water at 4.2 seconds.
(d) Find a cubic regression line and determine the height of the water at 4.2 seconds.
(e) Which of these seems to be the best fit?
(f) Using the function of best fit, find the height of the water when $t = 5$ seconds.
(g) Using the function of best fit, find the height of the water when $t = 13$ seconds.

2. A scientist counts 2,000 fish in a lake. The fish population increases at a rate of 1.5 fish per generation but the lake has space and food for only 2,000,000 fish. The following table gives the number of fish (in thousands) in each generation.

Generation	0	4	8	12	16	20	24	28
Number (thousands)	2	15	75	343	1139	1864	1990	1999

(a) Which function seems to best fit the dataL linear, quadratic, or exponential?
(b) Find the model for the function of best fit.
(c) Find the number of fish as a function of generation.
(d) Find the number of fish in generation 10.
(e) Find the number of fish in generation 25.

3. Using the golf ball example, find the maximum height the ball reaches.
4. Using the golf ball example, evaluate the height of the ball at 5.2 seconds.

Mixed Review

5. Evaluate $2 \div 6 \cdot 5 + 3^2 - 11 \cdot 9^{\frac{1}{2}}$.
6. 60 shirts cost \$812.00 to screen print. 115 shirts cost \$1,126.00 to screen print. Assuming the relationship between the number of shirts and the total cost is linear, write an equation in point-slope form.

   (a) What is the start-up cost (the cost to set up the screen)?
   (b) What is the slope? What does it represent?

7. Solve by graphing: $y = x^2 + 3x - 1$.
8. Simplify $\frac{\frac{6}{7}}{\frac{1}{2}}$.
9. Newton's Second Law states $F = m \cdot a$. Rewrite this equation to solve for m. Use it to determine the mass if the force is 300 Newtons and the acceleration is $70m/s^2$.
10. The area of a square game board is 256 square inches. What is the length of one side?
11. Write as a percent: $\frac{3}{1000}$.

5.10 Chapter Review

Define each term.

1. Vertex
2. Standard form for a quadratic equation
3. Model
4. Discriminant

Graph each function. List the vertex (round to the nearest tenth, if possible) and the range of the function.

5. $y = x^2 - 6x + 11$
6. $y = -4x^2 + 16x - 19$
7. $y = -x^2 - 2x + 1$
8. $y = \frac{1}{2}x^2 + 8x + 6$
9. $y = x^2 + 4x$
10. $y = -\frac{1}{4}x^2 + 8x - 4$
11. $y = (x + 4)^2 + 3$
12. $y = -(x - 3)^2 - 6$
13. $y = (x - 2)^2 + 2$
14. $y = -(x + 5)^2 - 1$

Rewrite in standard form.

15. $x - 24 = -5x$
16. $5 + 4a = a^2$
17. $-6 - 18a^2 = -528$
18. $y = -(x + 4)^2 + 2$

Solve each equation by graphing.

19. $x^2 - 8x + 87 = 9$
20. $23x + x^2 - 104 = 4$
21. $13 + 26x = -x^2 + 11x$
22. $x^2 - 9x = 119$
23. $-32 + 6x^2 - 4x = 0$

Solve each equation by taking its square roots.

24. $x^2 = 225$
25. $x^2 - 2 = 79$
26. $x^2 + 100 = 200$
27. $8x^2 - 2 = 262$
28. $-6 - 4x^2 = -65$
29. $703 = 7x^2 + 3$
30. $10 + 6x^2 = 184$
31. $2 + 6x^2 = 152$

Solve each equation by completing the square then taking its square roots.

32. $n^2 - 4n - 3 = 9$
33. $h^2 + 10h + 1 = 3$
34. $x^2 + 14x - 22 = 10$
35. $t^2 - 10t = -9$

Determine the maximum/minimum point by completing the square.

36. $x^2 - 20x + 28 = -8$
37. $a^2 + 2 - 63 = -5$

38. $x^2 + 6x - 33 = 4$

Solve each equation by using the Quadratic Formula.

39. $4x^2 - 3x = 45$
40. $-5x + 11x^2 = 15$
41. $-3r = 12r^2 - 3$
42. $2m^2 + 10m = 8$
43. $7c^2 + 14c - 28 = -7$
44. $3w^2 - 15 = -3w$

In 45-50, for each quadratic equation, determine:

(a) the discriminant

(b) the number of real solutions

(c) whether the real solutions are rational or irrational

45. $4x^2 - 4x + 1 = 0$
46. $2x^2 - x - 3 = 0$
47. $-2x^2 - x - 1 = -2$
48. $4x^2 - 8x + 4 = 0$
49. $-5x^2 + 10x - 5 = 0$
50. $4x^2 + 3x + 6 = 0$
51. Explain the difference between $y = x^2 + 4$ and $y = -x^2 + 4$.
52. Jorian wants to enclose his garden with fencing on all four sides. He has 225 feet of fencing. What dimensions would give him the largest area?
53. A ball is dropped off a cliff 70 meters high.
    (a) Using Newton's equation, model this situation.
    (b) What is the leading coefficient? What does this value tell you about the shape of the parabola?
    (c) What is the maximum height of the ball?
    (d) Where is the ball after 0.65 seconds?
    (e) When will the ball reach the ground?
54. The following table shows the number of hours spent per person playing video games for various years in the United States.

x	1995	1996	1997	1998	1999	2000
y	24	25	37	43	61	70

    (a) What seems to be the best function for this data?
    (b) Find the best fit function.
    (c) Using your equation, predict the number of hours someone will spend playing video games in 2012.
    (d) Does this value seem possible? Explain your thoughts.
55. The table shows the amount of money spent (in billions of dollars) in the U.S. on books for various years.

x	1990	1991	1992	1993	1994	1995	1996	1997	1998
y	16.5	16.9	17.7	18.8	20.8	23.1	24.9	26.3	28.2

    (a) Find a linear model for this data. Use it to predict the dollar amount spent in 2008.

(b) Find a quadratic model for this data. Use it to predict the dollar amount spent in 2008.
(c) Which model seems more accurate? Use the best model to predict the dollar amount spent in 2012.
(d) What could happen to change this value?

56. The data below shows the number of U.S. hospitals for various years.

x	1960	1965	1970	1980	1985	1990	1995	2000
y	6876	7123	7123	6965	6872	6649	6291	5810

(a) Find a quadratic regression line to fit this data.
(b) Use the model to determine the maximum number of hospitals.
(c) In which year was this?
(d) In what years were there approximately 7,000 hospitals?
(e) What seems to be the trend with this data?

57. A pendulum's distance is measured and recorded in the following table.

swing	1	2	3	4	5	6
length	25	16.25	10.563	6.866	4.463	2.901

(a) What seems to be the best model for this data?
(b) Find a quadratic regression line to fit this data. Approximate the length of the seventh swing.
(c) Find an exponential regression line to fit this data. Approximate the length of the seventh swing.

5.11 Chapter Test

1. *True or false?* The vertex determines the domain of the quadratic function.
2. Suppose the leading coefficient $a = -\frac{1}{3}$. What can you conclude about the shape of the parabola?
3. Find the discriminant of the equation and determine the number of real solutions: $0 = -2x^2 + 3x - 2$.
4. A ball is thrown upward from a height of four feet with an initial velocity of 45 *feet/second.*

(a) Using Newton's law, write the equation to model this situation.
(b) What is the maximum height of the ball?
(c) When will the ball reach 10 feet?
(d) Will the ball ever reach 36.7 feet?
(e) When will the ball hit the ground?

In 5–9, solve the equation using any method.

5. $2x^2 = 2x + 40$
6. $11j^2 = j + 24$
7. $g^2 = 1$
8. $11r^2 - 5 = -178$
9. $x^2 + 8x - 65 = -8$
10. What is the vertex of $y = -(x-6)^2 + 5$? Does the parabola open up or down? Is the vertex a maximum or a minimum?
11. Graph $y = (x+2)^2 - 3$.
12. Evaluate the discriminant. How many real solutions do the quadratic equation have? $-5x^2 - 6x = 1$
13. Suppose $D = -14$. What can you conclude about the solutions to the quadratic equation?
14. Rewrite in standard form: $y - 7 = -2(x+1)^2$.

15. Graph and determine the function's range and vertex: $y = -x^2 + 2x - 2$.
16. Graph and determine the function's range and $y-$intercept: $y = \frac{1}{2}x^2 + 4x + 5$.
17. The following information was taken from *USA Today* regarding the number of cancer deaths for various years.

Table 5.8:

Year	**Number of Deaths Per 100,000 men**
1980	205.3
1985	212.6
1989	217.6
1993	212.1
1997	201.9

Cancer Deaths of Men (*Source: USA Today*)

(a) Find a linear regression line to fit this data. Use it to predict the number of male deaths caused by cancer in 1999.

(b) Find a linear regression line to fit this data. Use it to predict the number of male deaths caused by cancer in 1999.

(c) Find an exponential regression line to fit this data. to predict the number of male deaths caused by cancer in 1999.

(d) Which seems to be the best fit for this data?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9620.

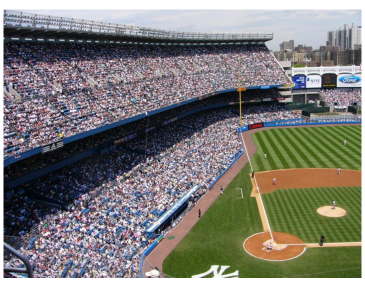
Chapter 6

Radicals

6.1 Radical Functions

Radicals in mathematics are important. By using radicals as inverse operations to exponents, you can solve almost any exponential equation. Radicals such as the square root have been used for thousands of years. Square roots are extremely useful in geometry by finding the hypotenuse of a right triangle or solving for the side length of a square.

In this chapter you will learn the basics of radicals and apply these basics to geometry concepts, such as Pythagorean's Theorem, the Distance Formula, and the Midpoint Formula. The last several sections of this chapter will discuss **data analysis**, a method used to analyze data by creating charts and graphs.

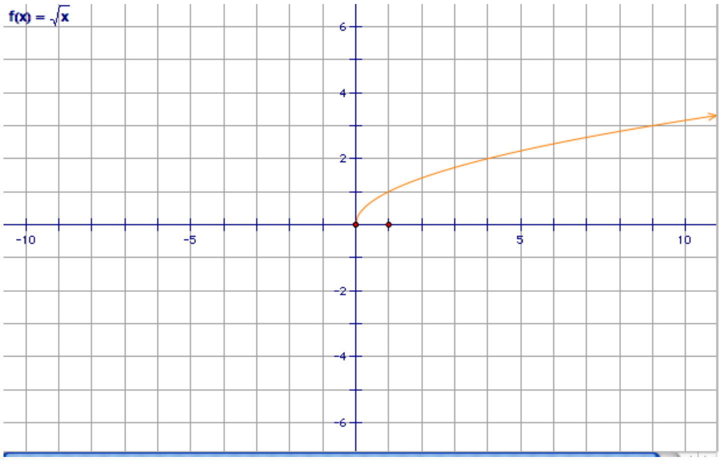


6.2 Graphs of Square Root Functions

You have used squared roots many times in this text: to simplify, to evaluate, and to solve. This lesson will focus on the graph of the square root function.

The square root function is defined by $f(x) = \sqrt{x-h} + k$, where $x - h \geq 0$ and (h, k) represents the origin of the curve.

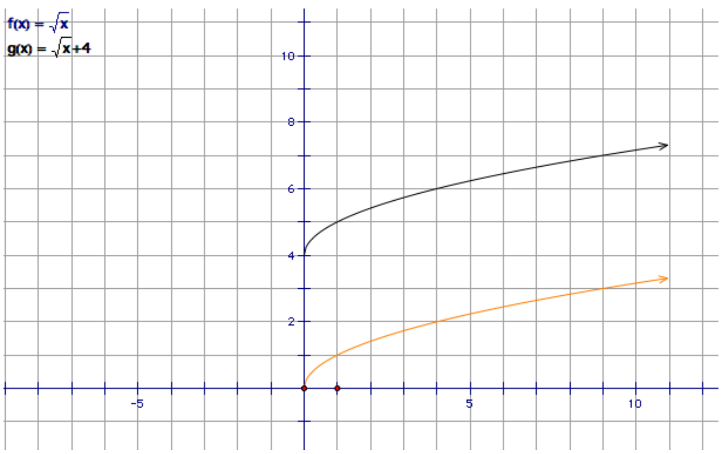
The graph of the **parent function** $f(x) = \sqrt{x}$ is shown below. The function is not defined for negative values of x; you cannot take the square root of a negative number and get a real value.



By shifting the square root function around the coordinate plane, you will change the origin of the curve.

Example: Graph $f(x) = \sqrt{x} + 4$ and compare it to the parent function.

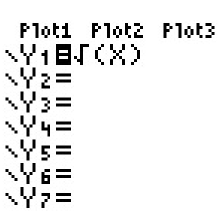
Solution: This graph has been shifted vertically upward four units from the parent function $f(x) = \sqrt{x}$. The graph is shown below.

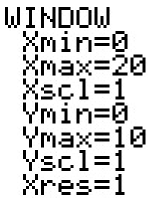


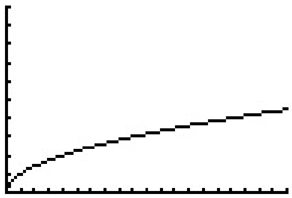
Graphing Square Root Functions Using a Calculator

Graphing square root functions is similar to graphing linear, quadratic, or exponential functions. Use the following steps:

These figures should be side by side. Due to the captions, they have moved in a vertical alignment.







Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Graphs of Square Root Functions (15:01)

Figure 6.1: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/815

1. In the definition of a square root function, why must $(x-h) \geq 0$?
2. What is the domain and range of the parent function $f(x) = \sqrt{x}$?

Identify the ordered pair of the origin of each square root function.

3. $f(x) = \sqrt{x-2}$
4. $g(x) = \sqrt{x+4}+6$
5. $h(x) = \sqrt{x-1}-1$
6. $y = \sqrt{x}+3$
7. $f(x) = \sqrt{2x}+4$

Graph the following functions on the same coordinate axes.

8. $y = \sqrt{x}$, $y = 2.5\sqrt{x}$, and $y = -2.5\sqrt{x}$
9. $y = \sqrt{x}$, $y = 0.3\sqrt{x}$, and $y = 0.6\sqrt{x}$
10. $y = \sqrt{x}$, $y = \sqrt{x-5}$, and $y = \sqrt{x+5}$

11. $y = \sqrt{x}$, $y = \sqrt{x} + 8$, and $y = \sqrt{x} - 8$

In 12-20, graph the function.

12. $y = \sqrt{2x - 1}$
13. $y = \sqrt{4x + 4}$
14. $y = \sqrt{5 - x}$
15. $y = 2\sqrt{x} + 5$
16. $y = 3 - \sqrt{x}$
17. $y = 4 + 2\sqrt{x}$
18. $y = 2\sqrt{2x + 3} + 1$
19. $y = 4 + 2\sqrt{2 - x}$
20. $y = \sqrt{x + 1} - \sqrt{4x - 5}$
21. The length between any two consecutive bases of a baseball diamond is 90 feet. How much shorter is it for the catcher to walk along the diagonal from home plate to second base than the runner running from second to home?
22. The units of acceleration of gravity are given in feet per second squared. It is $g = 32\ ft/s^2$ at sea level. Graph the period of a pendulum with respect to its length in feet. For what length in feet will the period of a pendulum be two seconds?
23. The acceleration of gravity on the Moon is 1.6 m/s^2. Graph the period of a pendulum on the Moon with respect to its length in meters. For what length, in meters, will the period of a pendulum be 10 seconds?
24. The acceleration of gravity on Mars is 3.69 m/s^2. Graph the period of a pendulum on Mars with respect to its length in meters. For what length, in meters, will the period of a pendulum be three seconds?
25. The acceleration of gravity on the Earth depends on the latitude and altitude of a place. The value of g is slightly smaller for places closer to the Equator than places closer to the Poles, and the value of g is slightly smaller for places at higher altitudes that it is for places at lower altitudes. In Helsinki, the value of $g = 9.819\ m/s^2$, in Los Angeles the value of $g = 9.796\ m/s^2$, and in Mexico City the value of $g = 9.779\ m/s^2$. Graph the period of a pendulum with respect to its length for all three cities on the same graph. Use the formula to find the length (in meters) of a pendulum with a period of 8 seconds for each of these cities.
26. The aspect ratio of a wide-screen TV is 2.39:1. Graph the length of the diagonal of a screen as a function of the area of the screen. What is the diagonal of a screen with area 150 in^2?

Graph the following functions using a graphing calculator.

27. $y = \sqrt{3x - 2}$
28. $y = 4 + \sqrt{2 - x}$
29. $y = \sqrt{x^2 - 9}$
30. $y = \sqrt{x} - \sqrt{x + 2}$

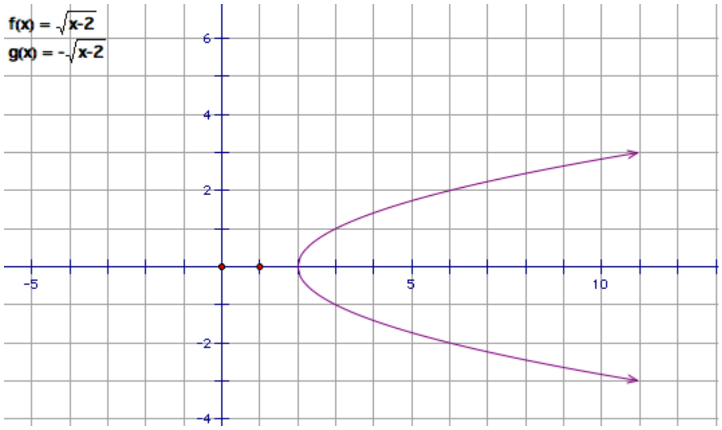
Mixed Review

31. Solve $16 = 2x^2 - 3x + 4$.
32. Write an equation for a line with slope of 0.2 containing the point (1, 10).
33. Are these lines parallel, perpendicular, or neither: $x + 5y = 16$ and $y = 5x - 3$?
34. Which of the following vertices minimizes the expression $20x + 32y$?

    (a) (50, 0)

(b) (0, 60)
(c) (15, 30)

35. Is the following graph a function? Explain your reasoning.



36. Between which two consecutive integers is $\sqrt{205}$?

6.3 Radical Expressions

Radicals are the roots of values. In fact, the word *radical* comes from the Latin word ”;radix,”; meaning ”;root.”; You are most comfortable with the square root symbol $\sqrt{x}$; however, there are many more radical symbols.

A **radical** is a mathematical expression involving a root by means of a radical sign.

$$\begin{array}{lll} \sqrt[3]{y} = x & \text{because } x^3 = y & \sqrt[3]{27} = 3, \ \textit{because } 3^3 = 27 \\ \sqrt[4]{y} = x & \text{because } x^4 = y & \sqrt[4]{16} = 2 \ \textit{because } 2^4 = 16 \\ \sqrt[n]{y} = x & \text{because } x^n = y & \end{array}$$

Some roots do not have real values; in this case, they are called **undefined**.

Even roots of negative numbers are **undefined**.

$\sqrt[n]{x}$ is undefined when n is an even whole number and $x < 0$.

Example 1: *Evaluate the following radicals:*

- $\sqrt[3]{64}$
- $\sqrt[4]{-81}$

Solution: $\sqrt[3]{64} = 4$ because $4^3 = 64$

$\sqrt[4]{-81}$ *is undefined because n is an even whole number and* $-81 < 0$.

In Chapter 8, you learned how to evaluate rational exponents:

$$a^{\frac{x}{y}} \textit{ where } x = \textit{power and } y = \textit{root}$$

This can be written in radical notation using the following property.

Rational Exponent Property: For integer values of x and whole values of y:

$$a^{\frac{x}{y}} = \sqrt[y]{a^x}$$

Example: *Rewrite $x^{\frac{5}{6}}$ using radical notation.*

Solution: This is correctly read as the sixth root of x to the fifth power. Writing in radical notation, $x^{\frac{5}{6}} = \sqrt[6]{x^5}$, *where* $x^5 > 0$.

Example 2: *Evaluate* $\sqrt[4]{4^2}$.

Solution: This is read, ";The fourth root of four to the second power.";

$$4^2 = 16$$

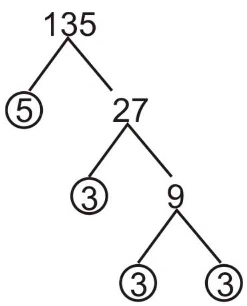
The fourth root of 16 is 2; therefore,

$$\sqrt[4]{4^2} = 2$$

In Chapter 1, Lesson 5, you learned how to simplify a square root. You can also simplify other radicals, like cube roots and fourth roots.

Example: *Simplify* $\sqrt[3]{135}$.

Solution: Begin by finding the prime factorization of 135. This is easily done by using a factor tree.



$$\sqrt[3]{135} = \sqrt[3]{3 \cdot 3 \cdot 3 \cdot 5} = \sqrt[3]{3^3} \cdot \sqrt[3]{5}$$
$$3\sqrt[3]{5}$$

Adding or Subtracting Radicals

To add or subtract radicals, they must have the same root and radicand.

$$a\sqrt[n]{x} + b\sqrt[n]{x} = (a+b)\sqrt[n]{x}$$

Example 3: *Add* $3\sqrt{5} + 6\sqrt{5}$.

Solution: The value "; $\sqrt{5}$"; is considered a *like term.* Using the rule above,

$$3\sqrt{5} + 6\sqrt{5} = (3+6)\sqrt{5} = 9\sqrt{5}$$

Example: *Simplify* $2\sqrt[3]{13} + 6\sqrt[3]{12}$.

Solution: The cube roots are not *like terms*, therefore there can be no further simplification.

In some cases, the radical may need to be reduced before addition/subtraction is possible.

Example 4: *Simplify* $4\sqrt{3} + 2\sqrt{12}$.

Solution: $\sqrt{12}$ *simplifies as* $2\sqrt{3}$.

$$4\sqrt{3} + 2\sqrt{12} \to 4\sqrt{3} + 2\left(2\sqrt{3}\right)$$
$$4\sqrt{3} + 4\sqrt{3} = 8\sqrt{3}$$

Multiplying or Dividing Radicals

To multiply radicands, the roots must be the same.

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

Example: *Simplify* $\sqrt{3} \cdot \sqrt{12}$.

Solution: $\sqrt{3} \cdot \sqrt{12} = \sqrt{36} = 6$

Dividing radicals is more complicated. A radical in the denominator of a fraction is not considered simplified by mathematicians. In order to simplify the fraction, you must **rationalize the denominator.**

To **rationalize the denominator** means to remove any radical signs from the denominator of the fraction using multiplication.

Remember: $$\sqrt{a} \times \sqrt{a} = \sqrt{a^2} = a$$

Example 1: *Simplify* $\frac{2}{\sqrt{3}}$.

Solution: We must clear the denominator of its radical using the property above. ***Remember***, what you do to one piece of a fraction, you must do to all pieces of the fraction.

$$\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3^2}} = \frac{2\sqrt{3}}{3}$$

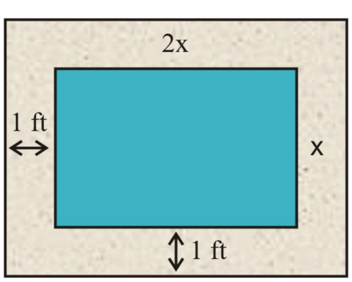
Example: *Simplify* $\frac{7}{\sqrt[3]{5}}$.

Solution: In this case, we need to make the number inside the cube root a perfect cube. We need to multiply the numerator and the denominator by $\sqrt[3]{5^2}$.

$$\frac{7}{\sqrt[3]{5}} \cdot \frac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}} = \frac{7\sqrt[3]{25}}{\sqrt[3]{5^3}} = \frac{7\sqrt[3]{25}}{5}$$

Real-World Radicals

Example: *A pool is twice as long as it is wide and is surrounded by a walkway of uniform width of 1 foot. The combined area of the pool and the walkway is* 400 square-feet. *Find the dimensions of the pool and the area of the pool.*



Solution:

1. ***Make a sketch.***
2. ***Let*** $x =$ the width of the pool.
3. ***Write an equation.*** $Area = length \cdot width$

Combined length of pool and walkway $= 2x + 2$

Combined width of pool and walkway $= x + 2$

$$\text{Area} = (2x+2)(x+2)$$

Since the combined area of pool and walkway is 400 ft^2 we can write the equation.

$$(2x+2)(x+2) = 400$$

4. Solve the equation:

	$(2x+2)(x+2) = 400$
Multiply in order to eliminate the parentheses.	$2x^2 + 4x + 2x + 4 = 400$
Collect like terms.	$2x^2 + 6x + 4 = 400$
Move all terms to one side of the equation.	$2x^2 + 6x - 396 = 0$
Divide all terms by 2.	$x^2 + 3x - 198 = 0$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-3 \pm \sqrt{3^2 - 4(1)(-198)}}{2(1)} \\ &= \frac{-3 \pm \sqrt{801}}{2} \approx \frac{-3 \pm 28.3}{2} \end{aligned}$$

Use the Quadratic Formula. $x \approx 12.65$ or -15.65 feet

5. We can disregard the negative solution since it does not make sense for this context. Thus, we can check our answer of 12.65 by substituting the result in the area formula.

$$\text{Area} = [2(12.65) + 2)](12.65 + 2) = 27.3 \cdot 14.65 \approx 400 \ ft^2.$$

The answer checks out.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following videos. Note that there is not always a match between the number of the practice exercise in the videos and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Radical Expressions with Higher Roots (8:46)

CK-12 Basic Algebra: More Simplifying Radical Expressions (7:57)

CK-12 Basic Algebra: How to Rationalize a Denominator (10:18)

Figure 6.2: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/816

Figure 6.3: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/817

Figure 6.4: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/818

1. For which values of n is $\sqrt[n]{-16}$ undefined?

Evaluate each radical expression.

2. $\sqrt{169}$
3. $\sqrt[4]{81}$
4. $\sqrt[3]{-125}$
5. $\sqrt[5]{1024}$

Write each expression as a rational exponent.

5. $\sqrt[3]{14}$
6. $\sqrt[4]{zw}$
7. $\sqrt{a}$
8. $\sqrt[9]{y^3}$

Write the following expressions in simplest radical form.

9. $\sqrt{24}$
10. $\sqrt{300}$
11. $\sqrt[5]{96}$
12. $\sqrt{\frac{240}{567}}$
13. $\sqrt[3]{500}$
14. $\sqrt[6]{64x^8}$
15. $\sqrt[3]{48a^3b^7}$
16. $\sqrt[3]{\frac{16x^5}{135y^4}}$
17. *True or false?* $\sqrt[7]{5} \cdot \sqrt[6]{6} = \sqrt[42]{30}$

Simplify the following expressions as much as possible.

17. $3\sqrt{8} - 6\sqrt{32}$
18. $\sqrt{180} + 6\sqrt{405}$
19. $\sqrt{6} - \sqrt{27} + 2\sqrt{54} + 3\sqrt{48}$
20. $\sqrt{8x^3} - 4x\sqrt{98x}$
21. $\sqrt{48a} + \sqrt{27a}$
22. $\sqrt[3]{4x^3} + x\sqrt[3]{256}$

Multiply the following expressions.

23. $\sqrt{6}\left(\sqrt{10} + \sqrt{8}\right)$
24. $\left(\sqrt{a} - \sqrt{b}\right)\left(\sqrt{a} + \sqrt{b}\right)$
25. $\left(2\sqrt{x} + 5\right)\left(2\sqrt{x} + 5\right)$

Rationalize the denominator.

26. $\frac{7}{\sqrt{15}}$

27. $\frac{9}{\sqrt{10}}$
28. $\frac{2x}{\sqrt{5x}}$
29. $\frac{\sqrt{5}}{\sqrt{3y}}$
30. The volume of a spherical balloon is $950 cm^3$. Find the radius of the balloon. (Volume of a sphere $= \frac{4}{3}\pi R^3$)
31. A rectangular picture is 9 inches wide and 12 inches long. The picture has a frame of uniform width. If the combined area of picture and frame is $180 in^2$, what is the width of the frame?
32. The volume of a soda can is 355 cm^3. The height of the can is four times the radius of the base. Find the radius of the base of the cylinder.

Mixed Review

33. An item originally priced $\$c$ is marked down 15%. The new price is \$612.99. What is c?
34. Solve $\frac{x+3}{6} = \frac{21}{x}$.
35. According to the Economic Policy Institute (EPI), minimum wage in 1989 was \$3.35 per hour. In 2009, it was \$7.25 per hour. What is the average rate of change?
36. What is the vertex of $y = 2(x+1)^2 + 4$? Is this a minimum or a maximum?
37. Using the minimum wage data (adjusted for inflation) compiled from EPI, answer the following questions.
    (a) Graph the data as a scatter plot.
    (b) Which is the best model for this data: linear, quadratic, or exponential?
    (c) Find the model of best fit and use it to predict minimum wage adjusted for inflation for 1999.
    (d) According to EPI, the 1999 minimum wage adjusted for inflation was \$6.58. How close was your model?
    (e) Use interpolation to find minimum wage in 1962.

Table 6.1:

Year	**Minimum Wage Adj. for Inflation**	**Year**	**Minimum Wage Adj. for Inflation**
1947	3.40	1952	5.36
1957	6.74	1960	6.40
1965	7.52	1970	7.81
1978	7.93	1981	7.52
1986	6.21	1990	6.00
1993	6.16	1997	6.81
2000	6.37	2004	5.80
2006	5.44	2008	6.48
2009	7.25		

6.4 Radical Equations

Solving radical equations is no different from solving linear or quadratic equations. Before you can begin to solve a radical equation, you must know how to cancel the radical. To do that, you must know its **inverse**.

Table 6.2:

Original Operation	Inverse Operation
Cube Root	Cubing (to the third power)
Square Root	Squaring (to the second power)
Fourth Root	Fourth power
";nth"; Root	";nth"; power

To solve a radical equation, you apply the solving equation steps you learned in previous chapters, including the inverse operations for roots.

Example 1: *Solve* $\sqrt{2x-1}=5$.

Solution: The first operation that must be removed is the square root. Square both sides.

$$\begin{aligned}\left(\sqrt{2x-1}\right)^2 &= 5^2\\ 2x-1 &= 25\\ 2x &= 26\\ x &= 13\end{aligned}$$

Remember to check your answer by substituting it into the original problem to see if it makes sense.

Example: *Solve* $\sqrt{x+15}=\sqrt{3x-3}$.

Solution: Begin by canceling the square roots by squaring both sides.

$$\begin{aligned}\left(\sqrt{x+15}\right)^2 &= \left(\sqrt{3x-3}\right)^2\\ x+15 &= 3x-3\\ \text{Isolate the } x-\text{variable}: \qquad 18 &= 2x\\ x &= 9\end{aligned}$$

Check the solution: $\sqrt{9+15}=\sqrt{3(9)-3}\rightarrow\sqrt{24}=\sqrt{24}$. The solution checks.

Extraneous Solutions

Not every solution of a radical equation will check in the original problem. This is called an **extraneous solution**. This means you can find a solution using algebra, but it will not work when checked. This is because of the rule in Lesson 11.2.

Even roots of negative numbers are undefined.

Example: Solve $\sqrt{x-3}-\sqrt{x}=1$.

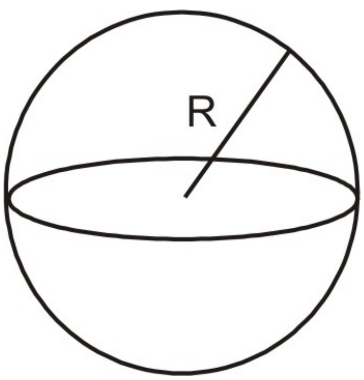
Solution:

$$
\begin{aligned}
\text{Isolate one of the radical expressions.} \quad & \sqrt{x-3} = \sqrt{x}+1 \\
\text{Square both sides.} \quad & \left(\sqrt{x-3}\right)^2 = \left(\sqrt{x}+1\right)^2 \\
\text{Remove parentheses.} \quad & x-3 = \left(\sqrt{x}\right)^2 + 2\sqrt{x}+1 \\
\text{Simplify.} \quad & x-3 = x+2\sqrt{x}+1 \\
\text{Now isolate the remaining radical.} \quad & -4 = 2\sqrt{x} \\
\text{Divide all terms by 2.} \quad & -2 = \sqrt{x} \\
\text{Square both sides.} \quad & x = 4
\end{aligned}
$$

Check: $\sqrt{4-3}-\sqrt{4} = \sqrt{1}-2 = 1-2 = -1$. The solution does not check out. The equation has no real solutions. Therefore, $x = 4$ is an extraneous solution.

Radical Equations in Real Life

Example: *A sphere has a volume of* $456\ cm^3$. *If the radius of the sphere is increased by 2 cm, what is the new volume of the sphere?*



Solution:

1. **Define variables.** Let $R =$ the radius of the sphere.
2. **Find an equation.** The volume of a sphere is given by the formula: $V = \frac{4}{3}\pi r^3$.

By substituting 456 for the volume variable, the equation becomes $456 = \frac{4}{3}\pi r^3$

$$
\begin{aligned}
\text{Multiply by 3.} \quad & 1368 = 4\pi r^3 \\
\text{Divide by } 4\pi. \quad & 108.92 = r^3 \\
\text{Take the cube root of each side.} \quad & r = \sqrt[3]{108.92} \Rightarrow r = 4.776\ cm \\
\text{The new radius is 2 centimeters more.} \quad & r = 6.776\ cm \\
\text{The new volume is :} \quad & V = \frac{4}{3}\pi(6.776)^3 = 1302.5\ cm^3
\end{aligned}
$$

Check by substituting the values of the radii into the volume formula.

$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(4.776)^3 = 456\ cm^3$. The solution checks out.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following videos. Note that there is not always a match between the number of the practice exercise in the videos and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Extraneous Solutions to Radical Equations (11:10)

Figure 6.5: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/819

CK-12 Basic Algebra: Radical Equation Examples (5:16)

Figure 6.6: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/820

CK-12 Basic Algebra: More Involved Radical Equation Example (11:54)

Figure 6.7: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/821

In 1-16, find the solution to each of the following radical equations. Identify extraneous solutions.

1. $\sqrt{x+2}-2=0$
2. $\sqrt{3x-1}=5$
3. $2\sqrt{4-3x}+3=0$

4. $\sqrt[3]{x-3} = 1$
5. $\sqrt[4]{x^2-9} = 2$
6. $\sqrt[3]{-2-5x} + 3 = 0$
7. $\sqrt{x} = x - 6$
8. $\sqrt{x^2-5x} - 6 = 0$
9. $\sqrt{(x+1)(x-3)} = x$
10. $\sqrt{x+6} = x + 4$
11. $\sqrt{x} = \sqrt{x-9} + 1$
12. $\sqrt{3x+4} = -6$
13. $\sqrt{10-5x} + \sqrt{1-x} = 7$
14. $\sqrt{2x-2} - 2\sqrt{x} + 2 = 0$
15. $\sqrt{2x+5} - 3\sqrt{2x-3} = \sqrt{2-x}$
16. $3\sqrt{x} - 9 = \sqrt{2x-14}$
17. The area of a triangle is 24 in^2 and the height of the triangle is twice as long and the base. What are the base and the height of the triangle?
18. The volume of a square pyramid is given by the formula $V = \frac{A(h)}{3}$, where $A =$ *area of the base* and $h =$ *height of the pyramid*. The volume of a square pyramid is 1,600 cubic meters. If its height is 10 meters, find the area of its base.
19. The volume of a cylinder is 245 cm^3 and the height of the cylinder is one-third the diameter of the cylinder's base. The diameter of the cylinder is kept the same, but the height of the cylinder is increased by two centimeters. What is the volume of the new cylinder? (Volume $= \pi r^2 \cdot h$)
20. The height of a golf ball as it travels through the air is given by the equation $h = -16t^2 + 256$. Find the time when the ball is at a height of 120 feet.

Mixed Review

21. Joy sells two types of yarn: wool and synthetic. Wool is \$12 per skein and synthetic is \$9 per skein. If Joy sold 16 skeins of synthetic and collected a total of 432, how many skeins of wool did she sell?
22. Solve $16 \geq |x-4|$.
23. Graph the solution $\begin{cases} y \leq 2x - 4 \\ y > -\frac{1}{4}x + 6 \end{cases}$.

6.5 Chapter Review

Explain the shift of each function from the parent function $f(x) = \sqrt{x}$.

1. $f(x) = \sqrt{x} + 7$
2. $f(x) = \sqrt{x+3}$
3. $g(x) = -\sqrt{x}$
4. $y = 3 + \sqrt{x-1}$

Graph the following square root functions. Identify the domain and range of each.

5. $f(x) = \sqrt{x-2} + 5$
6. $g(x) = -\sqrt{x+1}$
7. $f(x) = \sqrt{2x} - 2$

Simplify the following, if possible. Write your answer in its simplest form.

8. $\sqrt{\frac{3}{7}} \times \sqrt{\frac{14}{27}}$
9. $\sqrt{5} \cdot \sqrt{7}$
10. $\sqrt{11} \times \sqrt[3]{11}$
11. $\frac{\sqrt{18}}{\sqrt{2}}$
12. $8\sqrt[3]{4} + 11\sqrt[3]{4}$
13. $5\sqrt{80} - 12\sqrt{5}$
14. $\sqrt{10} + \sqrt{2}$
15. $\sqrt{24} - \sqrt{6}$
16. $\sqrt[3]{27} + \sqrt[4]{81}$
17. $4\sqrt{3} \cdot 2\sqrt{6}$
18. $\sqrt[3]{3} \times \sqrt{7}$
19. $6\sqrt{72}$
20. $7\sqrt{\left(\frac{40}{49}\right)}$
21. $\frac{5}{\sqrt{75}}$
22. $\frac{\sqrt{45}}{\sqrt{5}}$
23. $\frac{3}{\sqrt[3]{3}}$
24. $8\sqrt{10} - 3\sqrt{40}$
25. $\sqrt{27} + \sqrt{3}$

Solve each equation. If the answer is extraneous, say so.

26. $8 = \sqrt[3]{2k}$
27. $x = \sqrt{7x}$
28. $\sqrt{2+2m} = \sqrt{4-m}$
29. $\sqrt[4]{35-2x} = -1$
30. $14 = 6 + \sqrt{10-6x}$
31. $4 + \sqrt{\frac{n}{3}} = 5$
32. $\sqrt{-9-2x} = \sqrt{-1-x}$
33. $-2 = \sqrt[3]{t-6}$
34. $5\sqrt{10} = 6\sqrt{w}$
35. $\sqrt{x^2+3x} = 2$
36. $\sqrt[4]{t} = 5$

6.6 Chapter Test

1. Describe each type of visual display presented in this chapter. State one advantage and one disadvantage for each type of visual display.
2. Graph $f(x) = 7 + \sqrt{x-4}$. State its domain and range. What is the ordered pair of the origin?
3. *True or false?* The upper quartile is the mean of the upper half of the data.
4. What is the domain restriction of $y = \sqrt[4]{x}$?
5. Solve $-6 = 2\sqrt[3]{c+5}$.
6. Simplify $\frac{4}{\sqrt{48}}$.
7. Simplify and reduce: $\sqrt[3]{3} \times \sqrt[3]{81}$.

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9621.

Chapter 7

Rational Equations and Functions

7.1 Rational Equations and Functions

The final chapter of this text introduces the concept of **rational functions,** that is, equations in which the variable appears in the denominator of a fraction. A common rational function in the **inverse variation model,** similar to the direct variation model you studied in chapter 4 lesson 6. We finish the chapter with solving rational equations and using graphical representations to display data.

7.2 Inverse Variation Models

In Chapter 4, Lesson 6, you learned how to write direct variation models. In direct variation, the variables changed in the same way and the graph contained the origin. But what happens when the variables change in different ways? Consider the following situation.

A group of friends rent a beach house and decide to split the cost of the rent and food. Four friends pay \$170 each. Five friends pay \$162 each. Six friends pay \$157. If nine people were to share the expense, how much would each pay?

Let's look at this in a table.

Table 7.1:

n **(number of friends)**	t **(share of expense)**
4	170
5	162
6	157
9	???

As the number of friends gets larger, the cost per person gets smaller. This is an example of **inverse variation.**

An **inverse variation** function has the form $f(x) = \frac{k}{x}$, where k is called the constant of variation and must be a counting number and $x \neq 0$.

To show an inverse variation relationship, use either of the phrases:

- *Is inversely proportional to*
- *Varies inversely as*

Example 1: *Find the constant of variation of the beach house situation.*

Solution: Use the inverse variation equation to find k, the constant of variation.

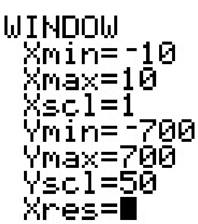
$$\begin{aligned} y &= \frac{k}{x} \\ 170 &= \frac{k}{4} \\ \text{Solve for } k: \qquad 170 \times 4 &= \frac{k}{4} \times 4 \\ k &= 680 \end{aligned}$$

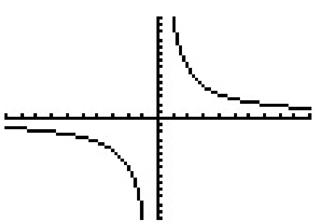
You can use this information to determine the amount of expense per person if nine people split the cost.

$$\begin{aligned} y &= \frac{680}{x} \\ y &= \frac{680}{9} = 75.56 \end{aligned}$$

If nine people split the expense, each would pay \$75.56.

Using a graphing calculator, look at a graph of this situation.





The graph of an inverse variation function $f(x) = \frac{k}{x}$ is a **hyperbola.** It has two branches in opposite quadrants.

If $k > 0$, the branches are in quadrants I and III.

If $k < 0$, the branches are in quadrants II and IV.

The graph appears to not cross the axes. In fact, this is true of any inverse variation equation of the form $y = \frac{k}{x^n}$. These lines are called **asymptotes.** Because of this, an inverse variation function has a special domain and range.

$$\begin{aligned} \textit{Domain} &: \neq 0 \\ \textit{Range} &: \neq 0 \end{aligned}$$

You will investigate these excluded values in later lessons of this chapter.

Example 2: ***The frequency,*** f, of sound varies inversely with wavelength, λ. A sound signal that has a wavelength of 34 meters has a frequency of 10 hertz. What frequency does a sound signal of 120 meters have?

Solution: Use the inverse variation equation to find k, the constant of variation.

$$f = \frac{k}{\lambda}$$

$$10 = \frac{k}{34}$$

Solve for k :

$$10 \times 34 = \frac{k}{34} \times 34$$

$$k = 340$$

Use k to answer the question:

$$f = \frac{340}{120} = 2.83 \ hertz$$

Practice SetSample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Proportionality (17:03)

Figure 7.1: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/833

1. Define *inverse variation.*
2. Using 4.6 as a reference, explain three main differences between direct variation and inverse variation.

Read each statement and decide if the relationship is *direct, inverse,* or *neither.*

3. The weight of a book ______________________________ as the number of pages it contains.
4. The temperature outside ______________________________ as the time of day.
5. The amount of prize money you receive from winning the lottery ________________ as the number of people who split the ticket cost.
6. The cost of a ferry ride ________________________________ as the number of times you ride.
7. The area of a square ____________________________________ as the length of its side.
8. The height from the ground ________________________________ as the number of seconds you have been on a roller coaster.
9. The time it takes to wash a car ________________________________ as the number of people helping.

10. The number of tiles it takes to tile a floor ____________________________ as the size of each tile.

Graph each inverse equation. State the domain and range.

11. $y = \frac{3}{x}$
12. $y = \frac{1}{x^2}$
13. $f(x) = -\frac{4}{x}$
14. $y = \frac{10}{x}$
15. $h(x) = -\frac{1}{x}$
16. $y = \frac{1}{4x}$
17. $g(x) = -\frac{2}{x^2}$
18. $y = \frac{4}{x^2}$
19. $y = \frac{5}{6x}$

In 20–25, model each situation with an inverse variation equation, finding k. Then answer the question.

20. y varies inversely as x. If $y = 24$ when $x = 3$, find y when $x = -1.5$.
21. d varies inversely as the cube of t. If $d = -23.5$ when $t = 3$, find d when $x = \frac{1}{4}$.
22. If z is inversely proportional to w and $z = 81$ when $w = 9$, find w when $z = 24$.
23. If y is inversely proportional to x and $y = 2$ when $x = 8$, find y when $x = 12$.
24. If a is inversely proportional to the square root of b, and $a = 32$ when $b = 9$, find b when $a = 6$.
25. If w is inversely proportional to the square of u and $w = 4$ when $u = 2$, find w when $u = 8$.
26. The law of the fulcrum states the distance from the fulcrum varies inversely as the weight of the object. Joey and Josh are on a seesaw. If Joey weighs 40 pounds and sits six feet from the fulcrum, how far would Josh have to sit to balance the seesaw? (Josh weighs 65 pounds.)
27. The intensity of light is inversely proportional to the square of the distance between the light source and the object being illuminated. A light meter that is 10 meters from a light source registers 35 lux. What intensity would it register 25 meters from the light source?
28. Ohm's Law states that current flowing in a wire is inversely proportional to the resistance of the wire. If the current is 2.5 amperes when the resistance is 20 ohms, find the resistance when the current is 5 amperes.
29. The number of tiles it takes to tile a bathroom floor varies inversely as the square of the side of the tile. If it takes 112 six-inch tiles to cover a floor, how many eight-inches tiles are needed?

Mixed Review

30. Solve and graph the solutions on a number line: $16 \geq -3x + 5$.
31. Graph on a coordinate plane: $x = \frac{7}{14}$.
32. Simplify $\sqrt[3]{320}$.
33. State the Commutative Property of Multiplication.
34. Draw the real number hierarchy and provide an example for each category.
35. Find 17.5% of 96.

Graphs of Rational FunctionsIn the previous lesson, you learned the basics of graphing an inverse variation function. The **hyperbola** forms two **branches** in opposite quadrants. The axes are **asymptotes** to the graph. This lesson will compare graphs of inverse variation functions. You will also learn how to graph other rational equations.

Example: Graph the function $f(x) = \frac{k}{x}$ for the following values of k:

$$k = -2, -1, -\frac{1}{2}, 1, 2, 4$$

Each graph is shown separately then on one coordinate plane.

As mentioned in the previous lesson, if k is positive, then the branches of the hyperbola are located in quadrants I and III. If k is negative, the branches are located in quadrants II and IV. Also notice how the hyperbola changes as k gets larger.

Rational FunctionsA **rational function** is a ratio of two polynomials (a polynomial divided by another polynomial). The formal definition is:

$$f(x) = \frac{g(x)}{h(x)}, \text{where } h(x) \neq 0$$

.

An **asymptote** is a value for which the equation or function is undefined. Asymptotes can be vertical, horizontal, or oblique. This text will focus on vertical asymptotes; other math courses will also show you how to find horizontal and oblique asymptotes. A function is undefined when the denominator of a fraction is zero. To find the asymptotes, find where the denominator of the rational function is zero. These are called **points of discontinuity** of the function.

The formal definition for asymptote is as follows.

An **asymptote** is a straight line to which, as the distance from the origin gets larger, a curve gets closer and closer but never intersects.

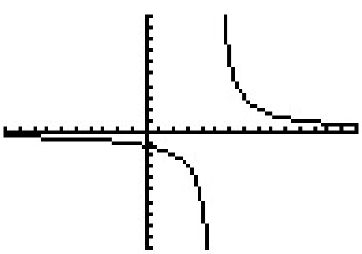
Example: Find the points of discontinuity and the asymptote for the function $y = \frac{6}{x-5}$.

Solution: Find the value of x for which the denominator of the rational function is zero.

$$0 = x - 5 \rightarrow x = 5$$

The point at which $x = 5$ is a point of discontinuity. Therefore, the asymptote has the equation $x = 5$.

Look at the graph of the function. There is a clear separation of the branches at the vertical line five units to the right of the origin.



The domain is ";all real numbers except five"; or symbolically written $x \neq 5$.

Example 1: Determine the asymptotes of $t(x) = \frac{2}{(x-2)(x+3)}$.

Solution: Using the Zero Product Property, there are two cases for asymptotes, where each set of parentheses equals zero.

$$x - 2 = 0 \rightarrow x = 2$$
$$x + 3 = 0 \rightarrow x = -3$$

The two asymptotes for this function are $x = 2$ and $x = -3$.

Check your solution by graphing the function.

The domain of the rational function above has two points of discontinuity. Therefore, its domain cannot include the numbers 2 or –3. The *domain:* $x \neq 2, x \neq -3$.

Horizontal AsymptotesRational functions can also have horizontal asymptotes. The equation of a horizontal asymptote is $y = c$, where c represents the vertical shift of the rational function.

Example: Identify the vertical and horizontal asymptotes of $f(x) = \frac{3}{(x-4)(x+8)} - 5$.

Solution: The vertical asymptotes occur where the denominator is equal to zero.

$$x - 4 = 0 \rightarrow x = 4$$
$$x + 8 = 0 \rightarrow x = -8$$

The vertical asymptotes are $x = 4$ and $x = -8$.

The rational function has been shifted down five units $f(x) = \frac{3}{(x-4)(x+8)} - 5$.

Therefore, the horizontal asymptote is $y = -5$.

Multimedia Link: For further explanation about asymptotes, read through this PowerPoint presentation presented by North Virginia Community College or watch this CK-12 Basic Algebra: Finding Vertical Asymptotes of Rational Functions - YouTube video.

Figure 7.2: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/834

Real-World Rational Functions

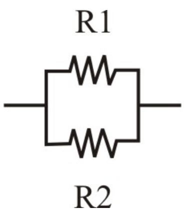
Electrical circuits are commonplace is everyday life. For instance, they are present in all electrical appliances in your home. The figure below shows an example of a simple electrical circuit. It consists of a battery that provides a voltage (V, measured in Volts), a resistor (R, measured in ohms, Ω) that resists the flow of electricity, and an ammeter that measures the current (I, measured in amperes, A) in the circuit. Your light bulb, toaster, and hairdryer are all basically simple resistors. In addition, resistors are used in an electrical circuit to control the amount of current flowing through a circuit and to regulate voltage levels. One important reason to do this is to prevent sensitive electrical components from burning out due to too much current or too high a voltage level. Resistors can be arranged in series or in parallel.

For resistors placed in a series, the total resistance is just the sum of the resistances of the individual resistors.

$$R_{tot} = R_1 + R_2$$

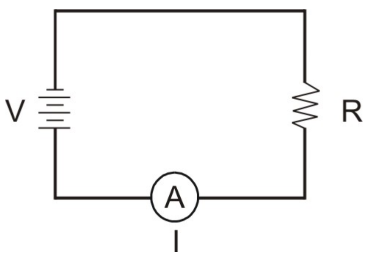
For resistors placed in parallel, the reciprocal of the total resistance is the sum of the reciprocals of the resistances of the individual resistors.

$$\frac{1}{R_c} = \frac{1}{R_1} + \frac{1}{R_2}$$

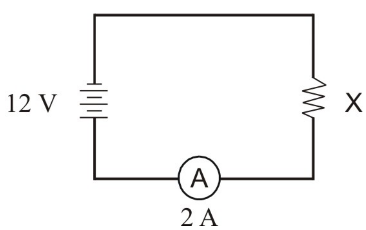


Ohm's Law gives a relationship between current, voltage, and resistance. It states that:

$$I = \frac{V}{R}$$



Example: Find the value of x marked in the diagram.



Solution: Using Ohm's Law, $I = \frac{V}{R}$, and substituting the appropriate information yields:

$$2 = \frac{12}{R}$$

Using the cross multiplication of a proportion yields:

$$2R = 12 \rightarrow R = 6 \ \Omega$$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following videos. Note that there is not always a match between the number of the practice exercise in the videos and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Asymptotes (21:06)

CK-12 Basic Algebra: Another Rational Function Graph Example (8:20)

CK-12 Basic Algebra: A Third Example of Graphing a Rational Function (11:31)

Figure 7.3: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/835

Figure 7.4: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/836

Figure 7.5: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/837

1. What is a rational function?
2. Define *asymptote.* How does an asymptote relate algebraically to an rational equation?
3. Which asymptotes are described in this lesson? What is the general equation for these asymptotes?

Identify the vertical and horizontal asymptotes of each rational function.

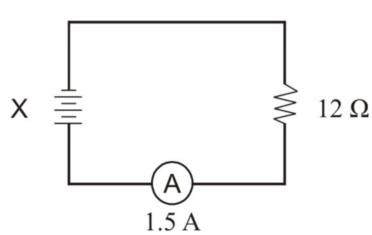
4. $y = \frac{4}{x+2}$
5. $f(x) = \frac{5}{2x-6} + 3$
6. $y = \frac{10}{x}$
7. $g(x) = \frac{4}{4x^2+1} - 2$
8. $h(x) = \frac{2}{x^2-9}$
9. $y = \frac{1}{x^2+4x+3} + \frac{1}{2}$
10. $y = \frac{3}{x^2-4} - 8$
11. $f(x) = \frac{-3}{x^2-2x-8}$

Graph each rational function. Show the vertical asymptote and horizontal asymptote as a dotted line.

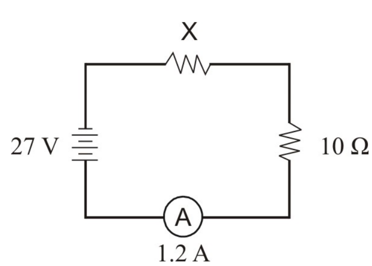
12. $y = -\frac{6}{x}$
13. $y = \frac{x}{2-x^2} - 3$
14. $f(x) = \frac{3}{x^2}$
15. $g(x) = \frac{1}{x-1} + 5$
16. $y = \frac{2}{x+2} - 6$
17. $f(x) = \frac{-1}{x^2+2}$
18. $h(x) = \frac{4}{x^2+9}$
19. $y = \frac{-2}{x^2+1}$
20. $j(x) = \frac{1}{x^2-1} + 1$
21. $y = \frac{2}{x^2-9}$
22. $f(x) = \frac{8}{x^2-16}$
23. $g(x) = \frac{3}{x^2-4x+4}$
24. $h(x) = \frac{1}{x^2-x-6} - 2$

Find the quantity labeled x in the following circuit.

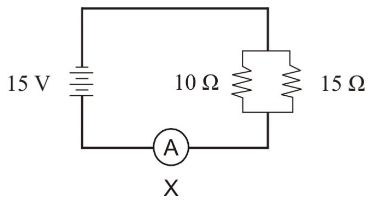
25.



26.



27.



Mixed Review

28. A building 350 feet tall casts a shadow $\frac{1}{2}mile$ long. How long is the shadow of a person five feet tall?
29. State the Cross Product Property.
30. Find the slope between (1, 1) and (–4, 5).
31. The amount of refund from soda cans in Michigan is directly proportional to the number of returned cans. If you earn \$12.00 refund for 120 cans, how much do you get per can?
32. You put the letters from VACATION into a hat. If you reach in randomly, what is the probability you will pick the letter A?
33. Give an example of a sixth-degree binomial.

7.3 Division of Polynomials

We will begin with a property that is the converse of the Adding Fractions Property presented in Chapter 2.

For all real numbers a, b, and c, and $c \neq 0$, $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$.

This property allows you to separate the numerator into its individual fractions. This property is used when dividing a polynomial by a monomial.

Example: Simplify $\frac{8x^2-4x+16}{2}$

Solution: Using the property above, separate the polynomial into its individual fractions.

$$\frac{8x^2}{2} - \frac{4x}{2} + \frac{16}{2}$$

Reduce.

$$4x^2 - 2x + 8$$

Example 1: Simplify $\frac{-3m^2-18m+6}{9m}$.

Solution: Separate the trinomial into its individual fractions and reduce.

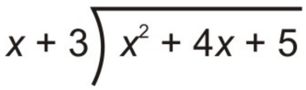
$$-\frac{3m^2}{9m} - \frac{18m}{9m} + \frac{6}{9m}$$
$$-\frac{m}{3} - 2 + \frac{2}{3m}$$

Polynomials can also be divided by binomials. However, instead of separating into its individual fractions, we use a process called long division.

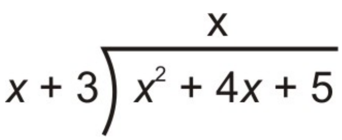
Example: Simplify $\frac{x^2+4x+5}{x+3}$.

Solution: When we perform division, the expression in the numerator is called the **dividend** and the expression in the denominator is called the **divisor.**

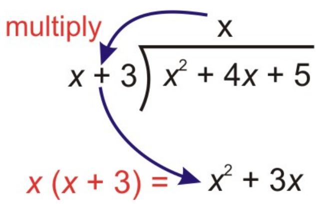
To start the division we rewrite the problem in the following form.



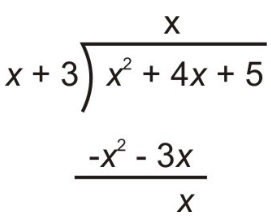
Start by dividing the first term in the dividend by the first term in the divisor $\frac{x^2}{x} = x$. Place the answer on the line above the x term.



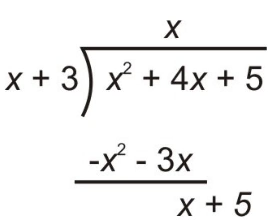
Next, multiply the x term in the answer by each of the $x+3$ terms in the divisor and place the result under the divided, matching like terms.



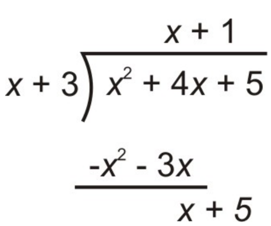
Now subtract x^2+3x from x^2+4x+5. It is useful to change the signs of the terms of x^2+3x to $-x^2-3x$ and add like terms vertically.



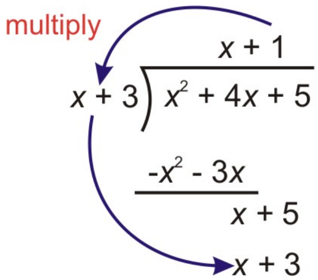
Now, bring down 5, the next term in the dividend.



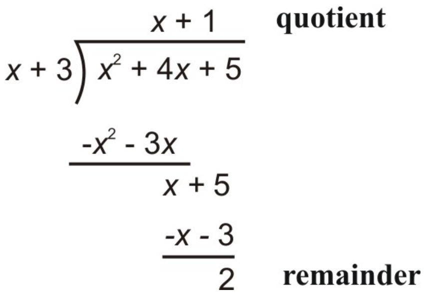
Repeat the process. First divide the first term of $x+5$ by the first term of the divisor $\left(\frac{x}{x}\right) = 1$. Place this answer on the line above the constant term of the dividend.



Multiply 1 by the divisor $x+3$ and write the answer below $x+5$, matching like terms.



Subtract $x+3$ from $x+5$ by changing the signs of $x+3$ to $-x-3$ and adding like terms.



Since there are no more terms from the dividend to bring down, we are done.

The answer is $x+1$ with a remainder of 2.

Multimedia Link: For more help with using long division to simplify rational expressions, visit this http://www.purplemath.com/modules/polydiv2.htm - website or watch this CK-12 Basic Algebra: 6 7 Polynomial long division with Mr. Nystrom - YouTube video.

Figure 7.6: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/838

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra: Polynomial Division (12:09)

Divide the following polynomials.

1. $\frac{2x+4}{2}$
2. $\frac{x-4}{x}$
3. $\frac{5x-35}{5x}$
4. $\frac{x^2+2x-5}{x}$

Figure 7.7: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/839

5. $\frac{4x^2+12x-36}{-4x}$
6. $\frac{2x^2+10x+7}{2x^2}$
7. $\frac{x^3-x}{-2x^2}$
8. $\frac{5x^4-9}{3x}$
9. $\frac{x^3-12x^2+3x-4}{12x^2}$
10. $\frac{3-6x+x^3}{-9x^3}$
11. $\frac{x^2+3x+6}{x+1}$
12. $\frac{x^2-9x+6}{x-1}$
13. $\frac{x^2+5x+4}{x+4}$
14. $\frac{x^2-10x+25}{x-5}$
15. $\frac{x^2-20x+12}{x-3}$
16. $\frac{3x^2-x+5}{x-2}$
17. $\frac{9x^2+2x-8}{x+4}$
18. $\frac{3x^2-4}{3x+1}$
19. $\frac{5x^2+2x-9}{2x-1}$
20. $\frac{x^2-6x-12}{5x+4}$
21. $\frac{x^4-2x}{8x+24}$
22. $\frac{x^3+1}{4x-1}$

Mixed Review

23. Boyle's Law states that the pressure of a compressed gas varies inversely as its volume. If the pressure of a 200-pound gas is 16.75 psi, find the pressure if the amount of gas is 60 pounds.
24. Is $5x^3 + x^2 - x^{-1} + 8$ an example of a polynomial? Explain your answer.
25. Find the slope of the line perpendicular to $y = -\frac{3}{4}x + 5$.
26. How many two-person teams can be made from a group of nine individuals?
27. What is a problem with face-to-face interviews? What do you think is a potential solution to this problem?
28. Solve for $m : -4 = \frac{\sqrt{m-3}}{-2}$.

7.4 Rational Expressions

You have gained experience working with rational functions so far this chapter. In this lesson, you will continue simplifying rational expressions by factoring.

To **simplify** a rational expression means to reduce the fraction into its lowest terms.

To do this, you will need to remember a property about multiplication.

For all real values a, b, and $b \neq 0$, $\frac{ab}{b} = a$.

Example: Simplify $\frac{4x-2}{2x^2+x-1}$.

Solution:

Both the numerator and denominator can be factored using methods learned in Chapter 9.

$$\frac{4x-2}{2x^2+x-1} \rightarrow \frac{2(2x-1)}{(2x-1)(x+1)}$$

The expression $(2x-1)$ appears in both the numerator and denominator and can be canceled. The expression becomes:

$$\frac{4x-2}{2x^2+x-1} = \frac{2}{x+1}$$

Example 1: Simplify $\frac{x^2-2x+1}{8x-8}$.

Solution: Factor both pieces of the rational expression and reduce.

$$\frac{x^2-2x+1}{8x-8} \rightarrow \frac{(x-1)(x-1)}{8(x-1)}$$
$$\frac{x^2-2x+1}{8x-8} = \frac{x-1}{8}$$

Finding Excluded Values of Rational Expressions

As stated in Lesson 2 of this chapter, excluded values are also called **points of discontinuity.** These are the values that make the denominator equal to zero and are not part of the domain.

Example 2: Find the excluded values of $\frac{2x+1}{x^2-x-6}$.

Solution: Factor the denominator of the rational expression.

$$\frac{2x+1}{x^2-x-6} = \frac{2x+1}{(x+2)(x-3)}$$

Find the value that makes each factor equal zero.

$$x = -2, x = 3$$

These are excluded values of the domain of the rational expression.

Real-Life Rational Expressions

The gravitational force between two objects in given by the formula $F = \frac{G(m_1 m_2)}{(d^2)}$. The gravitation constant is given by $G = 6.67 \times 10^{-11}(N \cdot m^2/kg^2)$. The force of attraction between the Earth and the Moon is $F = 2.0 \times 10^{20}\ N$ (with masses of $m_1 = 5.97 \times 10^{24}\ kg$ for the Earth and $m_2 = 7.36 \times 10^{22}\ kg$ for the Moon).

What is the distance between the Earth and the Moon?

Let's start with the Law of Gravitation formula.	$F = G\frac{m_1 m_2}{d^2}$
Now plug in the known values.	$2.0 \times 10^{20} N = 6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2} \cdot \frac{(5.97 \times 10^{24} kg)(7.36 \times 10^{22} kg)}{d^2}$
Multiply the masses together.	$2.0 \times 10^{20} N = 6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2} \cdot \frac{4.39 \times 10^{47} kg^2}{d^2}$
Cancel the kg^2 units.	$2.0 \times 10^{20} N = 6.67 \times 10^{-11} \frac{N \cdot m^2}{\cancel{kg^2}} \cdot \frac{4.39 \times 10^{47} \cancel{kg^2}}{d^2}$
Multiply the numbers in the numerator.	$2.0 \times 10^{20} N \frac{2.93 \times 10^{37}}{d^2} N \cdot m^2$
Multiply both sides by d^2.	$2.0 \times 10^{20} N \cdot d^2 = \frac{2.93 \times 10^{37}}{d^2} \cdot d^2 \cdot N \cdot m^2$
Cancel common factors.	$2.0 \times 10^{20} N \cdot d^2 = \frac{2.93 \times 10^{37}}{\cancel{d^2}} \cdot \cancel{d^2} \cdot N \cdot m^2$
Simplify.	$2.0 \times 10^{20} N \cdot d^2 = 2.93 \times 10^{37} N \cdot m^2$
Divide both sides by $2.0 \times 10^{20} N$.	$d^2 = \frac{2.93 \times 10^{37}}{2.0 \times 10^{20}} \frac{N \cdot m^2}{N}$
Simplify.	$d^2 = 1.465 \times 10^{17} m^2$
Take the square root of both sides.	$d = 3.84 \times 10^{8} m$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Simplifying Rational Expressions (15:22)

Figure 7.8: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/840

Reduce each fraction to lowest terms.

1. $\frac{4}{2x-8}$
2. $\frac{x^2+2x}{x}$
3. $\frac{9x+3}{12x+4}$

4. $\frac{6x^2+2x}{4x}$
5. $\frac{x-2}{x^2-4x+4}$
6. $\frac{x^2-9}{5x+15}$
7. $\frac{x^2+6x+8}{x^2+4x}$
8. $\frac{2x^2+10x}{x^2+10x+25}$
9. $\frac{x^2+6x+5}{x^2-x-2}$
10. $\frac{x^2-16}{x^2+2x-8}$
11. $\frac{3x^2+3x-18}{2x^2+5x-3}$
12. $\frac{x^3+x^2-20x}{6x^2+6x-120}$

Find the excluded values for each rational expression.

13. $\frac{2}{x}$
14. $\frac{4}{x+2}$
15. $\frac{2x-1}{(x-1)^2}$
16. $\frac{3x+1}{x^2-4}$
17. $\frac{x^2}{x^2+9}$
18. $\frac{2x^2+3x-1}{x^2-3x-28}$
19. $\frac{5x^3-4}{x^2+3x}$
20. $\frac{9}{x^3+11x^2+30x}$
21. $\frac{4x-1}{x^2+3x-5}$
22. $\frac{5x+11}{3x^2-2x-4}$
23. $\frac{x^2-1}{2x^3+x+3}$
24. $\frac{12}{x^2+6x+1}$
25. In an electrical circuit with resistors placed in parallel, the reciprocal of the total resistance is equal to the sum of the reciprocals of each resistance: $\frac{1}{R_c} = \frac{1}{R_1} + \frac{1}{R_2}$. If $R_1 = 25\Omega$ and the total resistance is $R_c = 10\Omega$, what is the resistance R_2?
26. Suppose that two objects attract each other with a gravitational force of 20 Newtons. If the distance between the two objects is doubled, what is the new force of attraction between the two objects?
27. Suppose that two objects attract each other with a gravitational force of 36 Newtons. If the mass of both objects was doubled, and if the distance between the objects was doubled, then what would be the new force of attraction between the two objects?
28. A sphere with radius r has a volume of $\frac{4}{3}\pi r^3$ and a surface area of $4\pi r^2$. Find the ratio of the surface area to the volume of the sphere.
29. The side of a cube is increased by a factor of two. Find the ratio of the old volume to the new volume.
30. The radius of a sphere is decreased by four units. Find the ratio of the old volume to the new volume.

Mixed Review

31. Name $4p^6 + 7p^3 - 9$.
32. Simplify $(4b^2 + b + 7b^3) + (5b^2 - 6b^4 + b^3)$. Write the answer in standard form.
33. State the Zero Product Property.
34. Why can't the Zero Product Property be used in this situation: $(5x + 1)(x - 4) = 2$?
35. Shelly earns \$4.85 an hour plus \$15 in tips. Graph her possible total earnings for one day of work.
36. Multiply and simplify: $(-4x^2 + 8x - 1)(-7x^2 + 6x + 8)$.
37. A rectangle's perimeter is 65 yards. The length is 7 more yards than its width. What dimensions would give the largest area?

7.5 Multiplication and Division of Rational Expressions

Because a rational expression is really a fraction, two (or more) rational expressions can be combined through multiplication and/or division in the same manner as numerical fractions. A reminder of how to multiply fractions is below.

For any rational expressions $a \neq 0, b \neq 0, c \neq 0, d \neq 0$,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$
$$\frac{a}{b} \div \frac{c}{d} \to \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Example: Multiply the following $\frac{a}{16b^8} \cdot \frac{4b^3}{5a^2}$

Solution:

$$\frac{a}{16b^8} \cdot \frac{4b^3}{5a^2} \to \frac{4ab^3}{80a^2b^8}$$

Simplify exponents using methods learned in chapter 8.

$$\frac{4ab^3}{80a^2b^8} = \frac{1}{20ab^5}$$

Example 1: Simplify $9c^2 \cdot \frac{4y^2}{21c^4}$.

Solution:

$$9c^2 \cdot \frac{4y^2}{21c^4} \to \frac{9c^2}{1} \cdot \frac{4y^2}{21c^4}$$
$$\frac{9c^2}{1} \cdot \frac{4y^2}{21c^4} = \frac{36c^2y^2}{21c^4}$$
$$\frac{36c^2y^2}{21c^4} = \frac{12y^2}{7c^2}$$

Multiplying Rational Expressions Involving Polynomials

When rational expressions become complex, it is usually easier to factor and reduce them **before** attempting to multiply the expressions.

Example: Multiply $\frac{4x+12}{3x^2} \cdot \frac{x}{x^2-9}$.

Solution: Factor all pieces of these rational expressions and reduce before multiplying.

$$\frac{4x+12}{3x^2} \cdot \frac{x}{x^2-9} \to \frac{4(x+3)}{3x^2} \cdot \frac{x}{(x+3)(x-3)}$$
$$\frac{4\cancel{(x+3)}}{3x^{\cancel{2}}} \cdot \frac{\cancel{x}}{\cancel{(x+3)}(x-3)}$$
$$\frac{4}{3x} \cdot \frac{1}{x-3} \to \frac{4}{3x^2-9x}$$

Example 1: Multiply $\frac{12x^2-x-6}{x^2-1} \cdot \frac{x^2+7x+6}{4x^2-27x+18}$.

Solution: Factor all pieces, reduce, and then multiply.

$$\begin{aligned}
\frac{12x^2-x-6}{x^2-1}\cdot\frac{x^2+7x+6}{4x^2-27x+18} &\rightarrow \frac{(3x+2)(4x-3)}{(x+1)(x-1)}\cdot\frac{(x+1)(x+6)}{(4x-3)(x-6)}\\
\frac{(3x+2)\cancel{(4x-3)}}{\cancel{(x+1)}(x-1)}\cdot\frac{\cancel{(x+1)}(x+6)}{\cancel{(4x-3)}(x-6)} &\rightarrow \frac{(3x+2)(x+6)}{(x-1)(x-6)}\\
\frac{12x^2-x-6}{x^2-1}\cdot\frac{x^2+7x+6}{4x^2-27x+18} &= \frac{3x^2+20x+12}{x^2-7x+6}
\end{aligned}$$

Dividing Rational Expressions Involving Polynomials

Division of rational expressions works in the same manner as multiplication. A reminder of how to divide fractions is below.

For any rational expressions $a \neq 0, b \neq 0, c \neq 0, d \neq 0$,

$$\frac{a}{b}\div\frac{c}{d}\rightarrow\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$$

Example: Simplify $\frac{9x^2-4}{2x-2}\div\frac{21x^2-2x-8}{1}$.

Solution:

$$\frac{9x^2-4}{2x-2}\div\frac{21x^2-2x-8}{1}\rightarrow\frac{9x^2-4}{2x-2}\cdot\frac{1}{21x^2-2x-8}$$

Repeat the process for multiplying rational expressions.

$$\begin{aligned}
\frac{9x^2-4}{2x-2}\cdot\frac{1}{21x^2-2x-8} &\rightarrow \frac{(3x-2)\cancel{(3x-2)}}{2(x-1)}\cdot\frac{1}{\cancel{(3x-2)}(7x+4)}\\
\frac{9x^2-4}{2x-2}\div\frac{21x^2-2x-8}{1} &= \frac{3x-2}{14x^2-6x-8}
\end{aligned}$$

Real-Life Application

Suppose Marciel is training for a running race. Marciel's speed (in miles per hour) of his training run each morning is given by the function x^3-9x, where x is the number of bowls of cereal he had for breakfast $(1 \leq x \leq 6)$. Marciel's training distance (in miles), if he eats x bowls of cereal, is $3x^2-9x$. What is the function for Marciel's time and how long does it take Marciel to do his training run if he eats five bowls of cereal on Tuesday morning?

$$\begin{aligned}
\text{time} &= \frac{\text{distance}}{\text{speed}}\\
\text{time} &= \frac{3x^2-9x}{x^3-9x}=\frac{3x(x-3)}{x(x^2-9)}=\frac{3x\cancel{(x-3)}}{x(x+3)\cancel{(x-3)}}\\
\text{time} &= \frac{3}{x+3}\\
\text{If } x &= 5, \text{then}\\
\text{time} &= \frac{3}{5+3}=\frac{3}{8}
\end{aligned}$$

.

Marciel will run for $\frac{3}{8}$ of an hour.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Multiplying and Dividing Rational Expressions (9:19)

Figure 7.9: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/841

In 1–20, perform the indicated operation and reduce the answer to lowest terms

1. $\frac{x^3}{2y^3} \cdot \frac{2y^2}{x}$
2. $2xy \div \frac{2x^2}{y}$
3. $\frac{2x}{y^2} \cdot \frac{4y}{5x}$
4. $2xy \cdot \frac{2y^2}{x^3}$
5. $\frac{4y^2-1}{y^2-9} \cdot \frac{y-3}{2y-1}$
6. $\frac{6ab}{a^2} \cdot \frac{a^3b}{3b^2}$
7. $\frac{x^2}{x-1} \div \frac{x}{x^2+x-2}$
8. $\frac{33a^2}{-5} \cdot \frac{20}{11a^3}$
9. $\frac{a^2+2ab+b^2}{ab^2-a^2b} \div (a+b)$
10. $\frac{2x^2+2x-24}{x^2+3x} \cdot \frac{x^2+x-6}{x+4}$
11. $\frac{3-x}{3x-5} \div \frac{x^2-9}{2x^2-8x-10}$
12. $\frac{x^2-25}{x+3} \div (x-5)$
13. $\frac{2x+1}{2x-1} \div \frac{4x^2-1}{1-2x}$
14. $\frac{x}{x-5} \cdot \frac{x^2-8x+15}{x^2-3x}$
15. $\frac{3x^2+5x-12}{x^2-9} \div \frac{3x-4}{3x+4}$
16. $\frac{5x^2+16x+3}{36x^2-25} \cdot (6x^2+5x)$
17. $\frac{x^2+7x+10}{x^2-9} \cdot \frac{x^2-3x}{3x^2+4x-4}$
18. $\frac{x^2+x-12}{x^2+4x+4} \div \frac{x-3}{x+2}$
19. $\frac{x^4-16}{x^2-9} \div \frac{x^2+4}{x^2+6x+9}$
20. $\frac{x^2+8x+16}{7x^2+9x+2} \div \frac{7x+2}{x^2+4x}$
21. Maria's recipe asks for $2\frac{1}{2}$ times more flour than sugar. How many cups of flour should she mix in if she uses $3\frac{1}{3}$ cups of sugar?
22. George drives from San Diego to Los Angeles. On the return trip, he increases his driving speed by 15 miles per hour. In terms of his initial speed, by what factor is the driving time decreased on the

return trip?

23. Ohm's Law states that in an electrical circuit $I = \frac{V}{R_c}$. The total resistance for resistors placed in parallel is given by $\frac{1}{R_{tot}} = \frac{1}{R_1} + \frac{1}{R_2}$. Write the formula for the electric current in term of the component resistances: R_1 and R_2.

Mixed Review

24. The time it takes to reach a destination varies inversely as the speed in which you travel. It takes 3.6 hours to reach your destination traveling 65 miles per hour. How long would it take to reach your destination traveling 78 miles per hour?
25. A local nursery makes two types of fall arrangements. One arrangement uses eight mums and five black-eyed susans. The other arrangement uses six mums and 9 black-eyed susans. The nursery can use no more than 144 mums and 135 black-eyed susans. The first arrangement sells for $49.99 and the second arrangement sells for 38.95. How many of each type should be sold to maximize revenue?
26. Solve for r and graph the solution on a number line: $-24 \geq |2r + 3|$.
27. What is true of any line parallel to $5x + 9y = -36$?
28. Solve for $d : 3 + 5d = -d - (3x - 3)$.
29. Graph and determine the domain and range: $y - 9 = -x^2 - 5x$.
30. Rewrite in vertex form by completing the square. Identify the vertex: $y^2 - 16y + 3 = 4$.

Quick Quiz

1. h is inversely proportional to t. If $t = -0.05153$ when $h = -16$, find t when $h = 1.45$.
2. Use $f(x) = \frac{-5}{x^2-25}$ for the following questions.

   (a) Find the excluded values.
   (b) Determine the vertical asymptotes.
   (c) Sketch a graph of this function.
   (d) Determine its domain and range.

3. Simplify $\frac{8c^4+12c^2-22c+1}{4}$.
4. Simplify $\frac{10a^2-30a}{a-3}$. What are its excluded values?
5. Fill the blank with *directly, inversely,* or *neither.* ";The amount of time it takes to mow the lawn varies ________________________ with the size of the lawn mower.";

7.6 Addition and Subtraction of Rational Expressions

Like numerical fractions, rational expressions represent a part of a whole quantity. Remember when adding or subtracting fractions, the denominators must be the same. Once the denominators are identical, the numerators are combined by adding or subtracting like terms.

Example 1: Simplify $\frac{4x^2-3}{x+5} + \frac{2x^2-1}{x+5}$.

Solution: The denominators are identical; therefore we can add the like terms of the numerator to simplify.

$$\frac{4x^2 - 3}{x + 5} + \frac{2x^2 - 1}{x + 5} = \frac{6x^2 - 4}{x + 5}$$

Not all denominators are the same however. In the case of **unlike denominators,** common denominators must be created through multiplication by finding the **least common multiple.**

The **least common multiple (LCM)** is the smallest number that is evenly divisible by every member of the set.

What is the least common multiple of $2, 4x$, and $6x^2$? The smallest number 2, 4, and 6 can divide into evenly is six. The largest exponent of x is 2. Therefore, the LCM of $2, 4x$, and $6x^2$ is $6x^2$.

Example 2: Find the least common multiple of $2x^2 + 8x + 8$ *and* $x^3 - 4x^2 - 12x$.

Solution: Factor the polynomials completely.

$$2x^2 + 8x + 8 = 2(x^2 + 4x + 4) = 2(x + 2)^2$$
$$x^3 - 4x^2 - 12x = x(x^2 - 4x - 12) = x(x - 6)(x + 2)$$

The LCM is found by taking each factor to the highest power that it appears in either expression. $LCM = 2x(x + 2)^2(x - 6)$

Use this approach to add rational expressions with unlike denominators.

Example: Add $\frac{2}{x+2} - \frac{3}{2x-5}$.

Solution: The denominators cannot be factored any further, so the LCM is just the product of the separate denominators.

$$LCD = (x + 2)(2x - 5)$$

The first fraction needs to be multiplied by the factor $(2x-5)$ and the second fraction needs to be multiplied by the factor $(x + 2)$.

$$\frac{2}{x+2} \cdot \frac{(2x-5)}{(2x-5)} - \frac{3}{2x-5} \cdot \frac{(x+2)}{(x+2)}$$

We combine the numerators and simplify.

$$\frac{2(2x-5) - 3(x+2)}{(x+2)(2x-5)} = \frac{4x - 10 - 3x - 6}{(x+2)(2x-5)}$$

Combine like terms in the numerator.

$$\frac{x-16}{(x+2)(2x-5)}\text{Answer}$$

Work Problems

These are problems where two objects work together to complete a job. Work problems often contain rational expressions. Typically we set up such problems by looking at the part of the task completed by each person or machine. The completed task is the sum of the parts of the tasks completed by each individual or each machine.

Part of task completed by first person + Part of task completed by second person = One completed task

To determine the part of the task completed by each person or machine, we use the following fact.

Part of the task completed = rate of work × time spent on the task

In general, it is very useful to set up a table where we can list all the known and unknown variables for each person or machine and then combine the parts of the task completed by each person or machine at the end.

Example: *Mary can paint a house by herself in 12 hours. John can paint a house by himself in 16 hours. How long would it take them to paint the house if they worked together?*

Solution: Let $t =$ *the time it takes Mary and John to paint the house together.*

Since Mary takes 12 hours to paint the house by herself, in one hour she paints $\frac{1}{12}$ of the house.

Since John takes 16 hours to paint the house by himself, in one hour he paints $\frac{1}{16}$ of the house.

Mary and John work together for t hours to paint the house together. Using the formula:

Part of the task completed = rate of work multiplied by the time spent on the task

we can write that Mary completed $\frac{t}{12}$ of the house and John completed $\frac{t}{16}$ of the house in this time and summarize the data in the following table.

Table 7.2:

Painter	Rate of work (per hour)	Time worked	Part of Task
Mary	$\frac{1}{12}$	t	$\frac{t}{12}$
John	$\frac{1}{16}$	t	$\frac{t}{16}$

Using the formula:

Part of task completed by first person + Part of task completed by second person = One completed task

Write an equation to model the situation.

$$\frac{t}{12} + \frac{t}{16} = 1.$$

Solve the equation by finding the least common multiple.

$$\begin{aligned}
LCM &= 48 \\
48 \cdot \frac{t}{12} + 48 \cdot \frac{t}{16} &= 48 \cdot 1 \\
\cancel{48}^4 \cdot \frac{t}{\cancel{12}} + \cancel{48}^3 \cdot \frac{t}{\cancel{16}} &= 48 \cdot 1 \\
4t + 3t &= 48 \\
7t = 48 \Rightarrow t &= \frac{48}{7} = 6.86 \textit{ hours Answer}
\end{aligned}$$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following videos. Note that there is not always a match between the number of the practice exercise in the videos and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Adding Rational Expressions Example 1 (3:47)

CK-12 Basic Algebra: Adding Rational Expressions Example 2 (6:40)

Figure 7.10: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/843

Figure 7.11: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/844

CK-12 Basic Algebra: Adding Rational Expressions Example 3 (6:23)

Figure 7.12: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/845

Perform the indicated operation and simplify. Leave the denominator in factored form.

1. $\frac{5}{24} - \frac{7}{24}$
2. $\frac{10}{21} + \frac{9}{35}$
3. $\frac{5}{2x+3} + \frac{3}{2x+3}$
4. $\frac{3x-1}{x+9} - \frac{4x+3}{x+9}$
5. $\frac{4x+7}{2x^2} - \frac{3x-4}{2x^2}$
6. $\frac{x^2}{x+5} - \frac{25}{x+5}$
7. $\frac{2x}{x-4} + \frac{x}{4-x}$
8. $\frac{10}{3x-1} - \frac{7}{1-3x}$
9. $\frac{5}{2x+3} - 3$
10. $\frac{5x+1}{x+4} + 2$
11. $\frac{1}{x} + \frac{2}{3x}$
12. $\frac{4}{5x^2} - \frac{2}{7x^3}$
13. $\frac{4x}{x+1} - \frac{2}{2(x+1)}$

14. $\frac{10}{x+5} + \frac{2}{x+2}$
15. $\frac{2x}{x-3} - \frac{3x}{x+4}$
16. $\frac{4x-3}{2x+1} + \frac{x+2}{x-9}$
17. $\frac{x^2}{x+4} - \frac{3x^2}{4x-1}$
18. $\frac{2}{5x+2} - \frac{x+1}{x^2}$
19. $\frac{x+4}{2x} + \frac{2}{9x}$
20. $\frac{5x+3}{x^2+x} + \frac{2x+1}{x}$
21. $\frac{4}{(x+1)(x-1)} - \frac{5}{(x+1)(x+2)}$
22. $\frac{2x}{(x+2)(3x-4)} + \frac{7x}{(3x-4)^2}$
23. $\frac{3x+5}{x(x-1)} - \frac{9x-1}{(x-1)^2}$
24. $\frac{1}{(x-2)(x-3)} + \frac{4}{(2x+5)(x-6)}$
25. $\frac{3x-2}{x-2} + \frac{1}{x^2-4x+4}$
26. $\frac{-x^2}{x^2-7x+6} + x - 4$
27. $\frac{2x}{x^2+10x+25} - \frac{3x}{2x^2+7x-15}$
28. $\frac{1}{x^2-9} + \frac{2}{x^2+5x+6}$
29. $\frac{-x+4}{2x^2-x-15} + \frac{x}{4x^2+8x-5}$
30. $\frac{4}{9x^2-49} - \frac{1}{3x^2+5x-28}$
31. One number is 5 less than another. The sum of their reciprocals is $\frac{13}{36}$. Find the two numbers.
32. One number is 8 times more than another. The difference in their reciprocals is $\frac{21}{20}$. Find the two numbers.
33. A pipe can fill a tank full of oil in 4 hours and another pipe can empty the tank in 8 hours. If the valves to both pipes are open, how long would it take to fill the tank?
34. Stefan could wash the cars by himself in 6 hours and Misha could wash the cars by himself in 5 hours. Stefan starts washing the cars by himself, but he needs to go to his football game after 2.5 hours. Misha continues the task. How long did it take Misha to finish washing the cars?
35. Amanda and her sister Chyna are shoveling snow to clear their driveway. Amanda can clear the snow by herself in three hours and Chyna can clear the snow by herself in four hours. After Amanda has been working by herself for one hour, Chyna joins her and they finish the job together. How long does it take to clear the snow from the driveway?
36. At a soda bottling plant, one bottling machine can fulfill the daily quota in ten hours and a second machine can fill the daily quota in 14 hours. The two machines started working together but after four hours the slower machine broke and the faster machine had to complete the job by itself. How many hours does the fast machine work by itself?

Mixed Review

37. Explain the difference between these two situations. Write an equation to model each situation. Assume the town started with 10,000 people. When will statement b become larger than statement a?
    (a) For the past seven years, the population grew by 500 people every year.
    (b) For the past seven years, the population grew by 5% every year.
38. Simplify. Your answer should have only positive exponents. $\frac{16x^2y^7}{-2x^8y} \cdot \frac{1}{2}x^{-10}$
39. Solve for $j : -12 = j^2 - 8j$. Which method did you use? Why did you choose this method?
40. Jimmy shot a basketball from a height of four feet with an upward velocity of 12 *feet/sec.*
    (a) Write an equation to model this situation.

(b) Will Jimmy's ball make it to the ten-foot-tall hoop?

41. The distance you travel varies directly as the speed at which you drive. If you can drive 245 miles in five hours, how long will it take you to drive 90 miles?
42. Two cities are 3.78 centimeters apart on an atlas. The atlas has a scale of $\frac{1}{2}cm = 14\ miles$. What is the true distance between these cities?

7.7 Solution of Rational Equations

You are now ready to solve rational equations! There are two main methods you will learn in this lesson to solve rational equations:

- Cross products
- Lowest common denominators

Solving a Rational Proportion

When two rational expressions are equal, a **proportion** is created and can be solved using its cross products. For example, to solve $\frac{x}{5} = \frac{(x+1)}{2}$, cross multiply and the products are equal.

$$\frac{x}{5} = \frac{(x+1)}{2} \rightarrow 2(x) = 5(x+1)$$

Solve for x:

$$\begin{aligned} 2(x) = 5(x+1) &\rightarrow 2x = 5x + 5 \\ 2x - 5x &= 5x - 5x + 5 \\ -3x &= 5 \\ x &= -\frac{5}{3} \end{aligned}$$

Example 1: Solve $\frac{2x}{x+4} = \frac{5}{x}$.

Solution:

$$\begin{aligned} \frac{2x}{x+4} = \frac{5}{x} &\rightarrow 2x^2 = 5(x+4) \\ 2x^2 = 5(x+4) &\rightarrow 2x^2 = 5x + 20 \\ 2x^2 - 5x - 20 &= 0 \end{aligned}$$

Notice that this equation has a degree of two, that is, it is a *quadratic equation.* We can solve it using the quadratic formula.

$$x = \frac{5 \pm \sqrt{185}}{4} \Rightarrow x \approx -2.15 \text{ or } x \approx 4.65$$

Solving by Clearing Denominators

When a rational equation has several terms, it may not be possible to use the method of cross products. A second method to solve rational equations is to **clear the fractions** by multiplying the entire equation by the least common multiple of the denominators.

Example: *Solve* $\frac{3}{x+2} - \frac{4}{x-5} = \frac{2}{x^2-3x-10}$.

Solution: Factor all denominators and find the least common multiple.

$$\frac{3}{x+2} - \frac{4}{x-5} - \frac{2}{(x+2)(x-5)}$$
$$LCM = (x+2)(x-5)$$

Multiply all terms in the equation by the LCM and cancel the common terms.

$$(x+2)(x-5)\cdot\frac{3}{x+2} - (x+2)(x-5)\cdot\frac{4}{x-5} = (x+2)(x-5)\cdot\frac{2}{(x+2)(x-5)}$$
$$\cancel{(x+2)}(x-5)\cdot\frac{3}{\cancel{x+2}} - (x+2)\cancel{(x-5)}\cdot\frac{4}{\cancel{x-5}} = \cancel{(x+2)(x-5)}\cdot\frac{2}{\cancel{(x+2)(x-5)}}$$

Now solve and simplify.

$$3(x-5) - 4(x+2) = 2$$
$$3x - 15 - 4x - 8 = 2$$
$$x = -25 \text{ Answer}$$

Check your answer.

$$\frac{3}{x+2} - \frac{4}{x-5} = \frac{3}{-25+2} - \frac{4}{-25-5} = 0.003$$
$$\frac{2}{x^2-3x-10} = \frac{2}{(-25)^2-3(-25)-10} = 0.003$$

Example: *A group of friends decided to pool their money together and buy a birthday gift that cost \$200. Later 12 of the friends decided not to participate any more. This meant that each person paid \$15 more than the original share. How many people were in the group to start?*

Solution: Let $x =$ the number of friends in the original group

Table 7.3:

	Number of People	**Gift Price**	**Share Amount**
Original group	x	200	$\frac{200}{x}$
Later group	$x-12$	200	$\frac{200}{x-12}$

Since each person's share went up by \$15 after 2 people refused to pay, we write the equation:

$$\frac{200}{x-12} = \frac{200}{x} + 15$$

Solve by clearing the fractions. Don't forget to check!

$$
\begin{aligned}
LCM &= x(x-12) \\
x(x-12)\cdot\frac{200}{x-12} &= x(x-12)\cdot\frac{200}{x} + x(x-12)\cdot 15 \\
\cancel{x(x-12)}\cdot\frac{200}{\cancel{x-12}} &= \cancel{x}(x-12)\cdot\frac{200}{\cancel{x}} + x(x-12)\cdot 15 \\
200x &= 200(x-12) + 15x(x-12) \\
200x &= 200x - 2400 + 15x^2 - 180x \\
0 &= 15x^2 - 180x - 2400 \\
x &= 20, x = -8
\end{aligned}
$$

The answer is 20 people. We discard the negative solution since it does not make sense in the context of this problem.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following videos. Note that there is not always a match between the number of the practice exercise in the videos and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.

CK-12 Basic Algebra: Solving Rational Equations (12:57)

Figure 7.13: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/846

CK-12 Basic Algebra: Two More Examples of Solving Rational Equations (9:58)

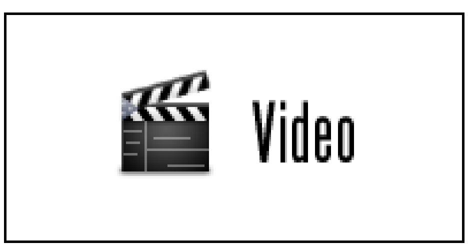


Figure 7.14: (Watch Youtube Video)
http://www.ck12.org/flexbook/embed/view/847

Solve the following equations.

1. $\frac{2x+1}{4} = \frac{x-3}{10}$
2. $\frac{4x}{x+2} = \frac{5}{9}$
3. $\frac{5}{3x-4} = \frac{2}{x+1}$
4. $\frac{7x}{x-5} = \frac{x+3}{x}$
5. $\frac{2}{x+3} - \frac{1}{x+4} = 0$
6. $\frac{3x^2+2x-1}{x^2-1} = -2$
7. $x + \frac{1}{x} = 2$
8. $-3 + \frac{1}{x+1} = \frac{2}{x}$
9. $\frac{1}{x} - \frac{x}{x-2} = 2$
10. $\frac{3}{2x-1} + \frac{2}{x+4} = 2$
11. $\frac{2x}{x-1} - \frac{x}{3x+4} = 3$
12. $\frac{x+1}{x-1} + \frac{x-4}{x+4} = 3$
13. $\frac{x}{x-2} + \frac{x}{x+3} = \frac{1}{x^2+x-6}$
14. $\frac{2}{x^2+4x+3} = 2 + \frac{x-2}{x+3}$
15. $\frac{1}{x+5} - \frac{1}{x-5} = \frac{1-x}{x+5}$
16. $\frac{x}{x^2-36} + \frac{1}{x-6} = \frac{1}{x+6}$
17. $\frac{2x}{3x+3} - \frac{1}{4x+4} = \frac{2}{x+1}$
18. $\frac{-x}{x-2} + \frac{3x-1}{x+4} = \frac{1}{x^2+2x-8}$
19. Juan jogs a certain distance and then walks a certain distance. When he jogs he averages seven miles per hour. When he walks, he averages 3.5 miles per hour. If he walks and jogs a total of six miles in a total of seven hours, how far does he jog and how far does he walk?
20. A boat travels 60 miles downstream in the same time as it takes it to travel 40 miles upstream. The boat's speed in still water is 20 miles/hour. Find the speed of the current.
21. Paul leaves San Diego driving at 50 miles/hour. Two hours later, his mother realizes that he forgot something and drives in the same direction at 70 miles/hour. How long does it take her to catch up to Paul?
22. On a trip, an airplane flies at a steady speed against the wind. On the return trip the airplane flies with the wind. The airplane takes the same amount of time to fly 300 miles against the wind as it takes to fly 420 miles with the wind. The wind is blowing at 30 miles/hour. What is the speed of the airplane when there is no wind?
23. A debt of $420 is shared equally by a group of friends. When five of the friends decide not to pay, the share of the other friends goes up by $25. How many friends were in the group originally?
24. A non-profit organization collected $2,250 in equal donations from their members to share the cost of improving a park. If there were thirty more members, then each member could contribute $20 less. How many members does this organization have?

Mixed Review

25. Divide $-2\frac{9}{10} \div -\frac{15}{8}$.
26. Solve for $g : -1.5\left(-3\frac{4}{5} + g\right) = \frac{201}{20}$.
27. Find the discriminant of $6x^2 + 3x + 4 = 0$ and determine the nature of the roots.
28. Simplify $\frac{6b}{2b+2} + 3$.
29. Simplify $\frac{8}{2x-4} - \frac{5x}{x-5}$.
30. Divide $(7x^2 + 16x - 10) \div (x + 3)$.
31. Simplify $(n - 1) * (3n + 2)(n - 4)$.

7.8 Chapter Review

Define the following terms used in this chapter.

1. Inverse variation
2. Asymptotes
3. Hyperbola
4. Points of discontinuity
5. Least common multiple
6. Random sampling
7. Stratified sampling
8. Biased
9. Cherry picking
10. What quadrants are the branches of the hyperbola located if $k < 0$?

Are the following examples of *direct variation* or *inverse variation?*

11. The number of slices n people get from sharing one pizza
12. The thickness of a phone book given n telephone numbers
13. The amount of coffee n people receive from a single pot
14. The total cost of pears given the nectarines cost \$0.99 per pound

For each variation equation:

1. Translate the sentence into an inverse variation equation.
2. Find k, the constant of variation.
3. Find the unknown value.

15. y varies inversely as x. When $x = 5, y = \frac{2}{15}$. Find y when $x = -\frac{1}{2}$.
16. y is inversely proportional to the square root of y. When $x = 16, y = 0.5625$. Find y when $x = \frac{1}{8}$.
17. Habitat for Humanity uses volunteers to build houses. The number of days it takes to build a house varies inversely as the number of volunteers. It takes eight days to build a house with twenty volunteers. How many days will it take sixteen volunteers to complete the same job?
18. The Law of the Fulcrum states the distance you sit to balance a seesaw varies inversely as your weight. If Gary weighs 20.43 kg and sits 1.8 meters from the fulcrum, how far would Shelley sit, assuming she weighs 36.32 kilograms?

For each function:

1. Graph it on a Cartesian plane.
2. State its domain and range.
3. Determine any horizontal and/or vertical asymptotes the function may have.

19. $y = \frac{4}{x}$
20. $f(x) = \frac{2}{4-x}$
21. $g(x) = \frac{-1}{x+1}$
22. $y = \frac{6}{3x+1} - 2$
23. $f(x) = \frac{3}{x} - 5$

Perform the indicated operation.

24. $\frac{5a}{6} - \frac{5b}{4b}$
25. $\frac{4}{3m} + \frac{4m}{5}$
26. $\frac{3x}{2xy} + \frac{4}{3}$
27. $\frac{2}{5n-2} + \frac{2n}{2}$
28. $\frac{2x+1}{3x+9} - \frac{x+5}{3x+9}$
29. $\frac{5m+n}{30n^4} - \frac{4m+n}{30n^4}$
30. $\frac{r-6}{4r^2-12r+8} - \frac{r+6}{4r^2-12r+8}$
31. $\frac{2}{16x^3y^2} + \frac{x-2y}{16x^3y^2}$
32. $\frac{n-6}{n+2} + \frac{2n}{5}$
33. $\frac{8}{4} - \frac{x+5}{x+8}$
34. $\frac{3x}{2(x+1)} + \frac{6}{7x-6}$
35. $\frac{11}{8} \cdot \frac{20x^2}{2}$
36. $\frac{17r}{16} \cdot \frac{7r^4}{16}$
37. $\frac{15}{18} \cdot \frac{14}{17t}$
38. $\frac{2(b-11)}{14b} \cdot \frac{b+5}{(b+5)(b-11)}$
39. $\frac{17w^2}{w+4} \cdot \frac{18(w+4)}{17w^2(w-9)}$
40. $\frac{10s^3-30s^2}{30s^2-10s^3} \cdot \frac{s-3}{8}$
41. $\frac{1}{f-5} \div \frac{f+3}{f^2+6f+9}$
42. $\frac{(a+8)(a+3)}{4(a+3)} \div \frac{10a^2(a+10)}{4}$
43. $\frac{1}{(h-10)(h+7)} \div \frac{(h-4)}{4h(h-10)}$
44. $\frac{2(5x-8)}{4x^2(8-5x)} \div \frac{6}{4x^2}$
45. $\frac{2(q-7)}{40q(q+1)} \div \frac{1}{40q(q+1)}$

Solve each equation.

46. $\frac{3}{3x^2} = \frac{1}{x} + \frac{1}{3x^2}$
47. $\frac{2}{5x^2} = -\frac{12}{x-3}$
48. $\frac{7x}{x-6} = \frac{3}{4x+16}$
49. $\frac{4}{c-2} = \frac{3}{c+4}$
50. $\frac{d-4}{4d^2} = \frac{1}{4d^2} + \frac{1}{4d}$
51. $\frac{1}{2} = \frac{2z-12}{z} - \frac{z+1}{4z}$
52. $\frac{1}{n} = \frac{1}{n^2} + \frac{6}{n}$
53. $\frac{1}{2a} = \frac{1}{2a^2} + \frac{1}{a}$
54. $\frac{k+4}{k^2} = \frac{5k-30}{3k^2} + \frac{1}{3k^2}$
55. It takes Jayden seven hours to paint a room. Andie can do it in five hours. How long will it take to paint the room if Jayden and Andie work together?
56. Kiefer can mow the lawn in 4.5 hours. Brad can do it in two hours. How long will it take if they worked together?
57. Melissa can mop the floor in 1.75 hours. With Brad's help, it took only 50 minutes. How long would it take Brad to mop it alone?
58. Working together, it took Frankie and Ricky eight hours to frame a room. It would take Frankie fifteen hours doing it alone. How long would it take Ricky to do it alone?

59. A parallel circuit has $R_1 = 50\Omega$ and $R_t = 16\Omega$. *Find* R_2.
60. A parallel circuit has $R_1 = 6\Omega$ and $R_2 = 9\Omega$. *Find* R_T.
61. A series circuit has $R_1 = 200\Omega$ and $R_t = 300\Omega$. *Find* R_2.
62. A series circuit has $R_1 = 11\Omega$ and $R_2 = 25\Omega$. *Find* R_T.
63. Write the formula for the total resistance for a parallel circuit with three individual resistors.

7.9 Chapter Test

1. *True or false?* A horizontal asymptote has the equation $y = c$ and represents where the denominator of the rational function is equal to zero.
2. Consider $f(x) = -\frac{4}{x}$. State its domain, range, asymptotes, and the locations of its branches.
3. *h varies inversly as r. When* $h = -2.25, r = 0.125$. *Find h when* $r = 12$.
4. Find the excluded values, the domain, the range, and the asymptotes of:
$$f(x) = -\frac{9}{x^2 - 16} + 4.$$

Perform the indicated operation.

10. $\frac{4}{21r^4} + \frac{4r+5t}{21r^4}$
11. $\frac{a-v}{12a^3} - \frac{a+5v}{12a^3}$
12. $\frac{8}{g+8} + \frac{g-3}{g-5}$
13. $\frac{4t}{5t-8} + \frac{24}{12}$
14. $\frac{4}{5} \cdot \frac{80}{48m}$
15. $\frac{1}{d-8} \div \frac{d+7}{2d+14}$
16. $\frac{1}{u-3} \div \frac{u-4}{2u-6}$

Solve.

17. $\frac{7w}{w-7} = \frac{7w}{w+5}$
18. $\frac{p-6}{3p^2-6p} = \frac{7}{3}$
19. $\frac{2}{x^2} = \frac{1}{2x^2} - \frac{x+1}{2x^2}$
20. $\frac{1}{2} - \frac{1}{4r} = \frac{3}{4}$
21. $\frac{y-5}{3y^2} = -\frac{1}{3y} + \frac{1}{y^2}$
22. Working together, Ashton and Matt can tile a floor in 25 minutes. Working alone, it would take Ashton two hours. How long would it take Matt to tile the floor alone?
23. Bethany can paint the deck in twelve hours. Melissa can paint the deck in five hours. How long would it take the girls to paint the deck, working together?
24. A parallel circuit has $R_t = 115\ \Omega$ *and* $R_2 = 75\ \Omega$. *Find* R_1.
25. A series circuit has $R_1 = 13\ \Omega$ *and* $R_t = 21\ \Omega$. *Find* R_2.

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9622.